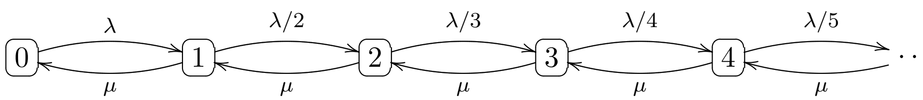
Solutions to Homework 10

6.262 Discrete Stochastic Processes
MIT, Spring 2011

Exercise 6.5:

Consider the Markov process illustrated below. The transitions are labelled by the rate q_{ij} at which those transitions occur. The process can be viewed as a single server queue where arrivals become increasingly discouraged as the queue lengthens. The word *time-average* below refers to the limiting time-average over each sample-path of the process, except for a set of sample paths of probability 0.



Part a) Find the time-average fraction of time p_i spent in each state $i > 0$ in terms of p_0 and then solve for p_0. Hint: First find an equation relating p_i to p_{i+1} for each i. It also may help to recall the power series expansion of e^x.

Solution: From equation (6.36) we know:

$$p_i \frac{\lambda}{i+1} = p_{i+1}\mu, \qquad \text{for } i \geq 0$$

By iterating over i we get:

$$p_{i+1} = \frac{\lambda}{\mu(i+1)} p_i = \left(\frac{\lambda}{\mu}\right)^{i+1} \frac{1}{(i+1)!} p_0, \qquad \text{for } i \geq 0$$

$$\begin{aligned} 1 &= \sum_{i=0}^{\infty} p_i \\ &= p_0 \left[1 + \sum_{i=1}^{\infty} \left(\frac{\lambda}{\mu}\right)^i \frac{1}{i!}\right] \\ &= p_0 e^{\lambda/\mu} \end{aligned}$$

Where the last derivation is in fact the Taylor expansion of the function e^x. Thus,

$$\begin{aligned} p_0 &= e^{-\lambda/\mu} \\ p_i &= \left(\frac{\lambda}{\mu}\right)^i \frac{1}{i!} e^{-\lambda/\mu}, \qquad \text{for } i \geq 1 \end{aligned}$$

Part b) Find a closed form solution to $\sum_i p_i v_i$ where v_i is the departure rate from state i. Show that the process is positive recurrent for all choices of $\lambda > 0$ and $\mu > 0$ and explain intuitively why this must be so.

Solution: We observe that $v_0 = \lambda$ and $v_i = \mu + \lambda/(i+1)$ for $i \geq 1$.

$$\begin{aligned}
\sum_{i\geq 0} p_i v_i &= \lambda e^{-\lambda/\mu} + \sum_{i\geq 1} \{[\mu + \lambda/(i+1)] \left(\frac{\lambda}{\mu}\right)^i \frac{1}{i!} e^{-\lambda/\mu}\} \\
&= \lambda e^{-\lambda/\mu} + \mu e^{-\lambda/\mu} \sum_{i\geq 1} \left(\frac{\lambda}{\mu}\right)^i \frac{1}{i!} + \mu e^{-\lambda/\mu} \sum_{i\geq 1} \left(\frac{\lambda}{\mu}\right)^{i+1} \frac{1}{(i+1)!} \\
&= \lambda e^{-\lambda/\mu} + \mu e^{-\lambda/\mu} \left(e^{\lambda/\mu} - 1\right) + \mu e^{-\lambda/\mu} \left(e^{\lambda/\mu} - 1 - \frac{\lambda}{\mu}\right) \\
&= 2\mu \left(1 - e^{-\lambda/\mu}\right)
\end{aligned}$$

The value of $\sum_i p_i v_i$ is finite for all values of $\lambda > 0$ and $\mu > 0$. So this process is positive recurrent for for all choices of transition rates λ and μ.

