LECTURE 11

LECTURE OUTLINE

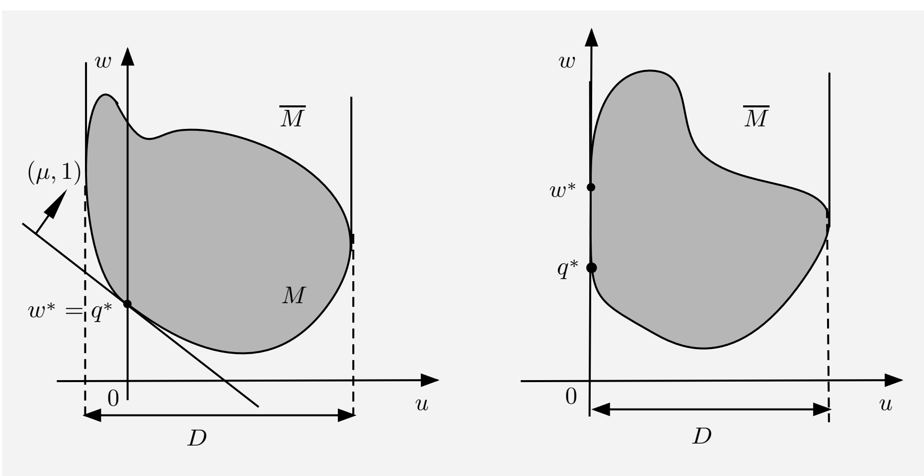
- Min Common/Max Crossing Th. III
- Nonlinear Farkas Lemma/Linear Constraints
- Linear Programming Duality
- Convex Programming Duality

Reading: Sections 4.5, 5.1-5.3

Recall the MC/MC Theorem II: If $-\infty < w^*$ and

$$0 \in D = \{u \mid \text{there exists } w \in \Re \text{ with } (u, w) \in M\}$$

then $q^* = w^*$ and there exists μ such that $q(\mu) = q^*$.



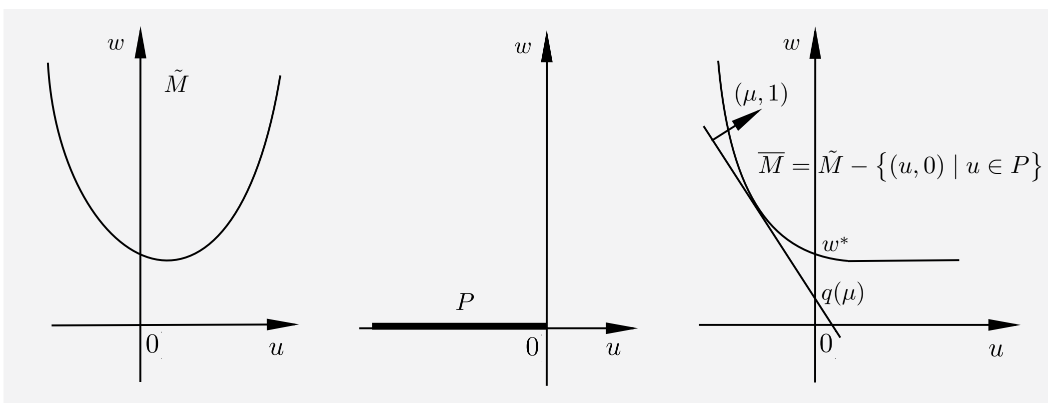
MC/MC TH. III - POLYHEDRAL

- Consider the MC/MC problems, and assume that $-\infty < w^*$ and:

(1) M is a "horizontal translation" of $\tilde{M}$ by $-P$,

$$M = \tilde{M} - \{(u, 0) \mid u \in P\},$$

where P: polyhedral and $\tilde{M}$: convex.



