Massachusetts Institute of Technology

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6.243j (Fall 2003): DYNAMICS OF NONLINEAR SYSTEMS

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Lecture 9: Local Behavior Near Trajectories[1]

This lecture presents results which describe local behavior of ODE models in a neigborhood of a given trajectory, with main attention paid to local stability of periodic solutions.

9.1 Smooth Dependence on Parameters

In this section we consider an ODE model

$$\dot{x}(t) = a(x(t), t, \mu), \quad x(t_0) = \bar{x}_0(\mu), \tag{9.1}$$

where μ is a parameter. When a and $\bar{x}_0$ are differentiable with respect to μ, the solution $x(t) = x(t, \mu)$ is differentiable with respect to μ as well. Moreover, the derivative of $x(t, \mu)$ with respect to μ can be found by solving linear ODE with time-varying coefficients.

Theorem 9.1 *Let $a: \mathbf{R}^n \times \mathbf{R} \times \mathbf{R}^k \mapsto \mathbf{R}^n$ be a continuous function, $\mu_0 \in \mathbf{R}^k$. Let $x_0: [t_0, t_1] \mapsto \mathbf{R}^n$ be a solution of (9.1) with $\mu = \mu_0$. Assume that a is continuously differentiable with respect to its first and third arguments on an open set X such that $(x_0(t), t, \mu_0) \in X$ for all $t \in [t_0, t_1]$. Then for all μ in a neigborhood of μ_0 the ODE in (9.1) has a unique solution $x(t) = x(t, \mu)$. This solution is a continuously differentiable function of μ, and its derivative with respect to μ at $\mu = \mu_0$ equals $\Delta(t)$, where $\Delta: [t_0, t_1] \mapsto \mathbf{R}^{n,k}$ is the n-by-k matrix-valued solution of the ODE*

$$\dot{\Delta}(t) = A(t)\Delta(t) + B(t), \quad \Delta(t_0) = \Delta_0, \tag{9.2}$$

where $A(t)$ is the derivative of the map $\bar{x} \mapsto a(\bar{x}, t, \mu_0)$ with respect to $\bar{x}$ at $\bar{x} = x_0(t)$, $B(t)$ is the derivative of the map $\mu \mapsto a(x_0(t), t, \mu)$ at $\mu = \mu_0$, and Δ_0 is the derivative of the map $\mu \mapsto \bar{x}_0(\mu)$ at $\mu = \mu_0$.

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Proof Existence and uniqueness of $x(t,\mu)$ and $\Delta(t)$ follow from Theorem 3.1. Hence, in order to prove differentiability and the formula for the derivative, it is sufficient to show that there exist a function $C: \mathbf{R}_+ \mapsto \mathbf{R}_+$ such that $C(r)/r \to 0$ as $r \to 0$ and $\epsilon > 0$ such that

$$|x(t,\mu) - \Delta(t)(\mu - \mu_0) - x_0(t)| \leq C(|\mu - \mu_0|)$$

whenever $|\mu - \mu_0| \leq \epsilon$. Indeed, due to continuous differentiability of a, there exist $C_1, \epsilon 0$ such that

$$|a(\bar{x}, t, \mu) - a(x_0(t), t, \mu_0) - A(t)(\bar{x} - x_0(t)) - B(t)(\mu - \mu_0)| \leq C_1(|\bar{x} - \bar{x}_0(t)| + |\mu - \mu_0|)$$

and

$$|\bar{x}_0(\mu) - \bar{x}_0(\mu_0) - \Delta_0(\mu - \mu_0)| \leq C_1(|\mu - \mu_0|)$$

whenever

$$|\bar{x} - \bar{x}_0(t)| + |\mu - \mu_0| \leq \epsilon, \quad t \in [t_0, t_1].$$

Hence, for

$$\delta(t) = x(t,\mu) - x_0(t) - \Delta(t)(\mu - \mu_0)$$

we have

$$|\dot{\delta}(t)| \leq C_2|\delta(t)| + C_3(|\mu - \mu_0|),$$

as long as $|\delta(t)| \leq \epsilon_1$ and $|\mu - \mu_0| \leq \epsilon_1$, where $\epsilon_1 > 0$ is sufficiently small. Together with

$$|\delta(t_0)| \leq C_4(|\mu - \mu_0|),$$

this implies the desired bound. ■

Example 9.1 Consider the differential equation

$$\dot{y}(t) = 1 + \mu \sin(y(t)), \quad y(0) = 0,$$

where μ is a small parameter. For $\mu = 0$, the equation can be solved explicitly: $y_0(t) = t$. Differentiating $y_\mu(t)$ with respect to μ at $\mu = 0$ yields $\Delta(t)$ satisfying

$$\dot{\Delta}(t) = \sin(t), \quad \Delta(0) = 0,$$

i.e. $\Delta(t) = 1 - \cos(t)$. Hence

$$y_\mu(t) = t + \mu(1 - \cos(t)) + O(\mu^2)$$

for small μ.

