6.642 Continuum Electromechanics
Fall 2008

6.642, Continuum Electromechanics, Fall 2004
Prof. Markus Zahn
Lecture 6: Stress Tensors

I. Maxwell Stress Tensor

A. Notation

$$F_x = \nabla \cdot \bar{\tau}_x, \quad \bar{\tau}_x = T_{xx}\,\bar{i}_x + T_{xy}\,\bar{i}_y + T_{xz}\,\bar{i}_z$$

$$F_y = \nabla \cdot \bar{\tau}_y, \quad \bar{\tau}_y = T_{yx}\,\bar{i}_x + T_{yy}\,\bar{i}_y + T_{yz}\,\bar{i}_z$$

$$F_z = \nabla \cdot \bar{\tau}_z, \quad \bar{\tau}_z = T_{zx}\,\bar{i}_x + T_{zy}\,\bar{i}_y + T_{zz}\,\bar{i}_z$$

$$\bar{\bar{T}} = \begin{bmatrix} T_{xx} & T_{xy} & T_{xz} \\ T_{yx} & T_{yy} & T_{yz} \\ T_{zx} & T_{zy} & T_{zz} \end{bmatrix}, \quad \bar{F} = \nabla \cdot \bar{\bar{T}}$$

$$f_x = \int_V F_x dV = \int_V \nabla \cdot \bar{\tau}_x dV = \oint_S \bar{\tau}_x \cdot \bar{n}\, da = \oint_S \left[T_{xx}n_x + T_{xy}n_y + T_{xz}n_z\right] dS$$

$$\bar{\tau}_x \cdot \bar{n} = T_{xx}n_x + T_{xy}n_y + T_{xz}n_z = T_{xn}n_n$$

$$\bar{\tau}_y \cdot \bar{n} = T_{yx}n_x + T_{yy}n_y + T_{yz}n_z = T_{yn}n_n$$

$$\bar{\tau}_z \cdot \bar{n} = T_{zx}n_x + T_{zy}n_y + T_{zz}n_z = T_{zn}n_n$$

$$f_i = \int_V \nabla \cdot \bar{\tau}_i dV = \oint_S \bar{\tau}_i \cdot \bar{n}\, dV = \oint_S T_{ij}n_j dS = \int_V F_i dV$$

$$\bar{F}_i = \nabla \cdot \bar{\tau}_i = \frac{\partial}{\partial x} T_{ix} + \frac{\partial}{\partial y} T_{iy} + \frac{\partial}{\partial z} T_{iz}$$

$$= \frac{\partial T_{ij}}{\partial x_j}$$

B. EQS Stress Tensor

$$\bar{F} = \rho_f \bar{E} - \frac{1}{2}\bar{E} \cdot \bar{E}\, \nabla\varepsilon + \nabla\left(\frac{1}{2}\bar{E} \cdot \bar{E} \frac{\partial\varepsilon}{\partial\rho}\rho\right)$$

$$= \nabla \cdot \left(\varepsilon\bar{E}\right)\bar{E} - \frac{1}{2}\left(\bar{E} \cdot \bar{E}\right)\nabla\varepsilon + \nabla\left(\frac{1}{2}\bar{E} \cdot \bar{E} \frac{\partial\varepsilon}{\partial\rho}\rho\right)$$

$$F_i = \frac{\partial\left(\varepsilon E_j\right)}{\partial x_j} E_i - \frac{1}{2} E_K E_K \frac{\partial \varepsilon}{\partial x_i} + \frac{\partial}{\partial x_i}\left(\frac{1}{2} E_K E_K \frac{\partial \varepsilon}{\partial \rho} \rho\right)$$

$$\nabla \times \bar{E} = 0 \Rightarrow \frac{\partial E_i}{\partial x_j} = \frac{\partial E_j}{\partial x_i}$$

$$F_i = \frac{\partial}{\partial x_j}\left(\varepsilon E_j E_i\right) - \varepsilon E_j \frac{\partial E_i}{\partial x_j} - \frac{1}{2} E_K E_K \frac{\partial \varepsilon}{\partial x_i} + \frac{\partial}{\partial x_i}\left(\frac{1}{2} E_K E_K \frac{\partial \varepsilon}{\partial \rho} \rho\right)$$