(2) We have $\mathrm{ri}(\tilde{D}) \cap P \neq \emptyset$, where

$$\tilde{D} = \{u \mid \text{there exists } w \in \Re \text{ with } (u, w) \in \tilde{M}\}$$

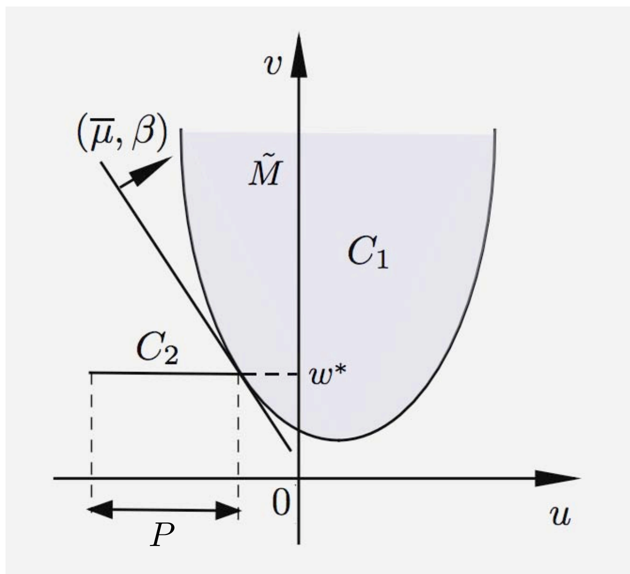
Then $q^* = w^*$, there is a max crossing solution, and all max crossing solutions μ satisfy $\mu' d \leq 0$ for all $d \in R_P$.

- **Comparison with Th. II:** Since $D = \tilde{D} - P$, the condition $0 \in \mathrm{ri}(D)$ of Theorem II is

$$\mathrm{ri}(\tilde{D}) \cap \mathrm{ri}(P) \neq \emptyset$$

PROOF OF MC/MC TH. III

- Consider the *disjoint* convex sets $C_1 = \{(u,v) \mid v > w \text{ for some } (u,w) \in \tilde{M}\}$ and $C_2 = \{(u,w^*) \mid u \in P\}$ [$u \in P$ and $(u,w) \in \tilde{M}$ with $w^* > w$ contradicts the definition of w^*]



- Since C_2 is polyhedral, there exists a separating hyperplane not containing C_1, i.e., a $(\mu,\beta) \neq (0,0)$ such that

$$\beta w^* + \mu' z \leq \beta v + \mu' x, \quad \forall\, (x,v) \in C_1,\ \forall\, z \in P$$

$$\inf_{(x,v)\in C_1} \{\beta v + \mu' x\} < \sup_{(x,v)\in C_1} \{\beta v + \mu' x\}$$

Since $(0,1)$ is a direction of recession of C_1, we see that $\beta \geq 0$. Because of the relative interior point assumption, $\beta \neq 0$, so we may assume that $\beta = 1$.

PROOF (CONTINUED)

- Hence,

$$w^* + \mu' z \le \inf_{(u,v) \in C_1} \{v + \mu' u\}, \qquad \forall\, z \in P,$$

so that

$$\begin{aligned} w^* &\le \inf_{(u,v) \in C_1,\, z \in P} \big\{v + \mu'(u - z)\big\} \\ &= \inf_{(u,v) \in \tilde{M} - P} \{v + \mu' u\} \\ &= \inf_{(u,v) \in M} \{v + \mu' u\} \\ &= q(\mu) \end{aligned}$$

Using $q^* \le w^*$ (weak duality), we have $q(\mu) = q^* = w^*$.

Proof that all max crossing solutions μ satisfy $\mu' d \le 0$ for all $d \in R_P$: follows from

$$q(\mu) = \inf_{(u,v) \in C_1,\, z \in P} \big\{v + \mu'(u - z)\big\}$$

so that $q(\mu) = -\infty$ if $\mu' d > 0$. **Q.E.D.**

- Geometrical intuition: every $(0, -d)$ with $d \in R_P$, is direction of recession of M.