9.2 Stability of periodic solutions

In the previous lecture, we were studying stability of equilibrium solutions of differential equations. In this section, stability of periodic solutions of nonlinear differential equations is considered. Our main objective is to derive an analog of the Lyapunov's first method, stating that a periodic solution is asymptotically stable if system's linearization around the solution is stable in a certain sense.

9.2.1 Periodic solutions of time-varying ODE

Consider system equations given in the form

$$\dot{x}(t) = f(x(t), t), \tag{9.3}$$

where $f : \mathbf{R}^n \times \mathbf{R} \mapsto \mathbf{R}^n$ is continuous. Assume that a is $(\tau, \hat{x})$-*periodic*, in the sense that there exist $\tau > 0$ and $\hat{x} \in \mathbf{R}^n$ such that

$$f(t+\tau, r) = f(t, r), \quad f(t, r+\hat{x}) = f(t, r) \ \ \forall\ t \in \mathbf{R},\ r \in \mathbf{R}^n. \tag{9.4}$$

Note that while the first equation in (9.4) means that f is periodic in t with a period τ, it is possible that $\hat{x} = 0$, in which case the second equation in (9.4) does not bring any additional information.

Definition A solution $x_0 : \mathbf{R} \mapsto \mathbf{R}^n$ of a $(\tau, \hat{x})$-periodic system (9.3) is called $(\tau, \hat{x})$ *periodic* if

$$x_0(t+\tau) = x_0(t) + \hat{x} \ \ \forall\ t \in \mathbf{R}. \tag{9.5}$$

Example 9.2 According to the definition, the solution $y(t) = t$ of the forced pendulum equation

$$\ddot{y}(t) + \dot{y}(t) + \sin(y(t)) = 1 + \sin(t) \tag{9.6}$$

as a periodic one (use $\tau = \hat{x} = 2\pi$). This is reasonable, since $y(t)$ in the pendulum equation represents an angle, so that shifting y by 2π does not change anything in the system equations.

Definition A solution $x_0 : [t_0, \infty) \mapsto \mathbf{R}^n$ of (9.3) is called *stable* if for every $\delta > 0$ there exists $\epsilon > 0$ such that

$$|x(t) - x_0(t)| \leq \delta\ \forall\ t \geq 0, \tag{9.7}$$

whenever $x(\cdot)$ is a solution of (9.3) such that $|x(0) - x_0(0)| < \epsilon$. $x_0(\cdot)$ is *asymptotically stable* if it is stable and the convergence $|x(t) - x_0(t)| \to 0$ is guaranteed as long as $|x(0) - x_0(0)|$ is small enough. $x_0(\cdot)$ *exponentially stable* if, in addition, there exist $\sigma, C > 0$ such that

$$\|x(t) - x_0(t)\| \leq C\exp(-\sigma t)|x(0) - x_0(0)| \ \ \forall\ t \geq 0 \tag{9.8}$$

whenever $|x(0) - x_0(0)|$ is small enough.

To derive a stability criterion for periodic solutions $x_0 : \mathbf{R} \mapsto \mathbf{R}^n$ of (9.3), assume continuous differentiability of function $f = f(\bar{x}, t)$ with respect to the first argument $\bar{x}$ for $|\bar{x} - x_0(t)| \leq \epsilon$, where $\epsilon > 0$ is small, and differentiate the solution as a function of initial conditions $x(0) \approx x_0(0)$.

