6.241 Spring 2011

Midterm Exam

March 27, 2011

Problem 1

Let $A \in \mathbb{C}^{n\times n}$, and $B \in \mathbb{C}^{m\times m}$. Show that $X(t) = e^{At}X(0)e^{Bt}$ is the solution to $\dot{X} = AX + XB$.

Solution — Recalling the definition of matrix exponential, $e^{At} = \sum_{i=0}^{\infty} \frac{1}{i!}(At)^i$, it is clear that, for any matrix A, $e^{At} = I$ for $t = 0$, and $de^{At}/dt = Ae^{At} = e^{At}A$.

Hence,

$$\begin{aligned}\frac{d}{dt}\left(e^{At}X(0)e^{Bt}\right) &= \left(\frac{d}{dt}e^{At}\right)X(0)e^{Bt} + e^{At}\left(\frac{d}{dt}X(0)\right)e^{Bt} + e^{At}X(0)\left(\frac{d}{dt}e^{Bt}\right)\\ &= A\left(e^{At}X(0)e^{Bt}\right) + 0 + \left(e^{At}X(0)e^{Bt}\right)B.\end{aligned}$$

Furthermore, for $t = 0$,

$$\left(e^{At}X(0)e^{Bt}\right)\big|_{t=0} = X(0).$$

Hence we can conclude that the proposed function is in fact the solution to the initial-value problem under consideration.

Problem 2

Given two non-zero vectors $v, w \in \mathbb{R}^n$. Does there exist a matrix A such that $v = Aw$ and

1. $\sigma_{\max}(A) = \sqrt{v^Tv/w^Tw}$?
2. $\|A\|_1 = \|v\|_\infty/\|w\|_\infty$?

Prove or disprove each case separately.

Solution — We have two cases:

1. The rank-one matrix $A = \frac{1}{w^Tw}vw^T$ has the required properties. Direct substitution shows that this matrix satisfies the condition $v = Aw$. Moreover, the only non-zero eigenvalue of the (rank-one) matrix $A^TA = \frac{1}{(w^Tw)^2}wv^Tvw^T = \frac{v^Tv}{(w^Tw)^2}ww^T$ is equal to $\lambda_{\max}(A^TA) = v^Tv/w^Tw$, from which we get $\sigma_{\max}(A) = \sqrt{\lambda_{\max}(A^TA)} = \sqrt{v^Tv/w^Tw}$.

2. There is no such matrix in general. Consider the following counter-example. Pick, e.g., $v = (1, 1)$, and $w = (1, 0)$. The matrix A must be such that all elements in its first column are equal to 1, and hence $\|A\|_1 \geq 2 > \|v\|_\infty / \|w\|_\infty = 1$.

Problem 3

Use the projection theorem to solve the problem:

$$\min_{x \in \mathbb{R}^n} \{x^T Q x : Ax = b\},$$

where Q is a positive-definite $n \times n$ matrix, A is a $m \times n$ real matrix, with rank $m < n$, and b is a real m-dimensional vector. Is the solution unique?

Solution — (Note that Q being positive-definite implies it is self-adjoint, i.e., Hermitian.) Let x_0 be such that $Ax_0 = b$, and consider the change of variables $z = x - x_0$. In the inner product space $\mathbb{R}^n$, with inner product $\langle u, v \rangle = u^T Q v$, it is desired to minimize $\|x\|^2 = x^T Q x = \|z + x_0\|^2$, subject to the constraint that z lies in the subspace $M := \{z \in \mathbb{R}^n : Az = 0\}$. Using the projection theorem, we know that an optimal solution $\hat{z} = \hat{x} - x_0$ must be such that $(\hat{z} + x_0 = \hat{x}) \perp M$, i.e., $\langle \hat{x}, y \rangle = \hat{x}^T Q y = 0$, for all $y \in M$. Summarizing, we know that

$$\begin{aligned} \hat{x}^T Q y &= 0, \qquad \forall Ay = 0 \\ Ax &= b. \end{aligned}$$

In order to satisfy the first equation for all y such that $Ay = 0$, $\hat{x}$ must be of the form $\hat{x} = Q^{-1} A^T v$, for some $v \in \mathbb{R}^m$. The vector v can be found using the constraint $Ax = b$, i.e.,

$$A\hat{x} = AQ^{-1}A^T v = b,$$

and hence

$$v = (AQ^{-1}A^T)^{-1} b.$$

Concluding,

$$\hat{x} = Q^{-1} A^T (AQ^{-1} A^T)^{-1} b.$$

Problem 4

Let $\|A\| < 1$. Show that $\|(I - A)^{-1}\| \geq \dfrac{1}{1 + \|A\|}$.

