LECTURE 9

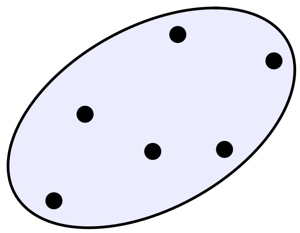
LECTURE OUTLINE

- Min common/max crossing duality
- Weak duality
- Special Cases
- Constrained optimization and minimax
- Strong duality

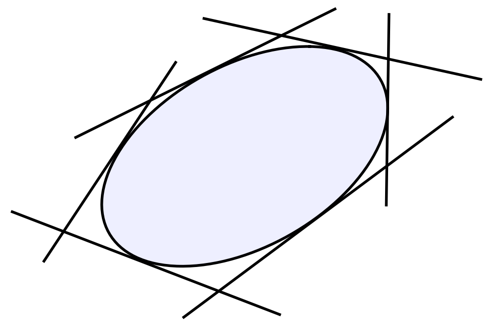
Reading: Sections 4.1,4.2, 3.4

EXTENDING DUALITY CONCEPTS

- From **dual descriptions of sets**

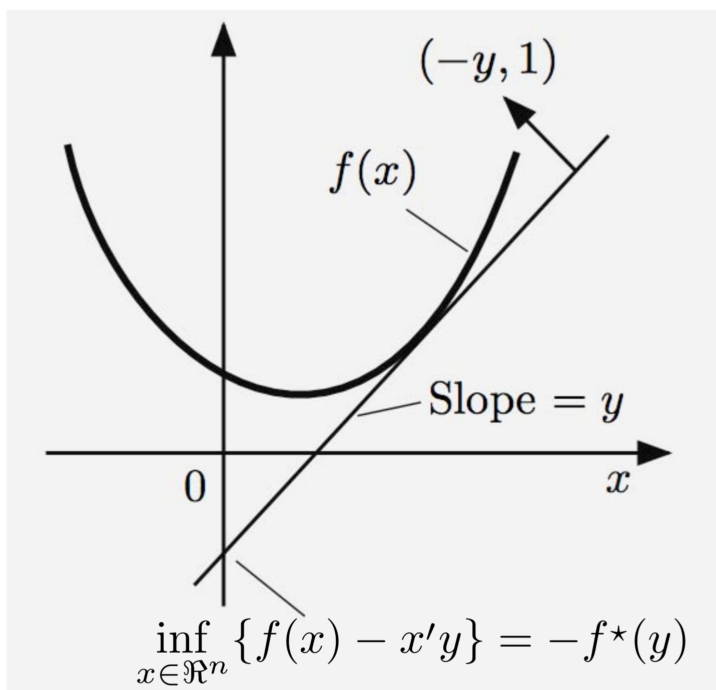


A union of points



An intersection of halfspaces

- To **dual descriptions of functions** (applying set duality to epigraphs)



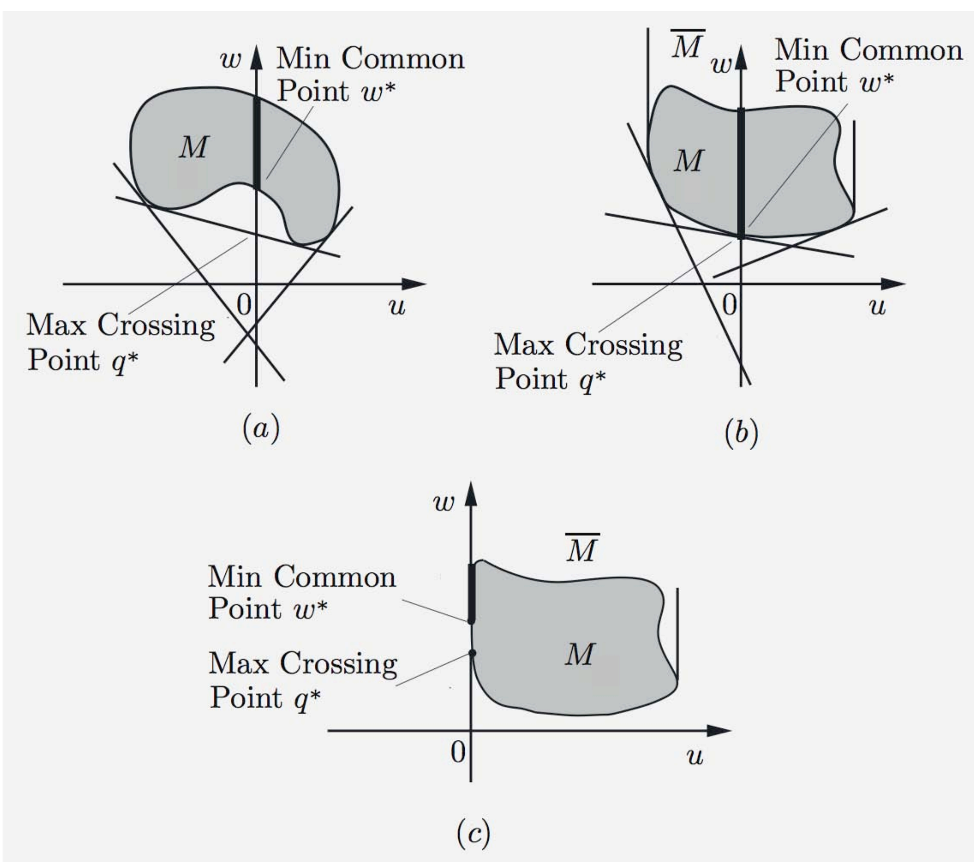
- We now go to **dual descriptions of problems**, by applying conjugacy constructions to a simple generic geometric optimization problem