$$F_i = \frac{\partial}{\partial x_j}\left(\varepsilon E_i E_j\right) - \underbrace{\varepsilon E_j \frac{\partial E_j}{\partial x_i}}_{\varepsilon \frac{\partial\left(\frac{1}{2} E_j E_j\right)}{\partial x_i}} - \frac{1}{2} E_K E_K \frac{\partial \varepsilon}{\partial x_i} + \frac{\partial}{\partial x_i}\left(\frac{1}{2} E_K E_K \frac{\partial \varepsilon}{\partial \rho} \rho\right)$$

$$F_i = \frac{\partial}{\partial x_j}\left(\varepsilon E_i E_j\right) - \frac{\partial}{\partial x_i}\left[\frac{1}{2} \varepsilon E_K E_K - \frac{1}{2} \rho \frac{\partial \varepsilon}{\partial \rho} E_K E_K\right]$$

$$\frac{\partial}{\partial x_j} = \delta_{ij} \frac{\partial}{\partial x_i}$$

$$\delta_{ij} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases} \quad \text{Kronecker Delta}$$

$$F_i = \frac{\partial}{\partial x_j}\left[\varepsilon E_i E_j - \frac{1}{2} \delta_{ij} E_K E_K \left(\varepsilon - \rho \frac{\partial \varepsilon}{\partial \rho}\right)\right] = \frac{\partial T_{ij}}{\partial x_j}$$

$$T_{ij} = \varepsilon E_i E_j - \frac{1}{2} \delta_{ij} E_K E_K \left(\varepsilon - \rho \frac{\partial \varepsilon}{\partial \rho}\right)$$

C. MQS Stress Tensor

$$\bar{F} = \bar{J} \times \bar{B} - \frac{1}{2} \bar{H} \cdot \bar{H}\, \nabla \mu + \nabla\left(\frac{1}{2} \rho \frac{\partial \mu}{\partial \rho} \bar{H} \cdot \bar{H}\right)$$

$$= \left(\nabla \times \bar{H}\right) \times \left(\mu \bar{H}\right) - \frac{1}{2} \bar{H} \cdot \bar{H}\, \nabla \mu + \nabla\left(\frac{1}{2} \rho \frac{\partial \mu}{\partial \rho} \bar{H} \cdot \bar{H}\right)$$

$$\left(\nabla \times \bar{H}\right) \times \bar{H} = \left(\bar{H} \cdot \nabla\right) \bar{H} - \frac{1}{2} \nabla\left(\bar{H} \cdot \bar{H}\right)$$

$$\bar{F} = \mu\left[\left(\bar{H} \cdot \nabla\right) \bar{H} - \frac{1}{2} \nabla\left(\bar{H} \cdot \bar{H}\right)\right] - \frac{1}{2} \bar{H} \cdot \bar{H}\, \nabla \mu + \nabla\left(\frac{1}{2} \rho \frac{\partial \mu}{\partial \rho} \bar{H} \cdot \bar{H}\right)$$

$$F_i = \mu\left[H_j \frac{\partial}{\partial x_j} H_i - \frac{1}{2}\frac{\partial}{\partial x_i}\left(H_k H_k\right)\right] - \frac{1}{2} H_k H_k \frac{\partial \mu}{\partial x_i} + \frac{\partial}{\partial x_i}\left(\frac{1}{2}\rho \frac{\partial \mu}{\partial \rho} H_k H_k\right)$$

