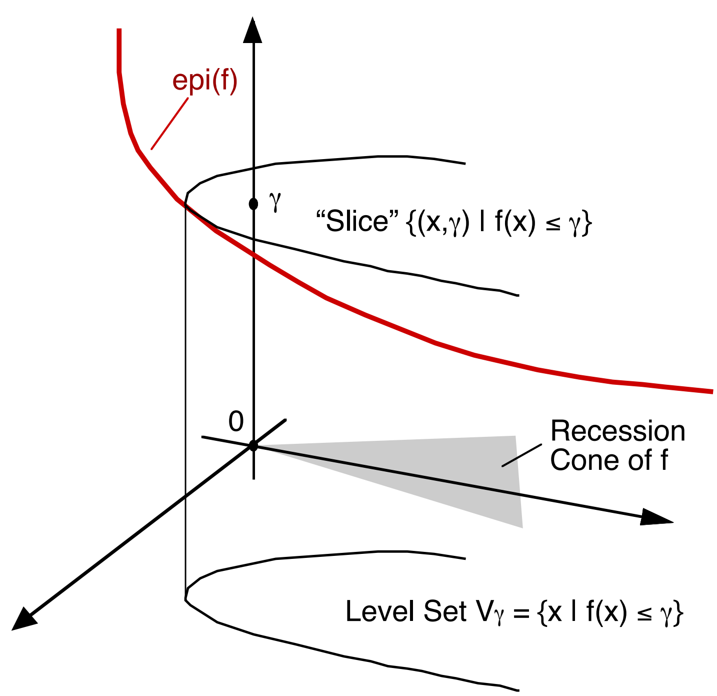
LECTURE 5

LECTURE OUTLINE

- Directions of recession of convex functions
- Local and global minima
- Existence of optimal solutions

DIRECTIONS OF RECESSION OF A FN

- We aim to characterize directions of monotonic decrease of convex functions.
- Some basic geometric observations:
  - The "horizontal directions" in the recession cone of the epigraph of a convex function f are directions along which the level sets are unbounded.
  - Along these directions the level sets $\{x \mid f(x) \leq \gamma\}$ are unbounded and f is monotonically nondecreasing.
- These are the *directions of recession* of f.



RECESSION CONE OF LEVEL SETS

- *Proposition*: Let $f : \Re^n \mapsto (-\infty, \infty]$ be a closed proper convex function and consider the level sets $V_\gamma = \{x \mid f(x) \leq \gamma\}$, where γ is a scalar. Then:

  (a) All the nonempty level sets V_γ have the same recession cone:

$$R_{V_\gamma} = \{d \mid (d, 0) \in R_{\text{epi}(f)}\}$$

  (b) If one nonempty level set V_γ is compact, then all level sets are compact.

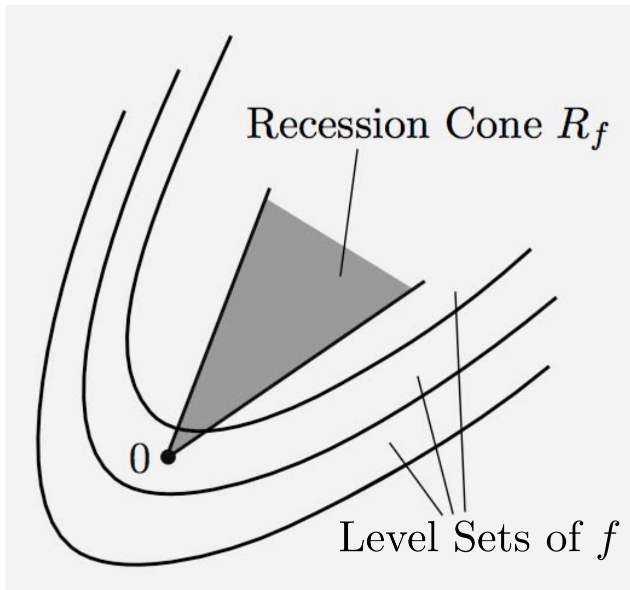
Proof: (a) Just translate to math the fact that

$R_{V_\gamma} =$ the "horizontal" directions of recession of epi(f)

(b) Follows from (a).