MIN COMMON / MAX CROSSING PROBLEMS

- We introduce a pair of fundamental problems:
- Let M be a nonempty subset of $\Re^{n+1}$

  (a) *Min Common Point Problem*: Consider all vectors that are common to M and the $(n+1)$st axis. Find one whose $(n+1)$st component is minimum.

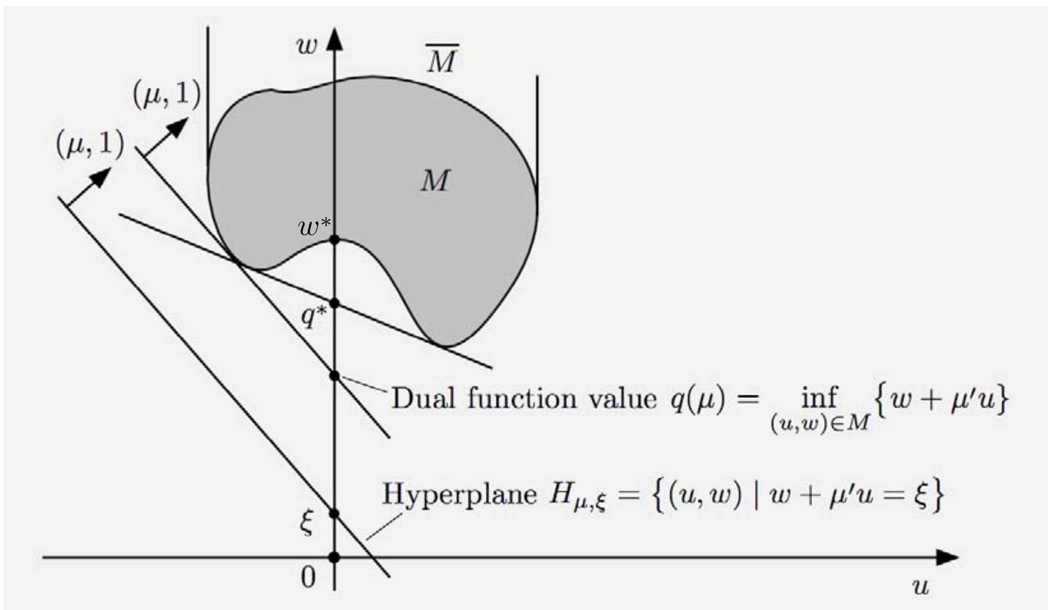
  (b) *Max Crossing Point Problem*: Consider nonvertical hyperplanes that contain M in their "upper" closed halfspace. Find one whose crossing point of the $(n+1)$st axis is maximum.



