LECTURE 4

LECTURE OUTLINE

- Algebra of relative interiors and closures
- Continuity of convex functions
- Closures of functions
- Recession cones and lineality space

CALCULUS OF REL. INTERIORS: SUMMARY

- The $\text{ri}(C)$ and $\text{cl}(C)$ of a convex set C "differ very little."
  - Any set "between" $\text{ri}(C)$ and $\text{cl}(C)$ has the same relative interior and closure.
  - The relative interior of a convex set is equal to the relative interior of its closure.
  - The closure of the relative interior of a convex set is equal to its closure.
- Relative interior and closure commute with Cartesian product and inverse image under a linear transformation.
- Relative interior commutes with image under a linear transformation and vector sum, but closure does not.
- Neither relative interior nor closure commute with set intersection.

CLOSURE VS RELATIVE INTERIOR

- *Proposition*:

  (a) We have $\mathrm{cl}(C) = \mathrm{cl}\big(\mathrm{ri}(C)\big)$ and $\mathrm{ri}(C) = \mathrm{ri}\big(\mathrm{cl}(C)\big)$.

  (b) Let C be another nonempty convex set. Then the following three conditions are equivalent:

    (i) C and C have the same rel. interior.

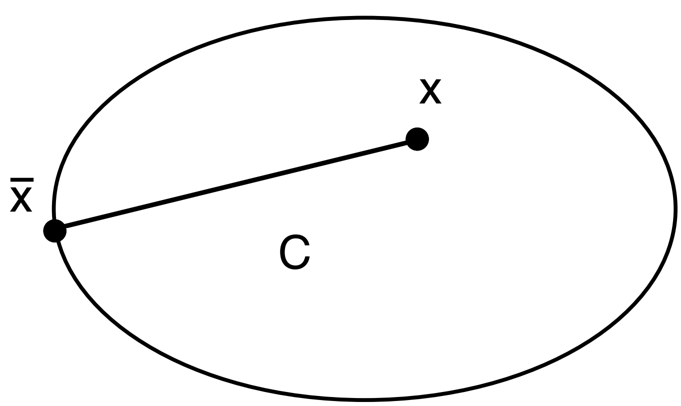
    (ii) C and C have the same closure.

    (iii) $\mathrm{ri}(C) \subset C \subset \mathrm{cl}(C)$.

Proof: (a) Since $\mathrm{ri}(C) \subset C$, we have $\mathrm{cl}\big(\mathrm{ri}(C)\big) \subset \mathrm{cl}(C)$. Conversely, let $x \in \mathrm{cl}(C)$. Let $x \in \mathrm{ri}(C)$. By the Line Segment Principle, we have

$$\alpha x + (1-\alpha)x \in \mathrm{ri}(C), \qquad \forall\, \alpha \in (0,1].$$

Thus, x is the limit of a sequence that lies in $\mathrm{ri}(C)$, so $x \in \mathrm{cl}\big(\mathrm{ri}(C)\big)$.



The proof of $\mathrm{ri}(C) = \mathrm{ri}\big(\mathrm{cl}(C)\big)$ is similar.

LINEAR TRANSFORMATIONS

- Let C be a nonempty convex subset of $\Re^n$ and let A be an $m \times n$ matrix.

  (a) We have $A \cdot \text{ri}(C) = \text{ri}(A \cdot C)$.

  (b) We have $A \cdot \text{cl}(C) \subset \text{cl}(A \cdot C)$. Furthermore, if C is bounded, then $A \cdot \text{cl}(C) = \text{cl}(A \cdot C)$.

Proof: (a) Intuition: Spheres within C are mapped onto spheres within $A \cdot C$ (relative to the affine hull).

(b) We have $A \cdot \text{cl}(C) \subset \text{cl}(A \cdot C)$, since if a sequence $\{x_k\} \subset C$ converges to some $x \in \text{cl}(C)$ then the sequence $\{Ax_k\}$, which belongs to $A \cdot C$, converges to Ax, implying that $Ax \in \text{cl}(A \cdot C)$.

To show the converse, assuming that C is bounded, choose any $z \in \text{cl}(A \cdot C)$. Then, there exists $\{x_k\} \subset C$ such that $Ax_k \to z$. Since C is bounded, $\{x_k\}$ has a subsequence that converges to some $x \in \text{cl}(C)$, and we must have $Ax = z$. It follows that $z \in A \cdot \text{cl}(C)$. **Q.E.D.**

Note that in general, we may have

$$A \cdot \text{int}(C) \neq \text{int}(A \cdot C), \qquad A \cdot \text{cl}(C) \neq \text{cl}(A \cdot C)$$

INTERSECTIONS AND VECTOR SUMS

- Let C_1 and C_2 be nonempty convex sets.