RECESSION CONE OF A CONVEX FUNCTION

- For a closed proper convex function $f : \Re^n \mapsto (-\infty, \infty]$, the (common) recession cone of the nonempty level sets $V_\gamma = \{x \mid f(x) \leq \gamma\}$, $\gamma \in \Re$, is the *recession cone of* f, and is denoted by R_f.



- Terminology:
  - $d \in R_f$: a *direction of recession of* f.
  - $L_f = R_f \cap (-R_f)$: the *lineality space of* f.
  - $d \in L_f$: a *direction of constancy* of f.
- **Example:** For the pos. semidefinite quadratic

$$f(x) = x'Qx + a'x + b,$$

the recession cone and constancy space are

$$R_f = \{d \mid Qd = 0,\ a'd \leq 0\}, \ \ L_f = \{d \mid Qd = 0,\ a'd = 0\}$$

RECESSION FUNCTION

- Function $r_f : \Re^n \mapsto (-\infty, \infty]$ whose epigraph is $R_{\text{epi}(f)}$ is the *recession function* of f.
- Characterizes the recession cone:

$$R_f = \big\{d \mid r_f(d) \leq 0\big\}, \quad L_f = \big\{d \mid r_f(d) = r_f(-d) = 0\big\}$$

since $R_f = \{(d, 0) \in R_{\text{epi}(f)}\}$.

- Can be shown that

$$r_f(d) = \sup_{\alpha > 0} \frac{f(x + \alpha d) - f(x)}{\alpha} = \lim_{\alpha \to \infty} \frac{f(x + \alpha d) - f(x)}{\alpha}$$

- Thus $r_f(d)$ is the "asymptotic slope" of f in the direction d. In fact,

$$r_f(d) = \lim_{\alpha \to \infty} \nabla f(x + \alpha d)' d, \qquad \forall\, x, d \in \Re^n$$