MATHEMATICAL FORMULATIONS

- **Optimal value of the min common problem:**

$$w^* = \inf_{(0,w)\in M} w$$



- **Math formulation of the max crossing problem:** Focus on hyperplanes with normals $(\mu, 1)$ whose crossing point ξ satisfies

$$\xi \le w + \mu' u, \qquad \forall\ (u,w) \in M$$

Max crossing problem is to maximize ξ subject to $\xi \le \inf_{(u,w)\in M}\{w + \mu' u\}$, $\mu \in \Re^n$, or

$$\text{maximize } \ q(\mu) \overset{\triangle}{=} \inf_{(u,w)\in M} \{w + \mu' u\}$$

$$\text{subject to } \ \mu \in \Re^n.$$

GENERIC PROPERTIES – WEAK DUALITY

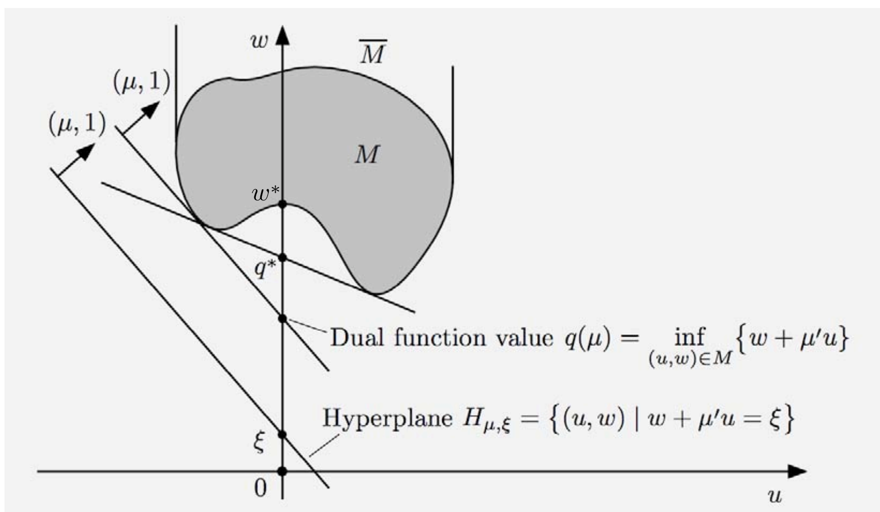
- Min common problem

$$\inf_{(0,w)\in M} w$$

- Max crossing problem

$$\text{maximize} \quad q(\mu) \stackrel{\triangle}{=} \inf_{(u,w)\in M} \{w + \mu' u\}$$

$$\text{subject to} \quad \mu \in \Re^n.$$



- Note that q is concave and upper-semicontinuous (inf of linear functions).

- **Weak Duality:** For all $\mu \in \Re^n$

$$q(\mu) = \inf_{(u,w)\in M} \{w + \mu' u\} \le \inf_{(0,w)\in M} w = w^*,$$

so maximizing over $\mu \in \Re^n$, we obtain $q^* \le w^*$.

- We say that **strong duality** holds if $q^* = w^*$.

CONNECTION TO CONJUGACY

- **An important special case:**

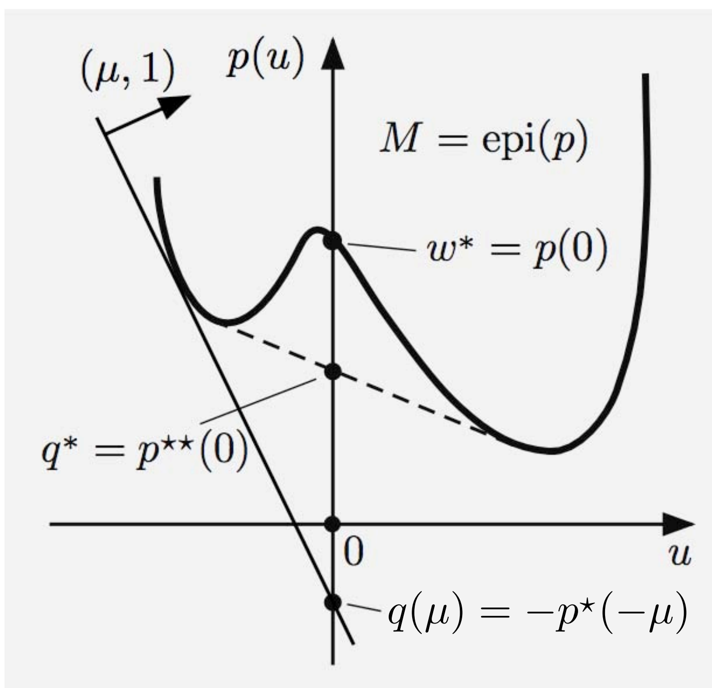
$$M = \mathrm{epi}(p)$$

where $p : \Re^n \mapsto [-\infty, \infty]$. Then $w^* = p(0)$, and

$$q(\mu) = \inf_{(u,w)\in \mathrm{epi}(p)} \{w+\mu' u\} = \inf_{\{(u,w)\mid p(u)\le w\}} \{w+\mu' u\},$$

and finally

$$q(\mu) = \inf_{u\in\Re^m} \big\{p(u) + \mu' u\big\}$$



- Thus, $q(\mu) = -p^\star(-\mu)$ and

$$q^* = \sup_{\mu\in\Re^n} q(\mu) = \sup_{\mu\in\Re^n} \big\{0\cdot(-\mu) - p^\star(-\mu)\big\} = p^{\star\star}(0)$$