MC/MC TH. III - A SPECIAL CASE

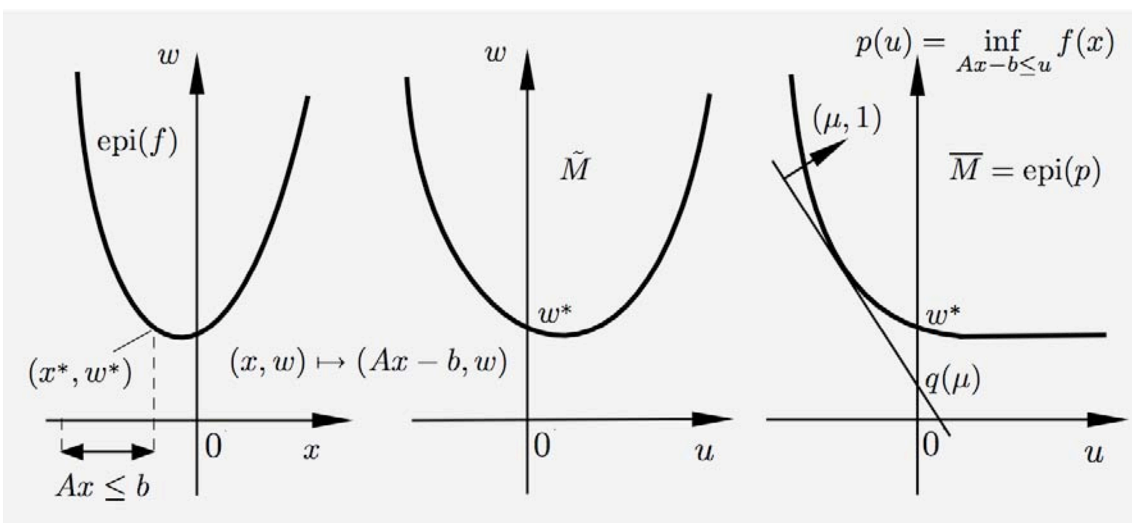
- Consider the MC/MC framework, and assume:

(1) For a convex function $f : \Re^m \mapsto (-\infty, \infty]$, an $r \times m$ matrix A, and a vector $b \in \Re^r$:

$$M = \big\{(u, w) \mid \text{for some } (x, w) \in \text{epi}(f),\ Ax - b \leq u\big\}$$

so $M = \tilde{M} +$ Positive Orthant, where

$$\tilde{M} = \big\{(Ax - b, w) \mid (x, w) \in \text{epi}(f)\big\}$$



(2) There is an $x \in \text{ri}(\text{dom}(f))$ s. t. $Ax - b \leq 0$.

Then $q^* = w^*$ and there is a $\mu \geq 0$ with $q(\mu) = q^*$.

- Also $M = M \approx \text{epi}(p)$, where $p(u) = \inf_{Ax-b\leq u} f(x)$.
- We have $w^* = p(0) = \inf_{Ax-b\leq 0} f(x)$.

NONL. FARKAS' L. - POLYHEDRAL ASSUM.

- Let $X \subset \Re^n$ be convex, and $f : X \mapsto \Re$ and $g_j : \Re^n \mapsto \Re$, $j = 1, \ldots, r$, be linear so $g(x) = Ax - b$ for some A and b. Assume that

