6.253: Convex Analysis and Optimization
Homework 2

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Problem 1

(a) Let C be a nonempty convex cone. Show that $cl(C)$ and $ri(C)$ is also a convex cone.
(b) Let $C = cone(\{x_1, \ldots, x_m\})$. Show that

$$ri(C) = \{\sum_{i=1}^{m} a_i x_i | a_i > 0, i = 1, \ldots, m\}.$$

Problem 2

Let C_1 and C_2 be convex sets. Show that

$$C_1 \cap ri(C_2) \neq \emptyset \qquad \text{if and only if} \qquad ri(C_1 \cap aff(C_2)) \cap ri(C_2) \neq \emptyset.$$

Problem 3

(a) Consider a vector x^* such that a given function $f : \mathbf{R}^n \mapsto \mathbf{R}$ is convex over a sphere centered at x^*. Show that x^* is a local minimum of f if and only if it is a local minimum of f along every line passing through x^* [i.e., for all $d \in \mathbf{R}^n$, the function $g : \mathbf{R} \mapsto \mathbf{R}$, defined by $g(\alpha) = f(x^* + \alpha d)$, has $\alpha^* = 0$ as its local minimum].
(b) Consider the nonconvex function $f : \mathbf{R}^2 \mapsto \mathbf{R}$ given by

$$f(x_1, x_2) = (x_2 - px_1^2)(x_2 - qx_1^2),$$

where p and q are scalars with $0 < p < q$, and $x^* = (0, 0)$. Show that $f(y, my^2) < 0$ for $y \neq 0$ and m satisfying $p < m < q$, so x^* is not a local minimum of f even though it is a local minimum along every line passing through x^*.

Problem 4

(a) Consider the quadratic program

$$\begin{aligned} \underset{x}{\text{minimize}} \quad & 1/2\ |x|^2 + c'x \\ \text{subject to} \quad & Ax = 0 \end{aligned}$$

where $c \in \mathbf{R}^n$ and A is an $m \times n$ matrix of rank m. Use the Projection Theorem to show that

$$x^* = -(I - A'(AA')^{-1}A)c$$

is the unique solution.
(b) Consider the more general quadratic program

$$\begin{aligned} \underset{x}{\text{minimize}} \quad & 1/2\ (x-\bar{x})'Q(x-\bar{x}) + c'(x-\bar{x}) \\ \text{subject to} \quad & Ax = b \end{aligned}$$

where c and A are as before, Q is a symmetric positive definite matrix, $b \in \mathbf{R}^m$, and $\bar{x}$ is a vector in $\mathbf{R}^n$, which is feasible, i.e., satisfies $A\bar{x} = b$. Use the transformation $y = Q^{1/2}(x - \bar{x})$ to write this problem in the form of part (a) and show that the optimal solution is

$$x^* = \bar{x} - Q^{-1}(c - A'\lambda),$$

where λ is given by

$$\lambda = (AQ^{-1}A')^{-1}AQ^{-1}c.$$

(c) Apply the result of part (b) to the program

$$\begin{aligned} \underset{x}{\text{minimize}} \quad & 1/2\ x'Qx + c'x) \\ \text{subject to} \quad & Ax = b \end{aligned}$$

and show that the optimal solution is

$$x^* = -Q^{-1}(c - A'\lambda - A'(AQ^{-1}A')^{-1}b).$$

Problem 5

Let X be a closed convex subset of $\mathbf{R}^n$, and let $f : \mathbf{R}^n \mapsto (-\infty, \infty]$ be a closed convex function such that $X \cap dom(f) \neq \emptyset$. Assume that f and X have no common nonzero direction of recession. Let X^* be the set of minima of f over X (which is nonempty and compact), and let $f^* = \inf_{x \in X} f(x)$. Show that:
(a) For every $\epsilon > 0$ there exists a $\delta > 0$ such that every vector $x \in X$ with $f(x) \leq f^* + \delta$ satisfies $\min_{x^* \in X^*} \|x - x^*\| \leq \epsilon$.
(b) If f is real-valued, for every $\delta > 0$ there exists an $\epsilon > 0$ such that every vector $x \in X$ with $\min_{x^* \in X^*} \|x - x^*\| \leq \epsilon$ satisfies $f(x) \leq f^* + \delta$.
(c) Every sequence $\{x_k\} \subset X$ satisfying $f(x_k) \to f^*$ is bounded and all its limit points belong to X^*.