Solution—First of all, for any vector x_0, with $\|x_0\| = 1$,

$$\|(I - A)x_0\| \geq \|x_0\| - \|Ax_0\| \geq 1 - \|A\| > 0,$$

which shows that the matrix $I - A$ is invertible, i.e., there is no vector x_0, with $\|x_0\| = 1$, such that $(I - A)x_0 = 0$.

Furthermore, the following chain of inequalities holds:

$$1 = \|I\| = \|(I-A)(I-A)^{-1}\| \leq \|I-A\| \cdot \|(I-A)^{-1}\| \\ \leq (\|I\| + \|A\|) \cdot \|(I-A)^{-1}\| = (1 + \|A\|) \cdot \|(I-A)^{-1}\|,$$

and the result follows. The definition of induced norm implies that $\|I\| = 1$. The first inequality is due to the submultiplicative property of induced norms. The second inequality can be derived from the triangle inequality.

Problem 5

Consider a single-input discrete-time LTI system, described by

$$x[k+1] = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x[k] + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u[k]$$

$$y[k] = x[k],$$

and the initial condition $x[0] = 0$. Given $M > 1$, what is the maximum value of $\|y[M]\|_2$ that can be attained with an input of "unit energy,", i.e., such that $u[0]^2 + u[2]^2 + \ldots + u[M-1]^2 = 1$? What is the input that attains such value? How would your answer change if you were to double M, i.e., $M \leftarrow 2M$?

You can solve this problem symbolically; if you want to get numerical results, it is suggested you use matlab or similar program.

Solution — Let us define

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Then,

$$\begin{aligned} y[1] = x[1] &= Ax[0] + Bu[0] = Bu[0], \\ y[2] = x[2] &= Ax[1] + Bu[1] = ABu[0] + Bu[1], \\ &\ldots \\ y[M] = x[M] &= Ax[M-1] + Bu[M-1] = A^{M-1}Bu[0] + A^{M-2}Bu[1] + \ldots + Bu[M-1], \end{aligned}$$

which can be written as

$$y[M] = \begin{bmatrix} A^{M-1}B & A^{M-2}B & \ldots & B \end{bmatrix} \begin{bmatrix} u[0] \\ u[1] \\ \ldots \\ u[M-1] \end{bmatrix} = \Gamma_M U_M,$$

where

$$\Gamma_M = \begin{bmatrix} A^{M-1}B & A^{M-2}B & \dots & B \end{bmatrix},$$

and

$$U_M = \begin{bmatrix} u[0] & u[1] & \dots & u[M-1] \end{bmatrix}^T.$$

The solution of the problem

$$\begin{array}{ll} \max\limits_{U_M} & \|y[M]\|_2 = \|\Gamma_M U_M\|_2 \\ \text{s.t.} & \|U_M\|_2 = 1 \end{array}$$

is given by $\sigma_{\max}(\Gamma_M)$, and is attained for $U_M = w_{\max}(\Gamma_M)$, where $w_{\max}$ refers to the (right) singular vector associated with the maximum singular value.

Numerically, e.g., for $M = 4$, $\sigma_{\max}(\Gamma_4) = 4.1$, $U_M = \begin{bmatrix} 0.7661 & 0.5452 & 0.3243 & 0.1035 \end{bmatrix}$, and $y[4] = \begin{bmatrix} 3.7129 & 1.7391 \end{bmatrix}$.