GENERAL OPTIMIZATION DUALITY

- Consider minimizing a function $f : \Re^n \mapsto [-\infty, \infty]$.
- Let $F : \Re^{n+r} \mapsto [-\infty, \infty]$ be a function with

$$f(x) = F(x, 0), \qquad \forall\, x \in \Re^n$$

- Consider the *perturbation function*

$$p(u) = \inf_{x \in \Re^n} F(x, u)$$

and the MC/MC framework with $M = \text{epi}(p)$

- The min common value w^* is

$$w^* = p(0) = \inf_{x \in \Re^n} F(x, 0) = \inf_{x \in \Re^n} f(x)$$

- The dual function is

$$q(\mu) = \inf_{u \in \Re^r} \big\{p(u) + \mu' u\big\} = \inf_{(x,u) \in \Re^{n+r}} \big\{F(x, u) + \mu' u\big\}$$

so $q(\mu) = -F^\star(0, -\mu)$, where $F^\star$ is the conjugate of F, viewed as a function of (x, u)

- Since

$$q^* = \sup_{\mu \in \Re^r} q(\mu) = -\inf_{\mu \in \Re^r} F^\star(0, -\mu) = -\inf_{\mu \in \Re^r} F^\star(0, \mu),$$

we have

$$w^* = \inf_{x \in \Re^n} F(x, 0) \geq -\inf_{\mu \in \Re^r} F^\star(0, \mu) = q^*$$

CONSTRAINED OPTIMIZATION

- Minimize $f : \Re^n \mapsto \Re$ over the set

$$C = \{x \in X \mid g(x) \leq 0\},$$

where $X \subset \Re^n$ and $g : \Re^n \mapsto \Re^r$.

- Introduce a "perturbed constraint set"

$$C_u = \{x \in X \mid g(x) \leq u\}, \qquad u \in \Re^r,$$

and the function

$$F(x,u) = \begin{cases} f(x) & \text{if } x \in C_u, \\ \infty & \text{otherwise,} \end{cases}$$

which satisfies $F(x,0) = f(x)$ for all $x \in C$.

- Consider *perturbation function*

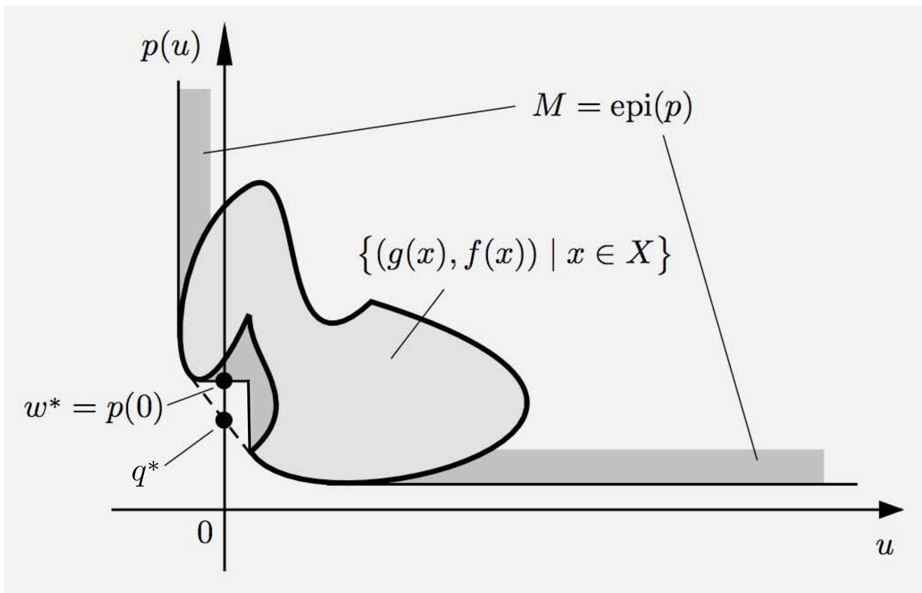
$$p(u) = \inf_{x \in \Re^n} F(x,u) = \inf_{x \in X,\, g(x) \leq u} f(x),$$

and the MC/MC framework with $M = \text{epi}(p)$.