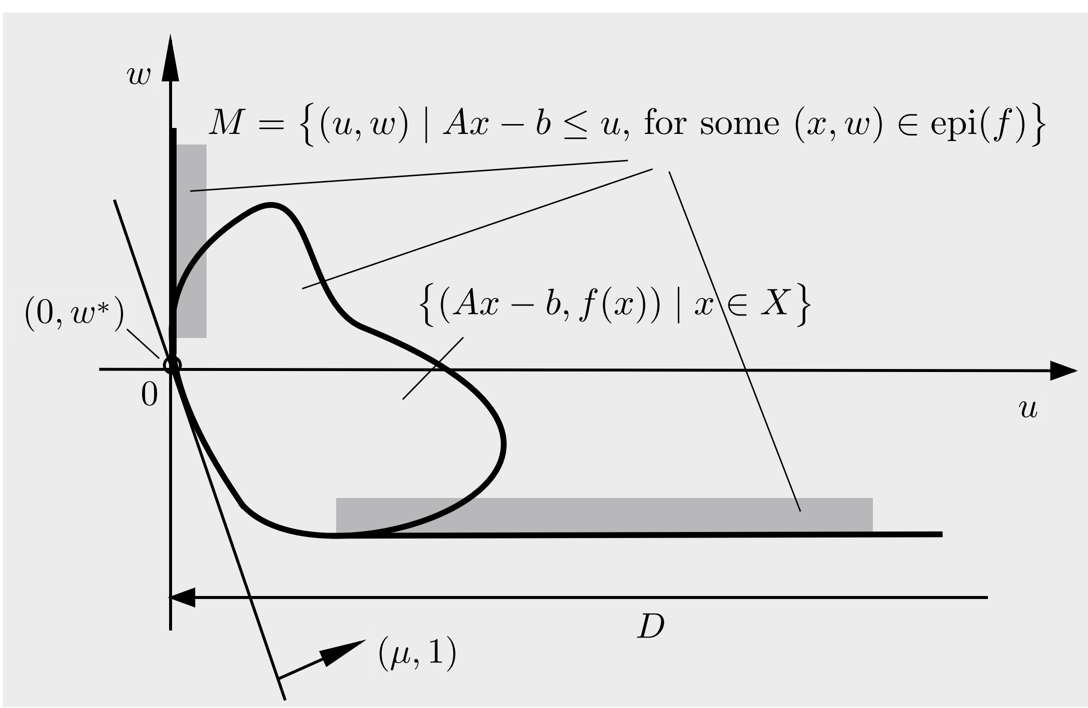
$$f(x) \geq 0, \qquad \forall\, x \in X \text{ with } Ax - b \leq 0$$

Let

$$Q^* = \big\{\mu \mid \mu \geq 0,\ f(x) + \mu'(Ax - b) \geq 0,\ \forall\, x \in X\big\}.$$

Assume that there exists a vector $\overline{x} \in \mathrm{ri}(X)$ such that $A\overline{x} - b \leq 0$. Then Q^* is nonempty.

Proof: As before, apply special case of MC/MC Th. III of preceding slide, using the fact $w^* \geq 0$, implied by the assumption.



(LINEAR) FARKAS' LEMMA

- Let A be an $m \times n$ matrix and $c \in \Re^m$. The system $Ay = c$, $y \geq 0$ has a solution if and only if

$$A'x \leq 0 \qquad \Rightarrow \qquad c'x \leq 0. \qquad (*)$$

- **Alternative/Equivalent Statement:** If $P = \text{cone}\{a_1, \ldots, a_n\}$, where $a_1, \ldots, a_n$ are the columns of A, then $P = (P^*)^*$ (Polar Cone Theorem).

Proof: If $y \in \Re^n$ is such that $Ay = c$, $y \geq 0$, then $y'A'x = c'x$ for all $x \in \Re^m$, which implies Eq. (*).

Conversely, apply the Nonlinear Farkas' Lemma with $f(x) = -c'x$, $g(x) = A'x$, and $X = \Re^m$. Condition (*) implies the existence of $\mu \geq 0$ such that

$$-c'x + \mu'A'x \geq 0, \qquad \forall\, x \in \Re^m,$$

or equivalently

$$(A\mu - c)'x \geq 0, \qquad \forall\, x \in \Re^m,$$

or $A\mu = c$.

LINEAR PROGRAMMING DUALITY

- Consider the linear program

$$\begin{aligned}&\text{minimize } \ c'x\\ &\text{subject to } \ a_j'x \geq b_j, \quad j = 1, \ldots, r,\end{aligned}$$

where $c \in \Re^n$, $a_j \in \Re^n$, and $b_j \in \Re$, $j = 1, \ldots, r$.