  (a) We have

$$\mathrm{ri}(C_1 + C_2) = \mathrm{ri}(C_1) + \mathrm{ri}(C_2),$$

$$\mathrm{cl}(C_1) + \mathrm{cl}(C_2) \subset \mathrm{cl}(C_1 + C_2)$$

  If one of C_1 and C_2 is bounded, then

$$\mathrm{cl}(C_1) + \mathrm{cl}(C_2) = \mathrm{cl}(C_1 + C_2)$$

  (b) If $\mathrm{ri}(C_1) \cap \mathrm{ri}(C_2) \neq \emptyset$, then

$$\mathrm{ri}(C_1 \cap C_2) = \mathrm{ri}(C_1) \cap \mathrm{ri}(C_2),$$

$$\mathrm{cl}(C_1 \cap C_2) = \mathrm{cl}(C_1) \cap \mathrm{cl}(C_2)$$

Proof of (a): $C_1 + C_2$ is the result of the linear transformation $(x_1, x_2) \mapsto x_1 + x_2$.

- Counterexample for (b):

$$C_1 = \{x \mid x \leq 0\}, \qquad C_2 = \{x \mid x \geq 0\}$$

CARTESIAN PRODUCT - GENERALIZATION

- Let C be convex set in $\Re^{n+m}$. For $x \in \Re^n$, let

$$C_x = \{y \mid (x, y) \in C\},$$

and let

$$D = \{x \mid C_x \neq \emptyset\}.$$

Then

$$\mathrm{ri}(C) = \big\{(x, y) \mid x \in \mathrm{ri}(D),\ y \in \mathrm{ri}(C_x)\big\}.$$

Proof: Since D is projection of C on x-axis,

$$\mathrm{ri}(D) = \big\{x \mid \text{there exists } y \in \Re^m \text{ with } (x, y) \in \mathrm{ri}(C)\big\},$$

so that

$$\mathrm{ri}(C) = \cup_{x \in \mathrm{ri}(D)} \Big(M_x \cap \mathrm{ri}(C)\Big),$$

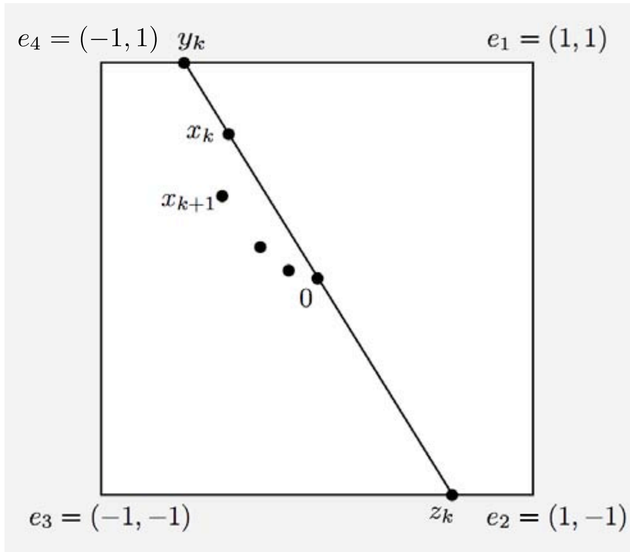
where $M_x = \big\{(x, y) \mid y \in \Re^m\big\}$. For every $x \in \mathrm{ri}(D)$, we have

$$M_x \cap \mathrm{ri}(C) = \mathrm{ri}(M_x \cap C) = \big\{(x, y) \mid y \in \mathrm{ri}(C_x)\big\}.$$

Combine the preceding two equations. **Q.E.D.**

CONTINUITY OF CONVEX FUNCTIONS

- If $f : \Re^n \mapsto \Re$ is convex, then it is continuous.



Proof: We will show that f is continuous at 0. By convexity, f is bounded within the unit cube by the max value of f over the corners of the cube.

