6.253: Convex Analysis and Optimization
Homework 5

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Problem 1

Consider the convex programming problem

$$\begin{aligned} &\underset{x}{\text{minimize}} && f(x) \\ &\text{subject to} && x \in X, \quad g(x) \leq 0, \end{aligned}$$

of Section 5.3, and assume that the set X is described by equality and inequality constraints as

$$X = \{x \mid h_i(x) = 0,\, i = 1, \ldots, \bar{m},\, g_j(x) \leq 0,\, j = r+1, \ldots, \bar{r}\}.$$

Then the problem can alternatively be described without an abstract set constraint, in terms of all of the constraint functions

$$h_i(x) = 0, \quad i = 1, \ldots, \bar{m}, \qquad g_j(x) \leq 0, \quad j = 1, \ldots, \bar{r}.$$

We call this the *extended representation* of (P). Show if there is no duality gap and there exists a dual optimal solution for the extended representation, the same is true for the original problem.

Solution.
Assume that there exists a dual optimal solution in the extended representation. Thus there exist nonnegative scalars $\lambda_1^*, \ldots, \lambda_m^*, \lambda_{m+1}^*, \ldots, \lambda_{\bar{m}}^*$ and $\mu_1^*, \ldots, \mu_r^*, \mu_{r+1}^*, \ldots, \mu_{\bar{r}}^*$ such that

$$f^* = \inf_{x \in R^n} \left\{ f(x) + \sum_{i=1}^{\bar{m}} \lambda_i^* h_i(x) + \sum_{j=1}^{\bar{r}} \mu_j^* g_j(x) \right\},$$

from which we have

$$f^* \leq f(x) + \sum_{i=1}^{\bar{m}} \lambda_i^* h_i(x) + \sum_{j=1}^{\bar{r}} \mu_j^* g_j(x), \qquad \forall\, x \in R^n.$$

For any $x \in X$, we have $h_i(x) = 0$ for all $i = 1, \ldots, \bar{m}$, and $g_j(x) \leq 0$ for all $j = r+1, \ldots, \bar{r}$, so that $\mu_j^* g_j(x) \leq 0$ for all $j = r+1, \ldots, \bar{r}$. Therefore, it follows from the preceding relation that

$$f^* \leq f(x) + \sum_{j=1}^{r} \mu_j^* g_j(x), \qquad \forall\, x \in X.$$

Taking the infimum over all $x \in X$, it follows that

$$
\begin{aligned}
f^* &\leq \inf_{x \in X} \left\{ f(x) + \sum_{j=1}^{r} \mu_j^* g_j(x) \right\} \\
&\leq \inf_{x \in X, g_j(x) \leq 0,\ j=1,\ldots,r} \left\{ f(x) + \sum_{j=1}^{r} \mu_j^* g_j(x) \right\} \\
&\leq \inf_{\substack{x \in X,\ h_i(x)=0,\ i=1,\ldots,m \\ g_j(x) \leq 0,\ j=1,\ldots,r}} f(x) \\
&= f^*.
\end{aligned}
$$

Hence, equality holds throughout above, showing that the scalars $\lambda_1^*, \ldots, \lambda_m^*$, $\mu_1^*, \ldots, \mu_r^*$ constitute a dual optimal solution for the original representation.

Problem 2

Consider the class of problems

$$
\begin{aligned}
&\underset{x}{\text{minimize}} && f(x) \\
&\text{subject to} && x \in X, \qquad g_j(x) \leq u_j, \quad j = 1, \ldots, r,
\end{aligned}
$$

where $u = (u_1, \ldots, u_r)$ is a vector parameterizing the right-hand side of the constraints. Given two distinct values $\bar{u}$ and $\tilde{u}$ of u, let $\bar{f}$ and $\tilde{f}$ be the corresponding optimal values, and assume that $\bar{f}$ and $\tilde{f}$ are finite. Assume further that $\bar{\mu}$ and $\tilde{\mu}$ are corresponding dual optimal solutions and that there is no duality gap. Show that

$$\tilde{\mu}'(\tilde{u} - \bar{u}) \leq \bar{f} - \tilde{f} \leq \bar{\mu}'(\tilde{u} - \bar{u}).$$