We saw that $P_{i+1} = \frac{\lambda P_i}{\mu(i+1)}$, so p_i must decrease rapidly in i for sufficiently large i. Thus the fraction of time spent in very high numbered states must be negligible. This suggests that the steady-state equations for the p_i must have a solution. Since ν_i is bounded between μ and $\mu + \lambda$ for all i, it is intuitively clear that $\sum_i \nu_i p_i$ is finite, so the embedded chain must be positive recurrent.

Part c) For the embedded Markov chain corresponding to this process, find the steady state probabilities π_i for each $i \geq 0$ and the transition probabilities P_{ij} for each i, j.

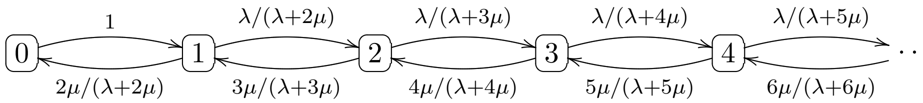
Solution: Since as shown in part (b), $\sum_{i\geq 0} p_i v_i = 1/\left(\sum_{i\geq 0} \pi_i/v_i\right) < \infty$, we know that for all $j \geq 0$, π_j is proportional to $p_j v_j$:

$$\pi_j = \frac{p_j v_j}{\sum_{k\geq 0} p_k v_k}, \qquad \text{for} j \geq 0$$

So $\pi_0 = \frac{\lambda e^{-\lambda/\mu}}{2\mu(1-e^{-\lambda/\mu})} = \frac{\rho}{2}\frac{1}{e^\rho - 1}$ and for $j \geq 1$:

$$\begin{aligned}
\pi_j &= \frac{[\mu + \lambda/(j+1)]\, \rho^j \frac{1}{j!} e^{-\rho}}{2\mu\,(1 - e^{-\rho})} \\
&= \frac{\mu\,[1 + \rho/(j+1)]\, \rho^j e^{-\rho}}{2\mu\,(1 - e^{-\rho})\, j!} \\
&= \frac{\rho^i}{j!} \left[1 + \frac{\rho}{j+1}\right] \frac{1}{2\,(e^\rho - 1)}, \qquad \text{for } j \geq 1
\end{aligned}$$

The embedded Markov chain will look like:



The transition probabilities are:

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\begin{aligned}
P_{01} &= 1 \\
P_{i,i-1} &= \frac{(i+1)\mu}{\lambda+(i+1)\mu}, \qquad \text{for } i \geq 1 \\
P_{i,i+1} &= \frac{\lambda}{\lambda+(i+1)\mu}, \qquad \text{for } i \geq 1
\end{aligned}
$$

Finding the steady state distribution of this Markov chain gives the same result as found above.

Part d) For each i, find both the time-average interval and the time-average number of overall state transitions between successive visits to i.

Solution: Looking at this process as a delayed renewal reward process where each entry to state i is a renewal and the inter-renewal intervals are independent. The reward is equal to 1 whenever the process is in state i.

Given that transition $n-1$ of the embedded chain enters state i, the interval U_n is exponential with rate v_i, so $\mathbb{E}[U_n|X_{n-1}=i] = 1/v_i$. During this U_n time, reward is 1 and then it is zero until the next renewal of the process.

The total average fraction of time spent in state i is p_i with high probability.

So in the steady state, the total fraction of time spent in state i (p_i) should be equal to the fraction of time spent in state i in one inter-renewal interval. The expected length of time spent in state i in one inter-renewal interval is $1/v_i$ and the expected inter renewal interval ($\overline{W_i}$) is what we want to know:

$$p_i = \frac{\overline{U}}{\overline{W_i}} = \frac{1}{v_i \overline{W_i}}$$

Thus $\overline{W_i} = \frac{1}{p_i v_i}$.