Consider sequence $x_k \to 0$ and the sequences $y_k = x_k/\|x_k\|_\infty$, $z_k = -x_k/\|x_k\|_\infty$. Then

$$f(x_k) \leq \big(1 - \|x_k\|_\infty\big) f(0) + \|x_k\|_\infty f(y_k)$$

$$f(0) \leq \frac{\|x_k\|_\infty}{\|x_k\|_\infty + 1} f(z_k) + \frac{1}{\|x_k\|_\infty + 1} f(x_k)$$

Take limit as $k \to \infty$. Since $\|x_k\|_\infty \to 0$, we have

$$\limsup_{k \to \infty} \|x_k\|_\infty f(y_k) \leq 0, \quad \limsup_{k \to \infty} \frac{\|x_k\|_\infty}{\|x_k\|_\infty + 1} f(z_k) \leq 0$$

so $f(x_k) \to f(0)$. **Q.E.D.**

- Extension to continuity over $\text{ri}(\text{dom}(f))$.

CLOSURES OF FUNCTIONS

- The *closure* of a function $f : X \mapsto [-\infty, \infty]$ is the function $\mathrm{cl}\, f : \Re^n \mapsto [-\infty, \infty]$ with

$$\mathrm{epi}(\mathrm{cl}\, f) = \mathrm{cl}\big(\mathrm{epi}(f)\big)$$

- The *convex closure of* f is the function $\check{\mathrm{cl}}\, f$ with

$$\mathrm{epi}(\check{\mathrm{cl}}\, f) = \mathrm{cl}\Big(\mathrm{conv}\big(\mathrm{epi}(f)\big)\Big)$$

- *Proposition*: For any $f : X \mapsto [-\infty, \infty]$

$$\inf_{x \in X} f(x) = \inf_{x \in \Re^n} (\mathrm{cl}\, f)(x) = \inf_{x \in \Re^n} (\check{\mathrm{cl}}\, f)(x).$$

Also, any vector that attains the infimum of f over X also attains the infimum of $\mathrm{cl}\, f$ and $\check{\mathrm{cl}}\, f$.

- *Proposition*: For any $f : X \mapsto [-\infty, \infty]$:

  (a) $\mathrm{cl}\, f$ (or $\check{\mathrm{cl}}\, f$) is the greatest closed (or closed convex, resp.) function majorized by f.

  (b) If f is convex, then $\mathrm{cl}\, f$ is convex, and it is proper if and only if f is proper. Also,

$$(\mathrm{cl}\, f)(x) = f(x), \qquad \forall\, x \in \mathrm{ri}\big(\mathrm{dom}(f)\big),$$

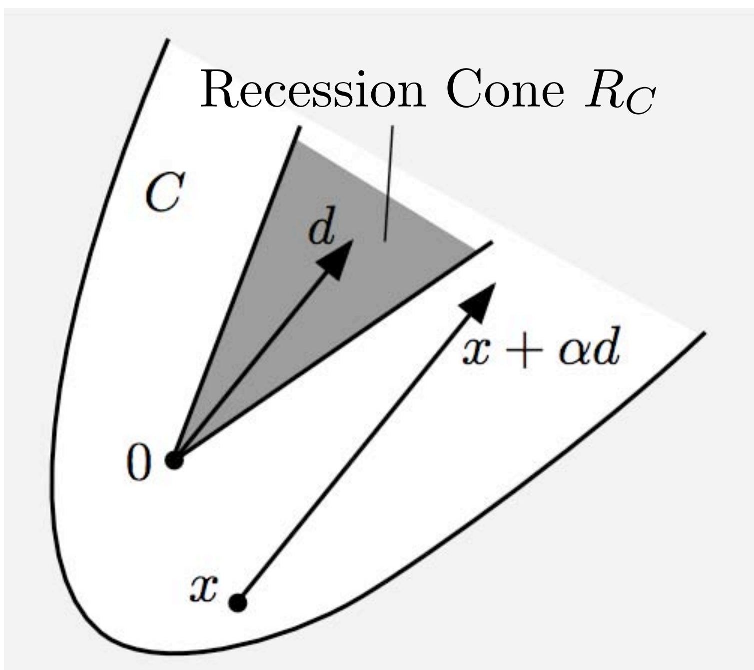
and if $x \in \mathrm{ri}\big(\mathrm{dom}(f)\big)$ and $y \in \mathrm{dom}(\mathrm{cl}\, f)$,

$$(\mathrm{cl}\, f)(y) = \lim_{\alpha \downarrow 0} f\big(y + \alpha(x - y)\big).$$

RECESSION CONE OF A CONVEX SET

- Given a nonempty convex set C, a vector d is a *direction of recession* if starting at **any** x in C and going indefinitely along d, we never cross the relative boundary of C to points outside C:

$$x + \alpha d \in C, \qquad \forall\, x \in C, \;\; \forall\, \alpha \geq 0$$



- *Recession cone* of C (denoted by R_C): The set of all directions of recession.