Solution.
We have

$$
\begin{aligned}
\bar{f} &= \inf_{x \in X} \{ f(x) + \bar{\mu}'(g(x) - \bar{u}) \}, \\
f &= \inf_{x \in X} \{ f(x) + \mu'(g(x) - u) \}.
\end{aligned}
$$

Let $\bar{q}(\mu)$ denote the dual function of the problem corresponding to $\bar{u}$:

$$\bar{q}(\mu) = \inf_{x \in X} \{ f(x) + \mu'(g(x) - \bar{u}) \}.$$

We have

$$
\begin{aligned}
\bar{f} - f &= \inf_{x \in X} \{ f(x) + \bar{\mu}'(g(x) - \bar{u}) \} - \inf_{x \in X} \{ f(x) + \mu'(g(x) - u) \} \\
&= \inf_{x \in X} \{ f(x) + \bar{\mu}'(g(x) - \bar{u}) \} - \inf_{x \in X} \{ f(x) + \mu'(g(x) - \bar{u}) \} + \mu'(u - \bar{u}) \\
&= \bar{q}(\bar{\mu}) - \bar{q}(\mu) + \mu'(u - \bar{u}) \\
&\geq \mu'(u - \bar{u}),
\end{aligned}
$$

where the last inequality holds because $\bar{\mu}$ maximizes $\bar{q}$.

This proves the left-hand side of the desired inequality. Interchanging the roles of $\bar{f}$, $\bar{u}$, $\bar{\mu}$, and f, u, μ, shows the desired right-hand side.

Problem 3

Let $g_j : R^n \mapsto R$, $j = 1, \ldots, r$, be convex functions over the nonempty convex subset of R^n. Show that the system

$$g_j(x) < 0, \qquad j = 1, \ldots, r,$$

has no solution within X if and only if there exists a vector $\mu \in R^r$ such that

$$\sum_{j=1}^{r} \mu_j = 1, \qquad \mu \geq 0,$$

$$\mu' g(x) \geq 0, \qquad \forall\, x \in X.$$

Hint: Consider the convex program

$$\begin{aligned} &\underset{x,y}{\text{minimize}} && y \\ &\text{subject to} && x \in X,, \quad y \in R, \qquad g_j(x) \leq y, \quad j = 1, \ldots, r. \end{aligned}$$

Solution.
The dual function for the problem in the hint is

$$\begin{aligned} q(\mu) &= \inf_{y \in R,\, x \in X} \left\{ y + \sum_{j=1}^{r} \mu_j (g_j(x) - y) \right\} \\ &= \begin{cases} \inf_{x \in X} \sum_{j=1}^{r} \mu_j g_j(x) & if \sum_{j=1}^{r} \mu_j = 1, \\ -\infty & if \sum_{j=1}^{r} \mu_j \neq 1. \end{cases} \end{aligned}$$

The problem in the hint satisfies the Slater condition, so the dual problem has an optimal solution μ^* and there is no duality gap.

Clearly the problem in the hint has an optimal value that is greater or equal to 0 if and only if the system of inequalities

$$g_j(x) < 0, \qquad j = 1, \ldots, r,$$

has no solution within X. Since there is no duality gap, we have

$$\max_{\mu \geq 0,\, \sum_{j=1}^{r} \mu_j = 1} q(\mu) \geq 0$$

if and only if the system of inequalities $g_j(x) < 0$, $j = 1, \ldots, r$, has no solution within X. This is equivalent to the statement we want to prove.

Problem 4

Consider the problem

$$\begin{aligned} &\underset{x}{\text{minimize}} && f(x) \\ &\text{subject to} && x \in X, \quad g(x) \leq 0, \end{aligned}$$

where X is a convex set, and f and g_j are convex over X. Assume that the problem has at least one feasible solution. Show that the following are equivalent.