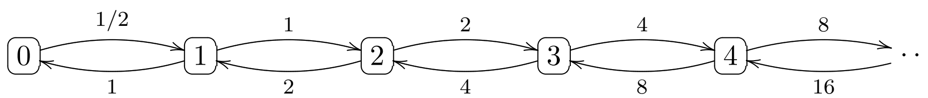
$$
\begin{aligned}
\overline{W_0} &= \frac{e^{\rho}}{\lambda} \\
\overline{W_i} &= \frac{(i+1)!}{\mu\rho^i(i+1+\rho)} e^{\rho}, \qquad \text{for } i \geq 1
\end{aligned}
$$

Applying Theorem 5.1.4 to the embedded chain, the expected number of transitions, $\mathsf{E}[T_{ii}]$ from one visit to state i to the next, is $\overline{T}_{ii} = 1/\pi_i$.

Exercise 6.9:
Let $q_{i,i+1} = 2^{i-1}$ for all $i \geq 0$ and let $q_{i,i-1} = 2^{i-1}$ for all $i \geq 1$. All other transition rates are 0.

Part a) Solve the steady-state equations and show that $p_i = 2^{-i-1}$ for all $i \geq 0$.

Solution: The defined Markov process can be shown as:



For each $i \geq 0$, $p_i q_{i,i+1} = p_{i+1} q_{i+1,i}$.

$$p_i = \frac{1}{2} p_{i-1}, \qquad \text{for} i \geq 1$$

$$1 = \sum_{j \geq 0} p_j = p_0 \left(1 + \frac{1}{2} + \frac{1}{4} + \cdots\right)$$

So $p_0 = \frac{1}{2}$ and

$$p_i = \frac{1}{2^{i+1}}, \qquad i \geq 0$$

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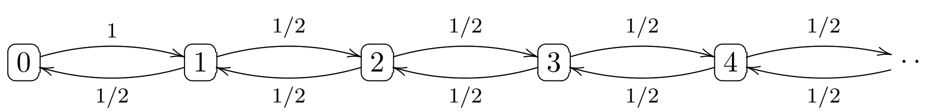
Part b) Find the transition probabilities for the embedded Markov chain and show that the chain is null-recurrent.

Solution:
The embedded Markov chain is:



The steady state probabilities satisfy $\pi_0 = 1/2\pi_1$ and $1/2\pi_i = 1/2\pi_{i+1}$ for $i \geq 1$. So $2\pi_0 = \pi_1 = \pi_2 = \pi_3 = \cdots$. This is a null-recurrent chain, as essentially shown in Exercise 5.2.

Part c) For any state i, consider the renewal process for which the Markov process starts in state i and renewals occur on each transition to state i. Show that, for each $i \geq 1$, the expected inter-renewal interval is equal to 2. Hint: Use renewal reward theory.

Solution:

As explained in Exercise 6.5 part (d), the expected inter-renewal intervals of recurrence of state i ($\overline{W_i}$) satisfies the equation $p_i = \frac{1}{v_i \overline{W_i}}$. Hence,

$$\overline{W_i} = \frac{1}{v_i p_i} = \frac{1}{2^{-(i+1)} 2^i} = 2$$

.

Where $v_i = q_{i,i+1} + q_{i,i-1} = 2^i$

Part d) Show that the expected number of transitions between each entry into state i is infinite. Explain why this does *not* mean that an infinite number of transitions can occur in a finite time.

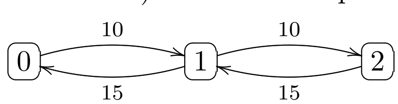
Solution: We have seen in part b) that the embedded chain is null-recurrent. This means that, given $X_0 = i$, for any given i, that a return to i must happen in a finite number of transitions (i.e., $\lim_{n\to\infty} F_{ii}(n) = 1$). We have seen many rv's that have an infinite expectation, but, being rv's, have a finite sample value WP1.

<u>**Exercise 6.14:**</u>

A small bookie shop has room for at most two customers. Potential customers arrive at a Poisson rate of 10 customers per hour; They enter if there is room and are turned away, never to return, otherwise. The bookie serves the admitted customers in order, requiring an exponentially distributed time of mean 4 minutes per customer.