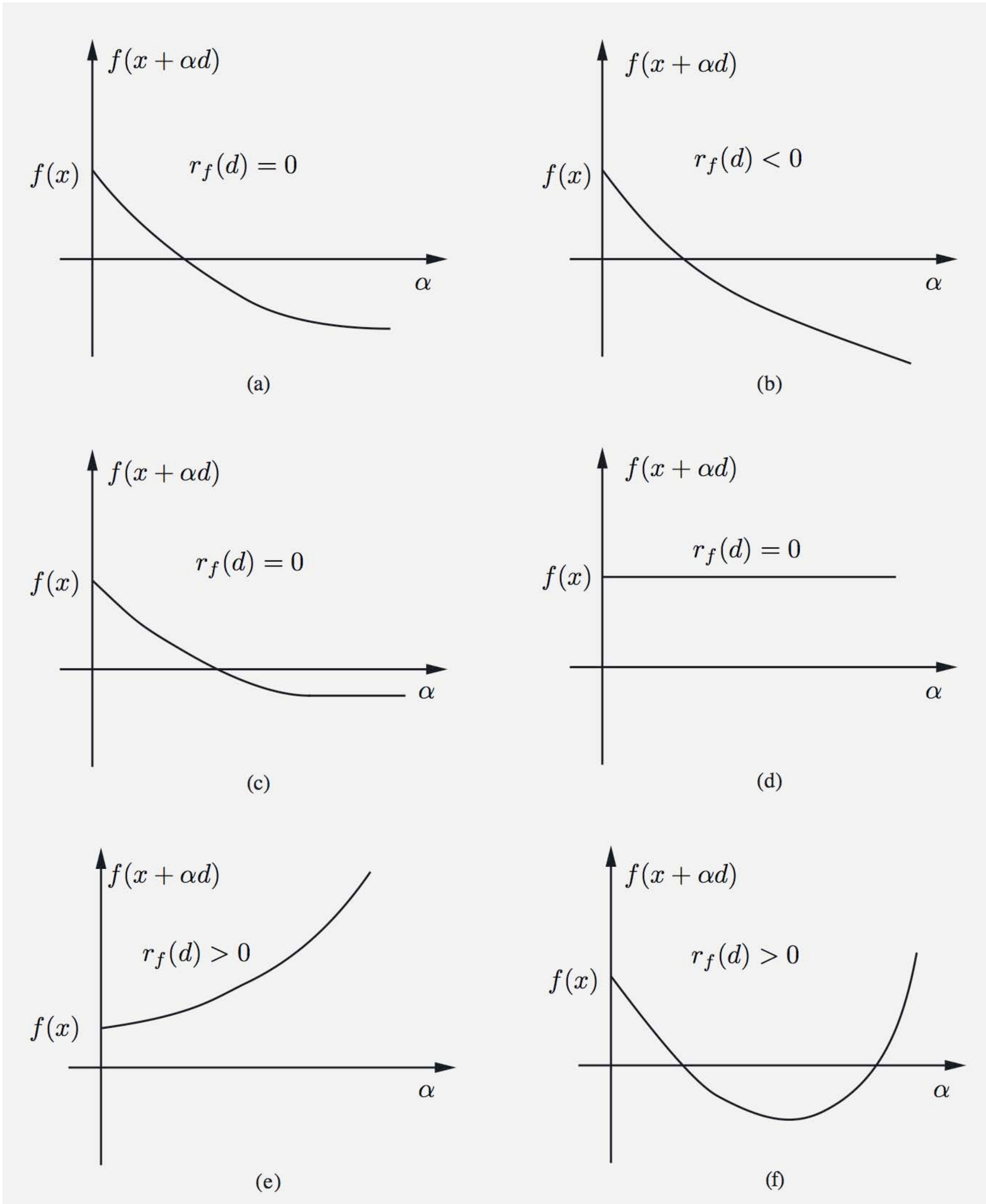
if f is differentiable.

- Calculus of recession functions:

$$r_{f_1 + \cdots + f_m}(d) = r_{f_1}(d) + \cdots + r_{f_m}(d),$$

$$r_{\sup_{i \in I} f_i}(d) = \sup_{i \in I} r_{f_i}(d)$$

DESCENT BEHAVIOR OF A CONVEX FN



- y is a direction of recession in (a)-(d).
- This behavior is *independent of the starting point* x, as long as $x \in \mathrm{dom}(f)$.

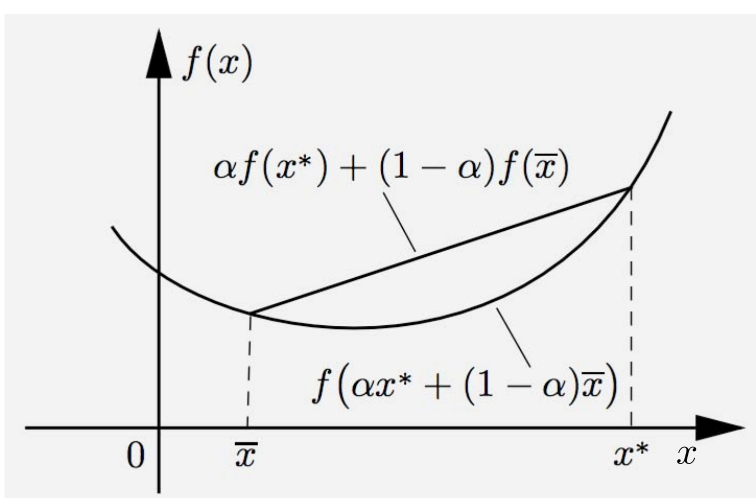
LOCAL AND GLOBAL MINIMA

- Consider minimizing $f : \Re^n \mapsto (-\infty, \infty]$ over a set $X \subset \Re^n$
- x is **feasible** if $x \in X \cap \mathrm{dom}(f)$
- x^* is a (global) **minimum** of f over X if x^* is feasible and $f(x^*) = \inf_{x \in X} f(x)$
- x^* is a **local minimum** of f over X if x^* is a minimum of f over a set $X \cap \{x \mid \|x - x^*\| \leq \epsilon\}$

Proposition: If X is convex and f is convex, then:

(a) A local minimum of f over X is also a global minimum of f over X.