The output after $2M$ steps can be written as

$$y[2M] = A^M \Gamma_M U'_M + \Gamma_M U''_M = \Gamma_{2M} U_{2M},$$

where the matrix Γ_{2M} is defined as

$$\Gamma_{2M} = \begin{bmatrix} A^M \Gamma_M & \Gamma_M \end{bmatrix}.$$

Clearly, $\sigma_{\max}(\Gamma_{2M}) \geq \sigma_{\max}(\Gamma_M)$, i.e., $\|y[2M]\|_2$ can be made at least as large as $\|y[M]\|_2$, e.g., by concentrating the energy of the input in the last M steps (and setting the previous ones to zero).

Problem 6

Consider a physical system whose behavior is modeled, in continuous time, by the differential equation

$$\dot{x} = Ax + Bu.$$

Assume that you have two sensors. The first sensor yields measurements $y_1 = C_1 x$ for $t = 0, 1, 2, 3, \dots$, and the second sensor yields measurements $y_2 = C_2 x$ for $t = 0, 2, 4, \dots$. Assuming that $u(t) = u(\lfloor t \rfloor)$, for all $t \geq 0$, derive a discrete-time state-space model for the system.

Solution — This is a sample-and-hold system, commonly used as a model for computer-controlled systems. In this particular model, the two sensors have different sampling rate. Even though the system is not time invariant, the sampling strategy is periodic—and we can find a time-invariant model for the system exploiting this periodicity.

Consider the following expression for the response of a continuous-time LTI system.

$$x(t_1) = e^{A(t_1-t_0)}x(t_0) + \int_{t_0}^{t_1} e^{A(t_1-\tau)}Bu(\tau)\,d\tau;$$

In particular, if t_0 is an integer, and $t_1 = t_0 + 1$,

$$x(t_0+1) = e^{A}x(t_0) + \int_{0}^{1} e^{A(1-\tau)}Bu(t_0)\,d\tau = A_{\mathrm{d}}x(t_0) + B_{\mathrm{d}}u(t_0),$$

where $A_{\mathrm{d}} = e^{A}$, and $B_{\mathrm{d}} = \left(\int_0^1 e^{A(1-\tau)}\,d\tau\right)B$.

Define the output signal for the discrete-time model as

$$y_{\mathrm{d}}[k] = \begin{bmatrix} y_1(2k-1) \\ y_1(2k) \\ y_2(2k) \end{bmatrix}.$$

Similarly, define the input signal for the discrete-time model as

$$u_{\mathrm{d}}[k] = \begin{bmatrix} u(2k-1) \\ u(2k) \end{bmatrix}.$$

Finally, define the state vector as

$$x_{\mathrm{d}}[k] = x(2k-1).$$

With these definitions in mind, one can write that

$$\begin{aligned} y(2k-1) &= x(2k-1) \\ y(2k) &= x(2k) = A_{\mathrm{d}}x(2k-1) + B_{\mathrm{d}}u(2k-1) \\ x(2k+1) &= A_{\mathrm{d}}^2x(2k-1) + A_{\mathrm{d}}B_{\mathrm{d}}u(2k-1) + B_{\mathrm{d}}u(2k) \end{aligned}$$

The desired state-space model is as follows:

$$\begin{aligned} x_{\mathrm{d}}[k+1] &= A_{\mathrm{d}}^2x_{\mathrm{d}}[k] + \begin{bmatrix} A_{\mathrm{d}}B_{\mathrm{d}} & B_{\mathrm{d}} \end{bmatrix} u[k] \\ y_{\mathrm{d}}[k] &= \begin{bmatrix} C_1 \\ C_1A_{\mathrm{d}} \\ C_2A_{\mathrm{d}} \end{bmatrix} x_{\mathrm{d}}[k] + \begin{bmatrix} 0 & 0 \\ C_1B_{\mathrm{d}} & 0 \\ C_2B_{\mathrm{d}} & 0 \end{bmatrix} u_{\mathrm{d}}[k]. \end{aligned}$$

Notice that this model is time-invariant, but is no longer strictly causal, since $D \neq 0$.

6.241J / 16.338J Dynamic Systems and Control
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