(i) The dual optimal value $q^* = \sup_{\mu \in R^r} q(\mu)$ is finite.

(ii) The primal function p is proper.

(iii) The set

$$M = \{(u, w) \in R^{r+1} \mid \text{there is an } x \in X \text{ such that } g(x) \leq u,\ f(x) \leq w\}$$

does not contain a vertical line.

Solution.
We note that $-q$ is closed and convex, and that

$$q(\mu) = \inf_{u \in R^r} \{p(u) + \mu' u\}, \qquad \forall\ \mu \in R^r.$$

Since $q(\mu) \leq p(0)$ for all $\mu \in R^r$, given the feasibility of the problem [i.e., $p(0) < \infty$], we see that q^* is finite if and only if q is proper. Since q is the conjugate of $p(-u)$ and p is convex, by the Conjugacy Theorem, q is proper if and only if p is proper. Hence (i) is equivalent to (ii).

We note that the epigraph of p is the closure of M. Hence, given the feasibility of the problem, (ii) is equivalent to the closure of M not containing a vertical line. Since M is convex, its closure does not contain a line if and only if M does not contain a line (since the closure and the relative interior of M have the same recession cone). Hence (ii) is equivalent to (iii).

Problem 5

Consider a proper convex function F of two vectors $x \in R^n$ and $y \in R^m$. For a fixed $(\bar{x}, \bar{y}) \in dom(F)$, let $\partial_x F(\bar{x}, \bar{y})$ and $\partial_y F(\bar{x}, \bar{y})$ be the subdifferentials of the functions $F(\cdot, \bar{y})$ and $F(\bar{x}, \cdot)$ at $\bar{x}$ and $\bar{y}$, respectively. (a) Show that

$$\partial F(\bar{x}, \bar{y}) \subset \partial_x F(\bar{x}, \bar{y}) \times \partial_y F(\bar{x}, \bar{y}),$$

and give an example showing that the inclusion may be strict in general. (b) Assume that F has the form

$$F(x, y) = h_1(x) + h_2(y) + h(x, y),$$

where h_1 and h_2 are proper convex functions, and h is convex, real-valued, and differentiable. Show that the formula of part (a) holds with equality.

Solution.
(a) We have $(g_x, g_y) \in \partial F(\bar{x}, \bar{y})$ if and only if

$$F(x, y) \geq F(\bar{x}, \bar{y}) + g_x'(x - \bar{x}) + g_y'(y - \bar{y}), \qquad \forall\ x \in R^n,\ y \in R^m.$$

By setting $y = \bar{y}$, we obtain that $g_x \in \partial_x F(\bar{x}, \bar{y})$, and by setting $x = \bar{x}$, we obtain that $g_y \in \partial_y F(\bar{x}, \bar{y})$, so that $(g_x, g_y) \in \partial_x F(\bar{x}, \bar{y}) \times \partial_y F(\bar{x}, \bar{y})$.

For an example where the inclusion is strict, consider any function whose subdifferential is not a Cartesian product at some point, such as $F(x, y) = |x + y|$ at points $(\bar{x}, \bar{y})$ with $\bar{x} + \bar{y} = 0$.
(b) Since F is the sum of functions of the given form, we have

$$\partial F(\bar{x}, \bar{y}) = \{(g_x, 0) \mid g_x \in \partial h_1(\bar{x})\} + \{(0, g_y) \mid g_y \in \partial h_2(\bar{y})\} + \{\nabla h(\bar{x}, \bar{y})\}$$

[the relative interior condition of the proposition is clearly satisfied]. Since

$$\nabla h(\bar{x}, \bar{y}) = (\nabla_x h(\bar{x}, \bar{y}), \nabla_y h(\bar{x}, \bar{y})),$$

$$\partial_x F(\bar{x}, \bar{y}) = \partial h_1(\bar{x}) + \nabla_x h(\bar{x}, \bar{y}),$$

$$\partial_y F(\bar{x}, \bar{y}) = \partial h_2(\bar{y}) + \nabla_y h(\bar{x}, \bar{y}),$$

the result follows.