- The dual problem is

$$\begin{aligned}&\text{maximize } \ b'\mu\\ &\text{subject to } \ \sum_{j=1}^{r} a_j \mu_j = c, \quad \mu \geq 0.\end{aligned}$$

- **Linear Programming Duality Theorem:**

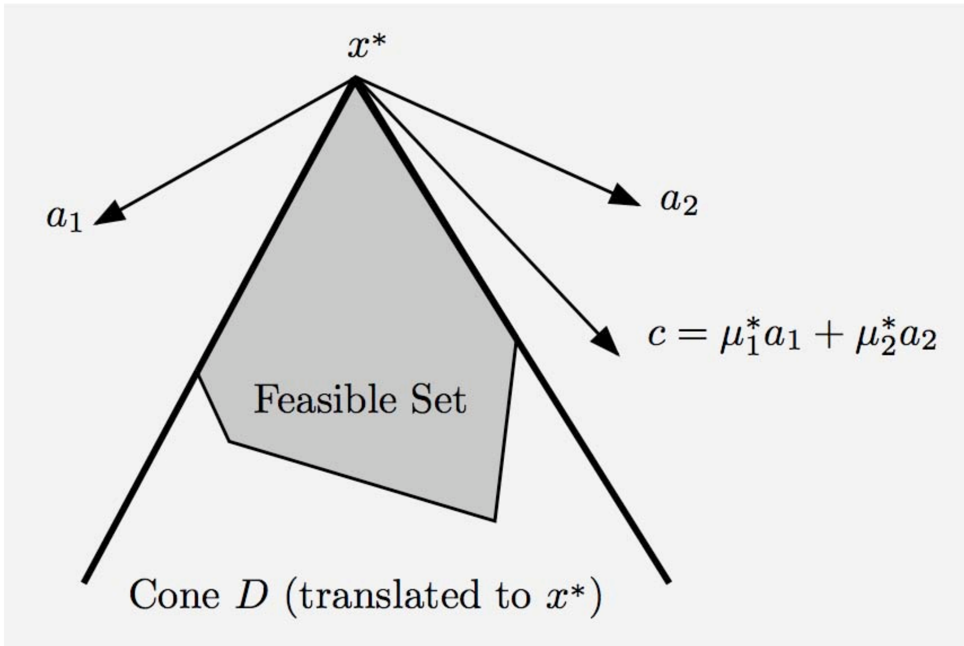
(a) If either f^* or q^* is finite, then $f^* = q^*$ and both the primal and the dual problem have optimal solutions.

(b) If $f^* = -\infty$, then $q^* = -\infty$.

(c) If $q^* = \infty$, then $f^* = \infty$.

Proof: (b) and (c) follow from weak duality. For part (a): If f^* is finite, there is a primal optimal solution x^*, by existence of solutions of quadratic programs. Use Farkas' Lemma to construct a dual feasible μ^* such that $c'x^* = b'\mu^*$ (next slide).

PROOF OF LP DUALITY (CONTINUED)



- Let x^* be a primal optimal solution, and let $J = \{j \mid a_j' x^* = b_j\}$. Then, $c'y \geq 0$ for all y in the cone of "feasible directions"

$$D = \{y \mid a_j' y \geq 0, \ \forall\, j \in J\}$$

By Farkas' Lemma, for some scalars $\mu_j^* \geq 0$, c can be expressed as

$$c = \sum_{j=1}^{r} \mu_j^* a_j, \quad \mu_j^* \geq 0, \ \forall\, j \in J, \quad \mu_j^* = 0, \ \forall\, j \notin J.$$

Taking inner product with x^*, we obtain $c'x^* = b'\mu^*$, which in view of $q^* \leq f^*$, shows that $q^* = f^*$ and that μ^* is optimal.

LINEAR PROGRAMMING OPT. CONDITIONS

A pair of vectors (x^*, μ^*) form a primal and dual optimal solution pair if and only if x^* is primal-feasible, μ^* is dual-feasible, and

$$\mu_j^*(b_j - a_j' x^*) = 0, \qquad \forall\, j = 1, \ldots, r. \qquad (*)$$

Proof: If x^* is primal-feasible and μ^* is dual-feasible, then

$$\begin{aligned} b'\mu^* &= \sum_{j=1}^{r} b_j \mu_j^* + \left(c - \sum_{j=1}^{r} a_j \mu_j^* \right)' x^* \\ &= c'x^* + \sum_{j=1}^{r} \mu_j^*(b_j - a_j' x^*) \end{aligned} \qquad (**)$$

So if Eq. (*) holds, we have $b'\mu^* = c'x^*$, and weak duality implies that x^* is primal optimal and μ^* is dual optimal.