(b) If f is strictly convex, then there exists at most one global minimum of f over X.



EXISTENCE OF OPTIMAL SOLUTIONS

- The set of minima of a proper $f : \Re^n \mapsto (-\infty, \infty]$ is the intersection of its nonempty level sets.

- The set of minima of f is nonempty and compact if the level sets of f are compact.

- **(An Extension of the) Weierstrass' Theorem:** The set of minima of f over X is nonempty and compact if X is closed, f is lower semicontinuous over X, and one of the following conditions holds:

  (1) X is bounded.

  (2) Some set $\{x \in X \mid f(x) \leq \gamma\}$ is nonempty and bounded.

  (3) For every sequence $\{x_k\} \subset X$ s. t. $\|x_k\| \to \infty$, we have $\lim_{k\to\infty} f(x_k) = \infty$. (Coercivity property).

Proof: In all cases the level sets of $f \cap X$ are compact. **Q.E.D.**

EXISTENCE OF SOLUTIONS - CONVEX C

- **Weierstrass' Theorem specialized to convex functions:** Let X be a closed convex subset of $\Re^n$, and let $f : \Re^n \mapsto (-\infty, \infty]$ be closed convex with $X \cap \mathrm{dom}(f) \neq \emptyset$. The set of minima of f over X is nonempty and compact if and only if X and f have no common nonzero direction of recession.

Proof: Let $f^* = \inf_{x \in X} f(x)$ and note that $f^* < \infty$ since $X \cap \mathrm{dom}(f) \neq \emptyset$. Let $\{\gamma_k\}$ be a scalar sequence with $\gamma_k \downarrow f^*$, and consider the sets

$$V_k = \big\{x \mid f(x) \leq \gamma_k\big\}.$$

Then the set of minima of f over X is

$$X^* = \cap_{k=1}^{\infty} (X \cap V_k).$$

The sets $X \cap V_k$ are nonempty and have $R_X \cap R_f$ as their common recession cone, which is also the recession cone of X^*, when $X^* \neq \emptyset$. It follows X^* is nonempty and compact if and only if $R_X \cap R_f = \{0\}$. **Q.E.D.**

EXISTENCE OF SOLUTION, SUM OF FNS

- Let $f_i : \Re^n \mapsto (-\infty, \infty]$, $i = 1, \ldots, m$, be closed proper convex functions such that the function

$$f = f_1 + \cdots + f_m$$

is proper. Assume that the recession function of a single function f_i satisfies $r_{f_i}(d) = \infty$ for all $d \neq 0$. Then the set of minima of f is nonempty and compact.

- **Proof:** The set of minima of f is nonempty and compact if and only if $R_f = \{0\}$, which is true if and only if $r_f(d) > 0$ for all $d \neq 0$. **Q.E.D.**

- **Example of application:** If one of the f_i is positive definite quadratic, the set of minima of the sum f is nonempty and compact.

- Also f has a unique minimum because the positive definite quadratic is strictly convex, which makes f strictly convex.

PROJECTION THEOREM

- Let C be a nonempty closed convex set in $\Re^n$.

  (a) For every $z \in \Re^n$, there exists a unique minimum of
  $$f(x) = \|z - x\|^2$$
  over all $x \in C$ (called the *projection of z on C*).

  (b) x^* is the projection of z if and only if
  $$(x - x^*)'(z - x^*) \leq 0, \qquad \forall\, x \in C$$

Proof: (a) f is strictly convex and has compact level sets.

(b) This is just the necessary and sufficient optimality condition

$$\nabla f(x^*)'(x - x^*) \geq 0, \qquad \forall\, x \in C.$$

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