- R_C is a cone containing the origin.

RECESSION CONE THEOREM

- Let C be a nonempty closed convex set.

  (a) The recession cone R_C is a closed convex cone.

  (b) A vector d belongs to R_C if and only if there exists *some* vector $x \in C$ such that $x + \alpha d \in C$ for all $\alpha \geq 0$.

  (c) R_C contains a nonzero direction if and only if C is unbounded.

  (d) The recession cones of C and $\mathrm{ri}(C)$ are equal.

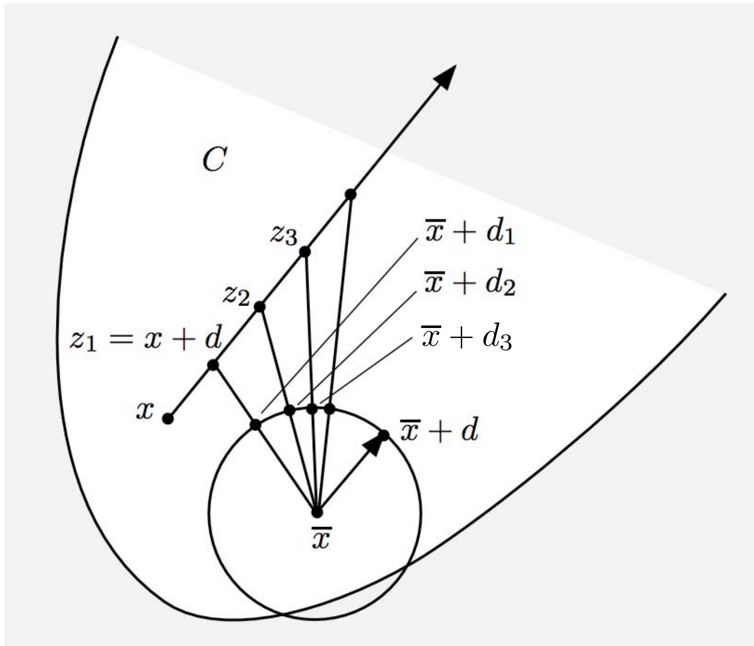
  (e) If D is another closed convex set such that $C \cap D \neq \emptyset$, we have

$$R_{C \cap D} = R_C \cap R_D$$

  More generally, for any collection of closed convex sets C_i, $i \in I$, where I is an arbitrary index set and $\cap_{i \in I} C_i$ is nonempty, we have

$$R_{\cap_{i \in I} C_i} = \cap_{i \in I} R_{C_i}$$

PROOF OF PART (B)



- Let $d \neq 0$ be such that there exists a vector $x \in C$ with $x + \alpha d \in C$ for all $\alpha \geq 0$. We fix $x \in C$ and $\alpha > 0$, and we show that $x + \alpha d \in C$. By scaling d, it is enough to show that $x + d \in C$.

For $k = 1, 2, \ldots$, let

$$z_k = x + kd, \qquad d_k = \frac{(z_k - x)}{\|z_k - x\|} \|d\|$$

We have

$$\frac{d_k}{\|d\|} = \frac{\|z_k - x\|}{\|z_k - x\|} \frac{d}{\|d\|} + \frac{x - x}{\|z_k - x\|}, \quad \frac{\|z_k - x\|}{\|z_k - x\|} \to 1, \quad \frac{x - x}{\|z_k - x\|} \to 0,$$

so $d_k \to d$ and $x + d_k \to x + d$. Use the convexity and closedness of C to conclude that $x + d \in C$.

LINEALITY SPACE

- The *lineality space* of a convex set C, denoted by L_C, is the subspace of vectors d such that $d \in R_C$ and $-d \in R_C$:

$$L_C = R_C \cap (-R_C)$$

- If $d \in L_C$, the entire line defined by d is contained in C, starting at any point of C.

- *Decomposition of a Convex Set*: Let C be a nonempty convex subset of $\Re^n$. Then,

$$C = L_C + (C \cap L_C^{\perp}).$$

- Allows us to prove properties of C on $C \cap L_C^{\perp}$ and extend them to C.

- True also if L_C is replaced by a subspace $S \subset L_C$.