Problem 6

This exercise shows how a duality gap results in nondifferentiability of the dual function. Consider the problem

$$\begin{aligned} &\underset{x}{\text{minimize}} && f(x) \\ &\text{subject to} && x \in X, \quad g(x) \leq 0, \end{aligned}$$

and assume that for all $\mu \geq 0$, the infimum of the Lagrangian $L(x, \mu)$ over X is attained by at least one $x_\mu \in X$. Show that if there is a duality gap, then the dual function $q(\mu) = \inf_{x \in X} L(x, \mu)$ is nondifferentiable at every dual optimal solution. *Hint*: If q is differentiable at a dual optimal solution μ^*, by the theory of Section 5.3, we must have $\partial q(\mu^*)/\partial \mu_j \leq 0$ and $\mu_j^* \partial q(\mu^*)/\partial \mu_j = 0$ for all j. Use optimality conditions for μ^*, together with any vector x_{μ^*} that minimizes $L(x, \mu^*)$ over X, to show that there is no duality gap.

Solution.
To obtain a contradiction, assume that q is differentiable at some dual optimal solution $\mu^* \in M$, where $M = \{\mu \in R^r \mid \mu \geq 0\}$. Then

$$\nabla q(\mu^*)(\mu^* - \mu) \geq 0, \qquad \forall\ \mu \geq 0.$$

If $\mu_j^* = 0$, then by letting $\mu = \mu^* + \gamma e_j$ for a scalar $\gamma \geq 0$, and the vector e_j whose jth component is 1 and the other components are 0, from the preceding relation we obtain $\partial q(\mu^*)/\partial \mu_j \leq 0$. Similarly, if $\mu_j^* > 0$, then by letting $\mu = \mu^* + \gamma e_j$ for a sufficiently small scalar γ (small enough so that $\mu^* + \gamma e_j \in M$), from the preceding relation we obtain $\partial q(\mu^*)/\partial \mu_j = 0$. Hence

$$\partial q(\mu^*)/\partial \mu_j \leq 0, \qquad \forall\ j = 1, \ldots, r,$$

$$\mu_j^* \partial q(\mu^*)/\partial \mu_j = 0, \qquad \forall\ j = 1, \ldots, r.$$

Since q is differentiable at μ^*, we have that

$$\nabla q(\mu^*) = g(x^*),$$

for some vector $x^* \in X$ such that $q(\mu^*) = L(x^*, \mu^*)$. This and the preceding two relations imply that x^* and μ^* satisfy the necessary and sufficient optimality conditions for an optimal primal and dual optimal solution pair. It follows that there is no duality gap, a contradiction.

Problem 7

Consider the problem

$$\begin{aligned} &\underset{x}{\text{minimize}} && f(x) = 10x_1 + 3x_2 \\ &\text{subject to} && 5x_1 + x_2 \geq 4, x_1, x_2 = 0 \text{ or } 1, \end{aligned}$$

(a) Sketch the set of constraint-cost pairs $\{(4 - 5x_1 - x_2, 10x_1 + 3x_2) | x_1, x_2 = 0 \text{ or } 1\}$.
(b)Describe the corresponding MC/MC framework as per Section 4.2.3.
(c) Solve the problem and its dual, and relate the solutions to your sketch in part (a).

Solution.
(a) The set of constraint-cost pairs contains 4 points: (-2,13), (-1,10), (3,3), (4,0).
(b) To each of these 4 points we add the first orphant and we get the $\bar{M}$ set.

(c) The primal optimal solution is $x^* = (1, 0)$ and the primal optimal cost is $p^* = 10$. The dual function is easily found to be:

$$q(\mu) = \begin{cases} 4\mu & if \mu \leq 2, \\ 10 - \mu & if 2 \leq \mu \leq 3, \\ 13 - 2\mu & if 3 \leq \mu. \end{cases}$$

Therefore $q^* = 8$. This is the intersection of the line segment connecting the points (4,0), (-1,10) with the y-axis.