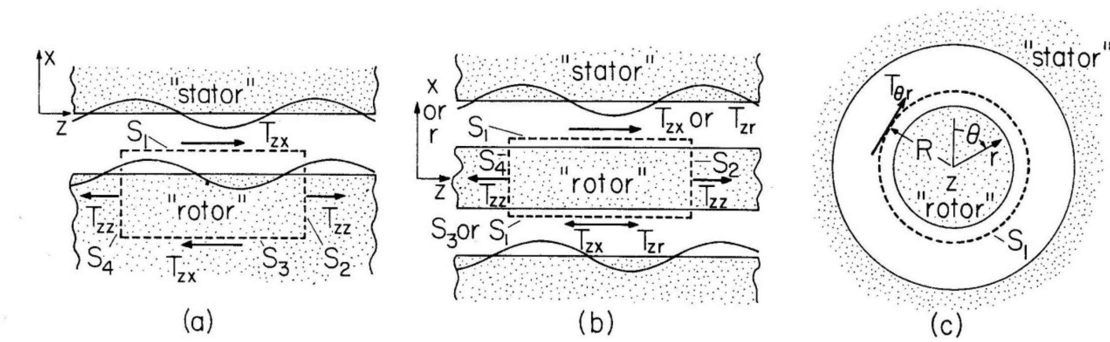
$$= \frac{\partial}{\partial x_j}\left(\mu H_i H_j\right) - \underbrace{H_i \frac{\partial}{\partial x_j}\left(\mu H_j\right)}_{\nabla \cdot \bar{B} = 0} - \underbrace{\frac{\mu}{2}\frac{\partial}{\partial x_i} H_k H_k - \frac{1}{2} H_k H_k \frac{\partial \mu}{\partial x_i}}_{-\frac{\partial}{\partial x_i}\left(\frac{1}{2}\mu H_k H_k\right)} + \frac{\partial}{\partial x_i}\left(\frac{1}{2}\rho \frac{\partial \mu}{\partial \rho} H_k H_k\right)$$

$$F_i = \frac{\partial}{\partial x_j}\left(\mu H_i H_j\right) - \frac{\partial}{\partial x_i}\left(\frac{1}{2}\mu H_k H_k - \rho \frac{\partial \mu}{\partial \rho} H_k H_k\right)$$

$$= \frac{\partial}{\partial x_j}\left(\mu H_i H_j\right) - \frac{1}{2}\delta_{ij} H_k H_k \left(\mu - \rho \frac{\partial \mu}{\partial \rho}\right) = \frac{\partial T_{ij}}{\partial x_j}$$

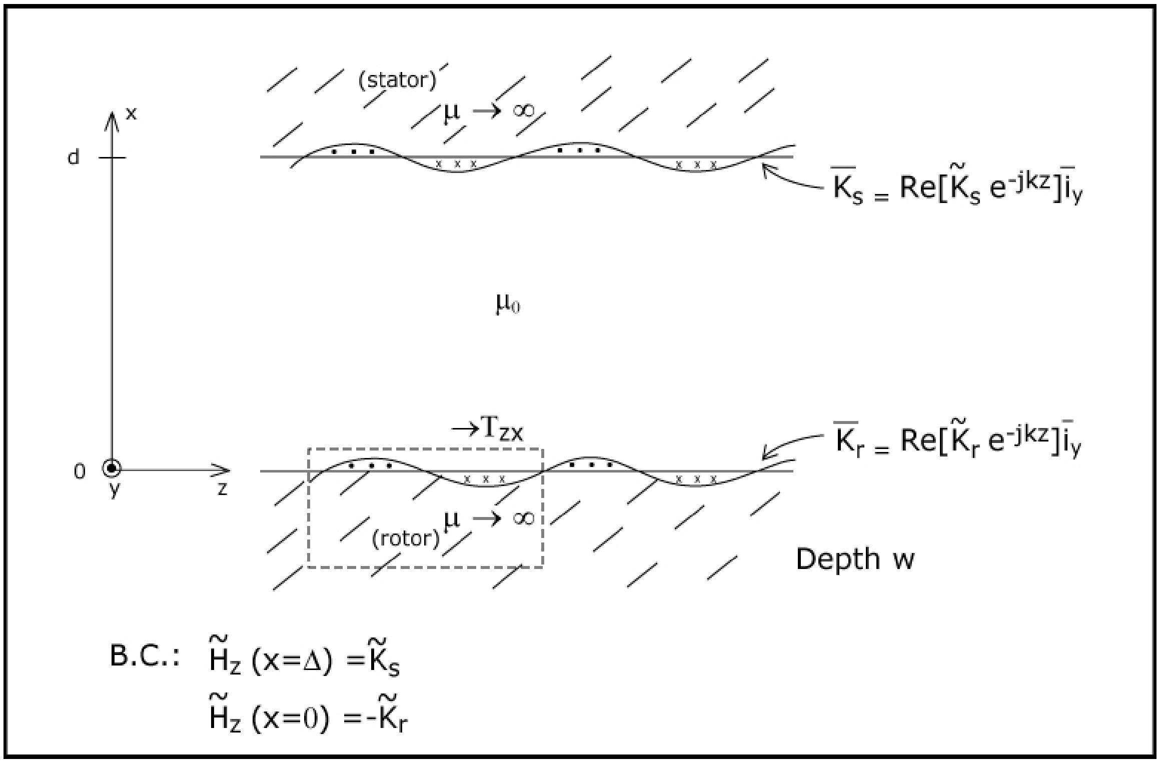
$$T_{ij} = \mu H_i H_j - \frac{1}{2}\delta_{ij} H_k H_k \left(\mu - \rho \frac{\partial \mu}{\partial \rho}\right)$$