CONSTR. OPT. - PRIMAL AND DUAL FNS

- Perturbation function (or *primal function*)

$$p(u) = \inf_{x \in \Re^n} F(x,u) = \inf_{x \in X,\, g(x) \le u} f(x),$$



- Introduce $L(x,\mu) = f(x) + \mu' g(x)$. Then

$$\begin{aligned} q(\mu) &= \inf_{u \in \Re^r} \left\{ p(u) + \mu' u \right\} \\ &= \inf_{u \in \Re^r,\, x \in X,\, g(x) \le u} \left\{ f(x) + \mu' u \right\} \\ &= \begin{cases} \inf_{x \in X} L(x,\mu) & \text{if } \mu \ge 0, \\ -\infty & \text{otherwise.} \end{cases} \end{aligned}$$

LINEAR PROGRAMMING DUALITY

- Consider the linear program

$$\begin{aligned} &\text{minimize} \;\; c'x \\ &\text{subject to} \;\; a_j'x \geq b_j, \quad j = 1, \ldots, r, \end{aligned}$$

where $c \in \Re^n$, $a_j \in \Re^n$, and $b_j \in \Re$, $j = 1, \ldots, r$.

- For $\mu \geq 0$, the dual function has the form

$$\begin{aligned} q(\mu) &= \inf_{x \in \Re^n} L(x, \mu) \\ &= \inf_{x \in \Re^n} \left\{ c'x + \sum_{j=1}^{r} \mu_j (b_j - a_j'x) \right\} \\ &= \begin{cases} b'\mu & \text{if } \sum_{j=1}^{r} a_j \mu_j = c, \\ -\infty & \text{otherwise} \end{cases} \end{aligned}$$

- Thus the dual problem is

$$\begin{aligned} &\text{maximize} \;\; b'\mu \\ &\text{subject to} \;\; \sum_{j=1}^{r} a_j \mu_j = c, \quad \mu \geq 0. \end{aligned}$$

MINIMAX PROBLEMS

Given $\phi : X \times Z \mapsto \Re$, where $X \subset \Re^n$, $Z \subset \Re^m$ consider

$$\begin{array}{ll} \text{minimize} & \sup_{z \in Z} \phi(x, z) \\ \text{subject to} & x \in X \end{array}$$

or

$$\begin{array}{ll} \text{maximize} & \inf_{x \in X} \phi(x, z) \\ \text{subject to} & z \in Z. \end{array}$$

- Some important contexts:
  - Constrained optimization duality theory
  - Zero sum game theory
- We always have

$$\sup_{z \in Z} \inf_{x \in X} \phi(x, z) \leq \inf_{x \in X} \sup_{z \in Z} \phi(x, z)$$

- **Key question:** When does equality hold?

CONSTRAINED OPTIMIZATION DUALITY

- For the problem

$$\begin{aligned} &\text{minimize} \;\; f(x) \\ &\text{subject to} \;\; x \in X, \qquad g(x) \leq 0 \end{aligned}$$

introduce the Lagrangian function

$$L(x, \mu) = f(x) + \mu' g(x)$$

- Primal problem (equivalent to the original)

$$\min_{x \in X} \sup_{\mu \geq 0} L(x, \mu) = \begin{cases} f(x) & \text{if } g(x) \leq 0, \\ \infty & \text{otherwise,} \end{cases}$$

- Dual problem

$$\max_{\mu \geq 0} \inf_{x \in X} L(x, \mu)$$

- Key duality question: Is it true that

$$\inf_{x \in \Re^n} \sup_{\mu \geq 0} L(x, \mu) = w^* \stackrel{?}{=} q^* = \sup_{\mu \geq 0} \inf_{x \in \Re^n} L(x, \mu)$$

ZERO SUM GAMES

- Two players: 1st chooses $i \in \{1, \ldots, n\}$, 2nd chooses $j \in \{1, \ldots, m\}$.

- If i and j are selected, the 1st player gives a_{ij} to the 2nd.

- Mixed strategies are allowed: The two players select probability distributions

$$x = (x_1, \ldots, x_n), \qquad z = (z_1, \ldots, z_m)$$

over their possible choices.

- Probability of (i, j) is $x_i z_j$, so the expected amount to be paid by the 1st player

$$x' A z = \sum_{i,j} a_{ij} x_i z_j$$

where A is the $n \times m$ matrix with elements a_{ij}.

