1.010 Uncertainty in Engineering
Fall 2008

1.010 - Brief Notes # 5
Functions of Random Variables and Vectors

(a) Functions of One Random Variable

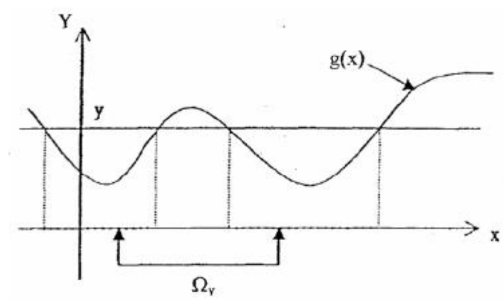
- **Problem**

  Given the CDF of the random variable $X, F_X(x)$, and a deterministic function $Y = g(x)$, find the (derived) distribution of the random variable Y.

- **General Solution**

  Let $\Omega_Y = \{x : g(x) \leq y\}$. Then:

  $F_Y(y) = P[Y \leq y] = P[x \in \Omega_Y] = \int_{\Omega_Y} f_X(x)dx$



- **Special Cases**

  - Linear Functions

    $Y = g(x) = a + bx$

    If $b > 0$:

$$X(y) = \frac{y-a}{b}; \quad \Omega_Y = \{x : a + bx \leq y\} = \left(-\infty, \frac{y-a}{b}\right]$$

$$F_Y(y) = P[x \in \Omega_Y] = F_X\left(\frac{y-a}{b}\right)$$

$$f_Y(y) = \frac{d}{dy}F_Y(y) = \frac{d}{dy}F_X\left(\frac{y-a}{b}\right) = \frac{1}{b}f_X\left(\frac{y-a}{b}\right)$$

<u>If $b < 0$:</u>

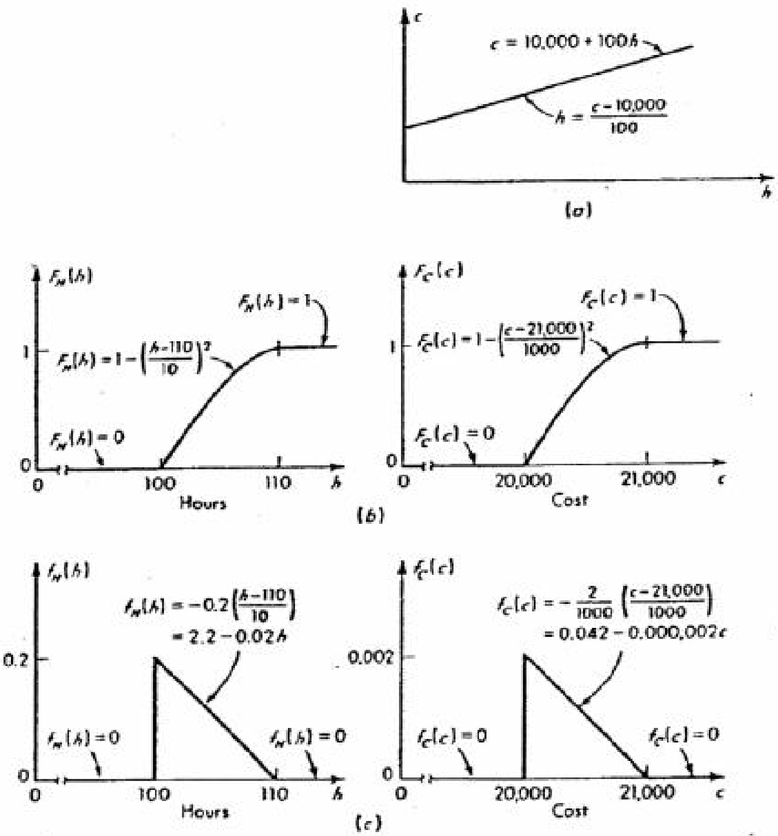
$$\Omega_Y = \left[\frac{y-a}{b}, \infty\right)$$

$$F_Y(y) = 1 - F_X\left(\frac{y-a}{b}\right)$$

$$f_Y(y) = -\frac{1}{b} f_X\left(\frac{y-a}{b}\right) = \frac{1}{|b|} f_X\left(\frac{y-a}{b}\right)$$