II. Air-Gap Magnetic Machine



Typical "air-gap" configurations in which a force or torque on a rigid "rotor" results from spatially periodic sources interacting with spatially periodic excitations on a rigid "stator." Because of the periodicity, the force or torque can be represented in terms of the electric or magnetic stress acting at the air-gap surfaces S_1: (a) planar geometry or developed model; (b) planar or cylindrical beam; (c) cylindrical rotor.

A. Generalized Description



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force on a wavelength

$$a(z) = \mathrm{Re}\left[\tilde{A}\,e^{-jkz}\right]$$

$$b(z) = \mathrm{Re}\left[\tilde{B}\,e^{-jkz}\right]$$

$$\frac{k}{2\pi}\int_0^{2\pi/k} a(z)\,b(z)\,dz = \frac{1}{2}\mathrm{Re}\left[\tilde{A}\tilde{B}^*\right] = \frac{1}{2}\mathrm{Re}\left[\tilde{A}^*\tilde{B}\right]$$

$$f_z = \frac{\pi w}{k}\mu_0 \mathrm{Re}\left[\tilde{H}_z^r \tilde{H}_x^{r\,*}\right]$$

$$= \frac{\pi w \mu_0}{k}\mathrm{Re}\left[-\tilde{K}_r \tilde{H}_x^{r\,*}\right]$$

$$\begin{bmatrix}\tilde{B}_x^s \\ \tilde{B}_x^r\end{bmatrix} = \mu_0 k \begin{bmatrix} -\coth kd & \dfrac{1}{\sinh kd} \\ -\dfrac{1}{\sinh kd} & \coth kd \end{bmatrix} \begin{bmatrix}\tilde{\chi}_s \\ \tilde{\chi}_r\end{bmatrix}$$

$$\tilde{H}_z = +jk\tilde{\chi} \Rightarrow \tilde{\chi}^s = \frac{1}{jk}\tilde{H}_z^s = \frac{\tilde{K}_s}{jk}$$

$$\tilde{\chi}^r = \frac{1}{jk}\tilde{H}_z^r = -\frac{\tilde{K}_r}{jk}$$

$$\mu_0 \tilde{H}_x^r = \mu_0 k \left[-\frac{\tilde{\chi}^s}{\sinh kd} + \tilde{\chi}^r \coth kd \right]$$

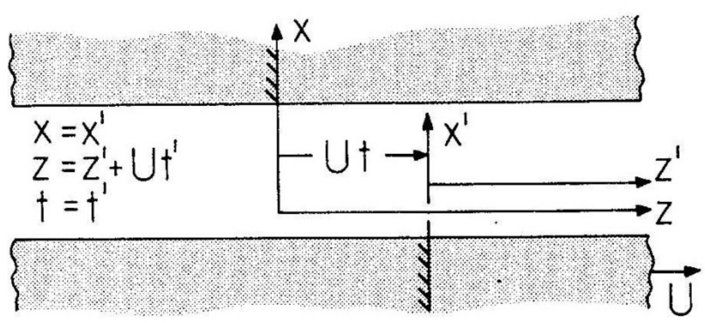
$$= \mu_0 k \left[-\frac{\tilde{K}_s}{jk \sinh kd} - \frac{\tilde{K}_r}{jk} \coth kd \right]$$

$$\text{Re}\left[-\tilde{K}_r^* \tilde{H}_x^r \mu_0 \right] = -\text{Re}\left[+\frac{j\mu_0 \cancel{k}}{\cancel{k}} \left(\frac{\tilde{K}_r^* \tilde{K}_s}{\sinh kd} + \tilde{K}_r^* \tilde{K}_r \coth kd \right) \right]$$

$$= \text{Re}\left[-\mu_0 j \tilde{K}_r^* \tilde{K}_s / \sinh kd \right]$$

$$f_z = -\frac{\pi w}{k} \frac{\mu_0}{\sinh kd} \text{Re}\left[j\tilde{K}_r^* \tilde{K}_s \right] \text{ (force on each wavelength)}$$

B. Synchronous Interaction



Rotor and stator reference frames z' and z.