Part a) Find the steady state distribution of the number of customers in the shop.

Solution: The arrival rate of the customers is 10 customers per hour and the service time is exponentially distributed with rate 15 customers per hour (or equivalently with mean 4 minutes per customer). The Markov process corresponding to this bookie store is:



To find the steady state distribution of this process we use the fact that $p_0 q_{0,1} = p_1 q_{1,0}$ and $p_1 q_{1,2} = p_2 q_{2,1}$. So:

$$\begin{aligned} 10p_0 &= 15p_1 \\ 10p_1 &= 15p_2 \\ 1 &= p_0 + p_1 + p_2 \end{aligned}$$

Thus, $p_1 = \frac{6}{19}$, $p_0 = \frac{9}{19}$, $p_2 = \frac{4}{19}$.

Part b) Find the rate at which potential customers are turned away.

Solution:
The customers are turned away when the process is in state 2 and when the process is in state 2, at rate $\lambda = 10$ the customers are turned away. So the overall rate at which the customers are turned away is $\lambda p_2 = \frac{40}{19}$.

Part c) Suppose the bookie hires an assistant; the bookie and assistant, working together, now serve each customer in an exponentially distributed time of mean 2 minutes, but there is only room for one customer (i.e., the customer being served) in the shop. Find the new rate at which customers are turned away.

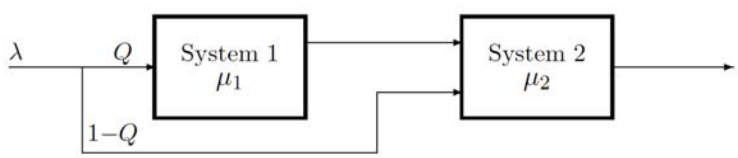
Solution:
The new Markov process will look like:

$$\begin{tikzcd} \boxed{0} \arrow[r, bend left, "10"] & \boxed{1} \arrow[l, bend left, "30"] \end{tikzcd}$$

The new steady state probabilities satisfy $10p_0 = 30p_1$ and $p_0 + p_1 = 1$. Thus, $p_1 = \frac{1}{4}$ and $p_0 = \frac{3}{4}$. The customers are turned away if the process is in state 1 and then this happens with rate $\lambda = 10$. Thus, the overall rate at which the customers are turned away is $\lambda p_1 = \frac{5}{2}$.

<u>**Exercise 6.16:**</u>

Consider the job sharing computer system illustrated below. Incoming jobs arrive from the left in a Poisson stream. Each job, independently of other jobs, requires pre-processing in system 1 with probability Q. Jobs in system 1 are served FCFS and the service times for successive jobs entering system 1 are IID with an exponential distribution of mean $1/\mu_1$. The jobs entering system 2 are also served FCFS and successive service times are IID with an exponential distribution of mean $1/\mu_2$. The service times in the two systems are independent of each other and of the arrival times. Assume that $\mu_1 > \lambda Q$ and that $\mu_2 > \lambda$. Assume that the combined system is in steady state.



Part a) Is the input to system 1 Poisson? Explain.

Solution: Yes. The incoming jobs from the left are Poisson process. This process is split in two processes independently where each job needs a preprocessing in system 1 with probability Q. We know that if a Poisson process is split into two processes, each of the processes are also Poisson. So the jobs entering the system 1 is Poisson with rate λQ.

Part b) Are each of the two input processes coming into system 2 Poisson?

Solution: By Burke's theorem, the output process of a M/M/1 queue is a Poisson process that has the same rate as the input process. So both sequences entering system 2 are Poisson, the first one has rate $Q\lambda$ and the second one has rate $(1-Q)\lambda$. The overall input is merged process of these two that is going to be a Poisson with rate λ (Since these processes are independent of each other.)

Part d) Give the joint steady-state PMF of the number of jobs in the two systems. Explain briefly.