- Each player optimizes his choice against the worst possible selection by the other player. So
  - 1st player minimizes $\max_z x' A z$
  - 2nd player maximizes $\min_x x' A z$

SADDLE POINTS

Definition: (x^*, z^*) is called a *saddle point* of ϕ if

$$\phi(x^*, z) \leq \phi(x^*, z^*) \leq \phi(x, z^*), \quad \forall\, x \in X,\, \forall\, z \in Z$$

Proposition: (x^*, z^*) is a saddle point if and only if the minimax equality holds and

$$x^* \in \arg\min_{x \in X} \sup_{z \in Z} \phi(x, z), \quad z^* \in \arg\max_{z \in Z} \inf_{x \in X} \phi(x, z) \quad (*)$$

Proof: If (x^*, z^*) is a saddle point, then

$$\begin{aligned} \inf_{x \in X} \sup_{z \in Z} \phi(x, z) &\leq \sup_{z \in Z} \phi(x^*, z) = \phi(x^*, z^*) \\ &= \inf_{x \in X} \phi(x, z^*) \leq \sup_{z \in Z} \inf_{x \in X} \phi(x, z) \end{aligned}$$

By the minimax inequality, the above holds as an equality throughout, so the minimax equality and Eq. (*) hold.

Conversely, if Eq. (*) holds, then

$$\begin{aligned} \sup_{z \in Z} \inf_{x \in X} \phi(x, z) &= \inf_{x \in X} \phi(x, z^*) \leq \phi(x^*, z^*) \\ &\leq \sup_{z \in Z} \phi(x^*, z) = \inf_{x \in X} \sup_{z \in Z} \phi(x, z) \end{aligned}$$

Using the minimax equ., (x^*, z^*) is a saddle point.

MINIMAX MC/MC FRAMEWORK

- Introduce perturbation function $p : \Re^m \mapsto [-\infty, \infty]$

$$p(u) = \inf_{x \in X} \sup_{z \in Z} \big\{ \phi(x, z) - u'z \big\}, \qquad u \in \Re^m$$

- Apply the MC/MC framework with $M = \text{epi}(p)$
- Introduce $\hat{\text{cl}}\, f$, the *concave closure of f*
- We have

$$\sup_{z \in Z} \phi(x, z) = \sup_{z \in \Re^m} (\hat{\text{cl}}\, \phi)(x, z),$$

so

$$w^* = p(0) = \inf_{x \in X} \sup_{z \in \Re^m} (\hat{\text{cl}}\, \phi)(x, z).$$

- The dual function can be shown to be

$$q(\mu) = \inf_{x \in X} (\hat{\text{cl}}\, \phi)(x, \mu), \qquad \forall\ \mu \in \Re^m$$

so if $\phi(x, \cdot)$ is concave and closed,

$$w^* = \inf_{x \in X} \sup_{z \in \Re^m} \phi(x, z), \qquad q^* = \sup_{z \in \Re^m} \inf_{x \in X} \phi(x, z)$$

PROOF OF FORM OF DUAL FUNCTION

- Write $p(u) = \inf_{x \in X} p_x(u)$, where

$$p_x(u) = \sup_{z \in Z} \big\{ \phi(x, z) - u'z \big\}, \qquad x \in X,$$

and note that

$$\inf_{u \in \Re^m} \big\{ p_x(u) + u'\mu \big\} = - \sup_{u \in \Re^m} \big\{ u'(-\mu) - p_x(u) \big\} = -p_x^\star(-\mu)$$

Except for a sign change, p_x is the conjugate of $(-\phi)(x, \cdot)$ [assuming $(-\hat{\text{cl}}\, \phi)(x, \cdot)$ is proper], so

$$p_x^\star(-\mu) = -(\hat{\text{cl}}\, \phi)(x, \mu).$$

Hence, for all $\mu \in \Re^m$,

$$\begin{aligned} q(\mu) &= \inf_{u \in \Re^m} \big\{ p(u) + u'\mu \big\} \\ &= \inf_{u \in \Re^m} \inf_{x \in X} \big\{ p_x(u) + u'\mu \big\} \\ &= \inf_{x \in X} \inf_{u \in \Re^m} \big\{ p_x(u) + u'\mu \big\} \\ &= \inf_{x \in X} \big\{ -p_x^\star(-\mu) \big\} \\ &= \inf_{x \in X} (\hat{\text{cl}}\, \phi)(x, \mu) \end{aligned}$$

6.253 Convex Analysis and Optimization
Spring 2010