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$$K_s = K_0^s \sin\left[\omega_s t - kz\right] = \text{Re}\left[-jK_0^s e^{j(\omega_s t - kz)} \right]$$

$$K_r = K_0^r \sin\left[\omega_r t - k\left(z' - \delta\right)\right]; \quad z' = z - Ut$$

$$= K_0^r \sin\left[\left(\omega_r + kU\right)t - k\left(z - \delta\right)\right]$$

$$= \mathrm{Re}\left[-jK_0^r\, e^{j(\omega_r + kU)t} e^{jk\delta}\right]$$

$$\tilde{K}_s = -jK_0^s\, e^{j\omega_s t}$$

$$\tilde{K}_r = -jK_0^r e^{jk\delta} e^{j(\omega_r + kU)t}$$

$$f_z = -\frac{\pi W}{k}\frac{\mu_0}{\sinh kd}\mathrm{Re}\left[j\left(-jK_0^s\right)e^{j\omega_s t}\left(jK_0^r e^{-jk\delta}\right)e^{-j(\omega_r + kU)t}\right]$$

$$= -\frac{\pi W}{k}\frac{\mu_0}{\sinh kd}K_0^s K_0^r \mathrm{Re}\left[je^{-jk\delta}e^{j(\omega_s - \omega_r - kU)t}\right]$$

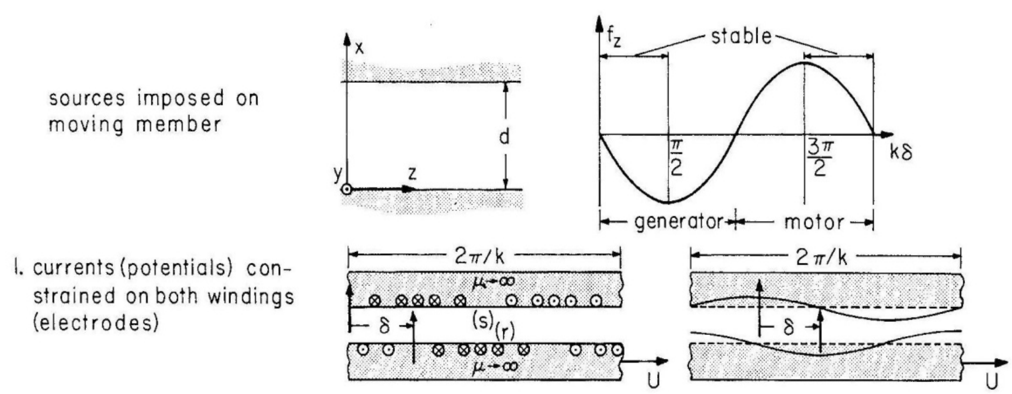
For time average force

$\Rightarrow \omega_s = \omega_r + kU$ (synchronous condition)

Usually $\omega_r = 0 \Rightarrow \omega_s = kU$

$$\left\langle f_z \right\rangle = -\frac{\pi W}{k}\frac{\mu_0}{\sinh kd}K_0^s K_0^r \sin k\delta$$

Basic configurations illustrating classes of electromechanical interactions and devices. MQS and EQS systems respectively in left and right columns.