Solution: We call the number of customers being served in system 1 at time t as $X_1(t)$ and number of customers being served in system 2 at time t, as $X_2(t)$.

The splitting of the input arrivals from the left is going to make two independent processes with rates $Q\lambda$ and $(1-Q)\lambda$. The first process goes into system 1 and defines $X_1(t)$. The output jobs of system 1 at time t is independent of its previous arrivals. Thus the input sequence of system 2 is independent of system 1. The two input processes of system 2 are also independent.

Thus, $X_1(t)$ is the number of customers in an M/M/1 queue with input rate $Q\lambda$ and service rate μ_1 and $X_2(t)$ is the number of customers in an M/M/1 queue with input rate λ and service rate μ_2. The total number of the customers in the system is $X(t) = X_1(t) + X_2(t)$ where $X_1(t)$ is independent of $X_2(t)$.

System one can be modeled as a birth death process where $q_{i,i+1} = Q\lambda$ and $q_{i,i-1} = \mu_1$. Thus, $\Pr\{X_1 = i\} = (1-\rho_1)\rho_1^i$ in the steady state where $\rho_1 = Q\lambda/\mu_1$. The same is true for system 2, thus, $\Pr\{X_2 = j\} = (1-\rho_2)\rho_2^j$ in the steady state where $\rho_2 = \lambda/\mu_2$.

Due to the independency of X_1 and X_2,

$$
\begin{aligned}
\Pr\{X = k\} &= \Pr\{X_1 + X_2 = k\} \\
&= \sum_{i=0}^{k} \Pr\{X_1 = i, X_2 = k - i\} \\
&= \sum_{i=0}^{k} \Pr\{X_1 = i\} \Pr\{X_2 = k - i\} \\
&= \sum_{i=0}^{k} (1 - \rho_1)\rho_1^i (1 - \rho_2)\rho_2^{k-i} \\
&= (1 - \rho_1)(1 - \rho_2)\rho_2^k \sum_{i=0}^{k} (\rho_1/\rho_2)^i \\
&= (1 - \rho_1)(1 - \rho_2)\frac{\rho_2^k - \rho_1^k}{\rho_2 - \rho_1}
\end{aligned}
$$

Part e) What is the probability that the first job to leave system 1 after time t is the same as the first job that entered the entire system after time t?

Solution:

The first job that enters the system after time t is the same as the first job to leave system 1 after time t if and only if $X_1(t) = 0$ (system 1 should be empty at time t, unless other jobs will leave system 1 before the specified job) and the first entering job to the whole system needs preprocessing and is routed to system 1 (and should need) which happens with probability Q. Since these two events are independent, the probability of the desired event will be $(1 - \rho_1)Q = (1 - \frac{Q\lambda}{\mu_1})Q$.

Part f) What is the probability that the first job to leave system 2 after time t both passed through system 1 and arrived at system 1 after time t?

Solution: This is the event that both systems are empty at time t and the first arriving job is routed to system 1 and is finished serving in system 1 before the first job without preprocesing enters system 2. These three events are independent of each other.

The probability that both systems are empty at time t in steady state is $\Pr\{X_1(t) = 0, X_2(t) = 0\} = (1 - \rho_1)(1 - \rho_2)$.

The probability that the first job is routed to system 1 is Q.

The service time of the first job in system 1 is called Y_1 which is exponentially distributed with rate μ_1 and the probability that the first job is finished before the first job without preprocessing enters system 2 is $\Pr\{Y_1 < Z\}$ where Z is the r.v. which is the arrival time of the first job that does not need preprocessing. it is also exponentially distributed with rate $(1 - Q)\lambda$. Thus, $\Pr\{Y_1 < Z\} = \frac{\mu_1}{\mu_1 + (1-Q)\lambda}$.

Hence, the total probability of the described event is $(1 - \frac{Q\lambda}{\mu_1})(1 - \frac{\lambda}{\mu_2})Q\frac{\mu_1}{\mu_1+(1-Q)\lambda}$.