Theorem 9.2 *Let $f : \mathbf{R}^n \times \mathbf{R} \mapsto \mathbf{R}^n$ be a continuous $(\tau, \hat{x})$-periodic function. Let $x_0 : \mathbf{R} \mapsto \mathbf{R}^n$ be a $(\tau, \hat{x})$-periodic solution of (9.3). Assume that there exists $\epsilon > 0$ such that f is continuously differentiable with respect to its first argument for $|\bar{x} - x_0(t)| < \epsilon$ and $t \in \mathbf{R}$. For*

$$A(t) = \frac{df}{dx}|_{x=x_0(t)}, \tag{9.9}$$

define $\Delta : [0, \tau] \mapsto \mathbf{R}^{n,n}$ be the n-by-n matrix solution of the linear ODE

$$\dot{\Delta}(t) = A(t)\Delta(t), \quad \Delta(0) = I. \tag{9.10}$$

Then

(a) $x_0(\cdot)$ is exponentially stable if $\Delta(\tau)$ is a Schur matrix (i.e. if all eigenvalues of $\Delta(\tau)$ have absolute value less than one);

(b) $x_0(\cdot)$ is not exponentially stable if $\Delta(\tau)$ has an eigenvalue with absolute value greater or equal than 1;

(c) $x_0(\cdot)$ is not stable if $\Delta(\tau)$ has an eigenvalue with absolute value greater than 1.

The matrix-valued function $\Delta = \Delta(t)$ is called the *evolution matrix* of linear system (9.10). The proof of Theorem 9.2 follows the same path as the proof of a similar theorem for stability of equilibria, using time-varying quadratic Lyapunov functions.

9.2.2 Stable limit cycles time-invariant ODE

Consider system equations given in the form

$$\dot{x}(t) = a(x(t)), \tag{9.11}$$

where $a : \mathbf{R}^n \times \mathbf{R} \mapsto \mathbf{R}^n$ is continuous. Let $\hat{x} \in \mathbf{R}^n$ be such that

$$a(\bar{x} + \hat{x}) = a(\bar{x}) \ \ \forall \ \bar{x} \in \mathbf{R}^n$$

(in particular, $\hat{x} = 0$ always satisfies this condition).

Definition Let $\tau > 0$. A non-constant $(\tau, \hat{x})$-periodic solution $x_0 : \mathbf{R} \mapsto \mathbf{R}^n$ of system (9.11) is called a *stable limit cycle* if

(a) for every $\epsilon > 0$ there exists $\delta > 0$ such that $\text{dist}(x(t), x_0(\cdot)) < \epsilon$ for all $t \geq 0$ and all solutions $x = x(t)$ of (9.11) such that $\text{dist}(x(0), x_0(\cdot)) < \delta$, where

$$\text{dist}(\bar{x}, x_0(\cdot)) = \min_{t \in \mathbf{R}} |\bar{x} - x_0(t)|;$$

(b) there exists $\epsilon > 0$ such that $\text{dist}(x(t), x_0(\cdot)) \to 0$ as $t \to \infty$ for every solution of (9.11) such that $\text{dist}(x(0), x_0(\cdot)) < \delta$.

Note that a non-constant periodic solution $x_0 = x_0(t)$ of time-invariant ODE equations is *never* asymptotically stable, because, as $\delta \to 0$, the initial conditions for the solution $x_\delta(t) = x_0(t + \delta)$ approach the initial conditions for $x_0(\cdot)$, but the difference $x_\delta(t) - x_0(t)$ does not converge to 0 as $t \to \infty$ unless $x_\delta \equiv x_0$. Therefore, the notion of a stable limit cycle is a relaxed version of asymptotic stability of a solution.

Theorem 9.3 *Let $a: \mathbf{R}^n \mapsto \mathbf{R}^n$ be a continuous $(\hat{x})$-periodic function. Let $x_0: \mathbf{R} \mapsto \mathbf{R}^n$ be a non-constant $(\tau, \hat{x})$-periodic solution of (9.11). Assume that there exists $\epsilon > 0$ such that a is continuously differentiable on the set*

$$X = \{\bar{x} \in \mathbf{R}^n : \ |\bar{x} - x_0(t)| < \epsilon \text{ for some } t \in \mathbf{R}.$$

Let $\Delta: \ [0, \tau] \mapsto \mathbf{R}^{n,n}$ be defined by (9.9),(9.10). Then

(a) if all eigenvalues of $\Delta(\tau)$ except one have absolute value less than 1, $x_0(\cdot)$ is a stable limit cycle;

(b) if one eigenvalue of $\Delta(\tau)$ has absolute value greater than 1, $x_0(\cdot)$ is not a stable limit cycle.