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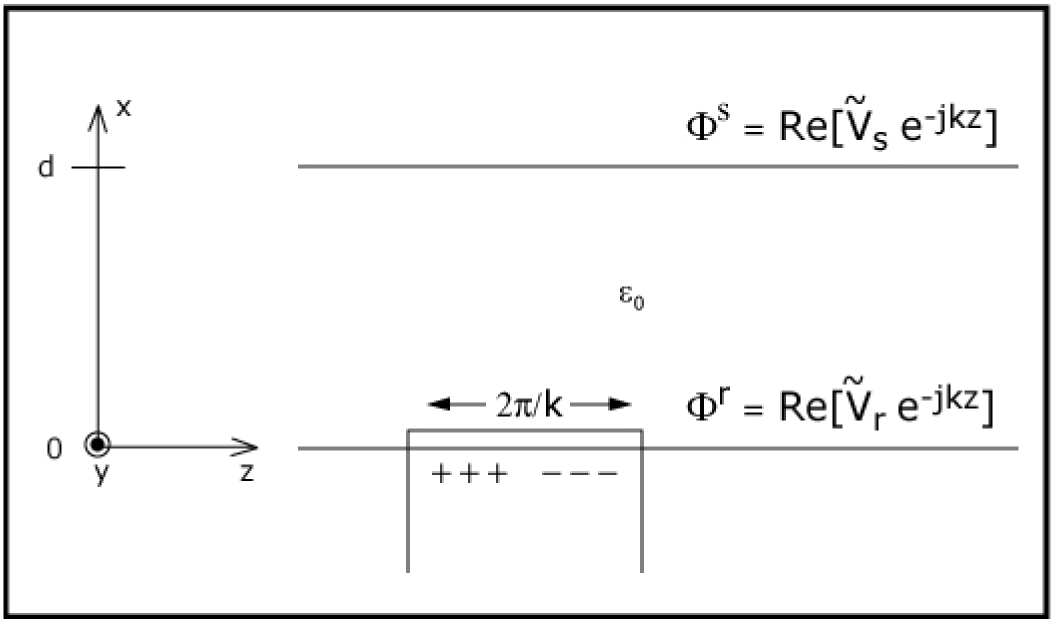
III. Electrostatic Machine



$$f_z = w\frac{2\pi}{k}\int_0^{2\pi/k} T_{zx}\big|_{x=0}\, dz = \frac{2\pi w}{k}\int_0^{2\pi/k} \varepsilon_0 E_z E_x\big|_{x=0} dz$$

$$\tilde{E}_z^r = jk\tilde{V}_r$$

$$f_z = \frac{1}{2}\frac{2\pi w}{k}\,\mathrm{Re}\left[\varepsilon_0 \tilde{E}_z^{r\,*}\tilde{E}_x^r\right]$$

$$= \frac{\pi w}{k}\,\mathrm{Re}\left[\varepsilon_0\left(-jk\tilde{V}_r^*\right)\tilde{E}_x^r\right]$$

$$\begin{bmatrix}\tilde{D}_x^s \\ \tilde{D}_x^r\end{bmatrix} = \varepsilon_0 k \begin{bmatrix}-\coth kd & \dfrac{1}{\sinh kd} \\ -\dfrac{1}{\sinh kd} & \coth kd\end{bmatrix}\begin{bmatrix}\tilde{V}_s \\ \tilde{V}_r\end{bmatrix}$$

$$\varepsilon_0 \tilde{E}_x^r = \varepsilon_0 k\left[\frac{-\tilde{V}_s}{\sinh kd} + \tilde{V}_r \coth kd\right]$$

$$\mathrm{Re}\left[-jk\varepsilon_0 \tilde{V}_r^* \tilde{E}_x^r\right] = \mathrm{Re}\left[-jk^2\varepsilon_0 \tilde{V}_r^*\left(\frac{-\tilde{V}_s}{\sinh kd} + \tilde{V}_r \coth kd\right)\right]$$

$$= \mathrm{Re}\left[+jk^2\varepsilon_0 \tilde{V}_s \tilde{V}_r^* \big/ \sinh kd\right]$$

$$f_z = \frac{\pi w}{k}\frac{k^2\varepsilon_0}{\sinh kd}\,\mathrm{Re}\left[j\tilde{V}_s \tilde{V}_r^*\right]$$

$$V_s = V_0^s \cos\left(\omega_s t - kz\right)$$
$$V_r = -V_0^r \cos\left(\omega_r t - k\left(z' - \delta\right)\right);\; z' = z - Ut$$
$$\tilde{V}^r = -V_0^r e^{j(\omega_r + kU)t} e^{jk\delta}$$
$$\tilde{V}^s = V_0^s e^{j\omega_s t}$$

$$\left\langle f_z \right\rangle = \frac{\pi w k \varepsilon_0}{\sinh kd} \mathrm{Re}\left[-j\, V_0^s\; V_0^r\; e^{-jk\delta}\; e^{j(\omega_s - \omega_r - kU)t}\right]$$

$$\omega_s = \omega_r + kU$$

$$\left\langle f_z \right\rangle = -\frac{\pi w k \varepsilon_0}{\sinh kd} V_0^s\; V_0^r \sin k\delta$$