Conversely, if (x^*, μ^*) form a primal and dual optimal solution pair, then x^* is primal-feasible, μ^* is dual-feasible, and by the duality theorem, we have $b'\mu^* = c'x^*$. From Eq. (**), we obtain Eq. (*).

CONVEX PROGRAMMING

Consider the problem

$$\begin{aligned}&\text{minimize} \quad f(x)\\&\text{subject to} \quad x \in X,\ g_j(x) \le 0,\ j = 1, \ldots, r,\end{aligned}$$

where $X \subset \Re^n$ is convex, and $f : X \mapsto \Re$ and $g_j : X \mapsto \Re$ are convex. Assume f^*: finite.

- Consider the Lagrangian function

$$L(x, \mu) = f(x) + \mu' g(x),$$

the dual function

$$q(\mu) = \begin{cases} \inf_{x \in X} L(x, \mu) & \text{if } \mu \ge 0, \\ -\infty & \text{otherwise} \end{cases}$$

and the dual problem of maximizing $\inf_{x \in X} L(x, \mu)$ over $\mu \ge 0$.

- Recall this is the max crossing problem in the MC/MC framework where $M = \text{epi}(p)$ with

$$p(u) = \inf_{x \in X,\, g(x) \le u} f(x)$$

STRONG DUALITY THEOREM

- Assume that f^* is finite, and that one of the following two conditions holds:

  (1) There exists $x \in X$ such that $g(x) < 0$.

  (2) The functions g_j, $j = 1, \ldots, r$, are affine, and there exists $x \in \mathrm{ri}(X)$ such that $g(x) \leq 0$.

Then $q^* = f^*$ and the set of optimal solutions of the dual problem is nonempty. Under condition (1) this set is also compact.

- Replace $f(x)$ by $f(x) - f^*$ so that $f(x) - f^* \geq 0$ for all $x \in X$ w/ $g(x) \leq 0$. Apply Nonlinear Farkas' Lemma. Then, there exist $\mu_j^* \geq 0$, s.t.

$$f^* \leq f(x) + \sum_{j=1}^{r} \mu_j^* g_j(x), \qquad \forall\, x \in X$$

- It follows that

$$f^* \leq \inf_{x \in X} \{ f(x) + \mu^{*\prime} g(x) \} \leq \inf_{x \in X,\, g(x) \leq 0} f(x) = f^*.$$

Thus equality holds throughout, and we have

$$f^* = \inf_{x \in X} \left\{ f(x) + \sum_{j=1}^{r} \mu_j^* g_j(x) \right\} = q(\mu^*)$$

QUADRATIC PROGRAMMING DUALITY

- Consider the quadratic program

$$\begin{aligned} &\text{minimize} \quad \tfrac{1}{2}x'Qx + c'x \\ &\text{subject to} \quad Ax \le b, \end{aligned}$$

where Q is positive definite.

- If f^* is finite, then $f^* = q^*$ and there exist both primal and dual optimal solutions, since the constraints are linear.

- Calculation of dual function:

$$q(\mu) = \inf_{x \in \Re^n} \{ \tfrac{1}{2}x'Qx + c'x + \mu'(Ax - b) \}$$

The infimum is attained for $x = -Q^{-1}(c + A'\mu)$, and, after substitution and calculation,

$$q(\mu) = -\tfrac{1}{2}\mu' AQ^{-1}A'\mu - \mu'(b + AQ^{-1}c) - \tfrac{1}{2}c'Q^{-1}c$$

- The dual problem, after a sign change, is

$$\begin{aligned} &\text{minimize} \quad \tfrac{1}{2}\mu' P\mu + t'\mu \\ &\text{subject to} \quad \mu \ge 0, \end{aligned}$$

where $P = AQ^{-1}A'$ and $t = b + AQ^{-1}c$.

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