<u>For any $b \neq 0$:</u>

$$f_Y(y) = \frac{1}{|b|} f_X\left(\frac{y-a}{b}\right)$$



Example of linear transformation (b > 0): derived distributions, construction-cost illustration, $C = 10{,}000 + 100H$. (a) Functional relationship between cost and time; (b) cumulative distribution function of H, given, and C, derived; (c) probability density function of H, given, and C, derived.

- **General monotonic (one-to-one) functions**

  - *Monotonically increasing functions*

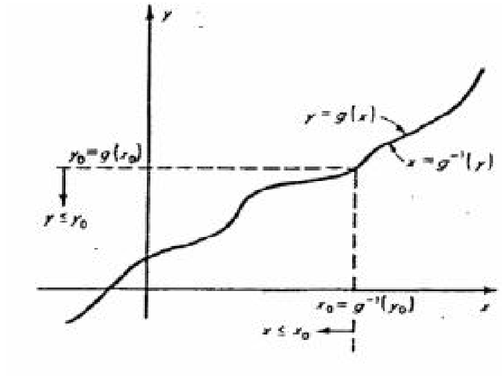
    $F_Y(y) = F_X[x(y)]$

    $$f_Y(y) = \frac{dF_Y(y)}{dy} = \frac{dx(y)}{dy} \cdot f_X[x(y)]$$

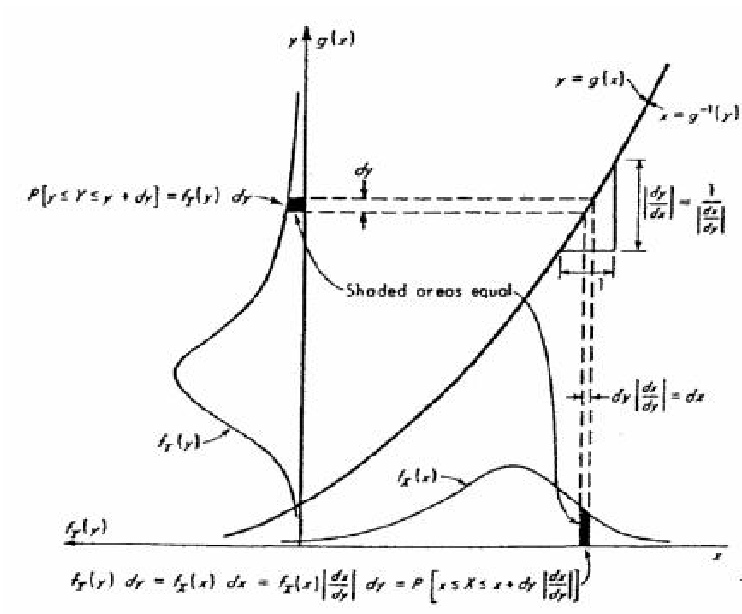
  - *Monotonically decreasing functions*

    $F_Y(y) = 1 - F_X[x(y)]$

    $$f_Y(y) = \left|\frac{dx(y)}{dy}\right| \cdot f_X[x(y)]$$



A monotonically increasing one-to-one function relating Y to X.



Graphical interpretation of $f_Y(y) = \frac{dx}{dy} f_X(x)$.

- **Examples of Monotonic Transformations**

Consider an exponential variable $X \sim EX(\lambda)$ with cumulative distribution function $F_X(x) = 1 - e^{-\lambda x},\ x \geq 0$.

Exponential, Power and Log Functions

- **Exponential Functions**

Suppose $Y = e^X, \Rightarrow x = ln(y), y \geq 0$. This is a monotonic increasing function, and $F_Y(y) = F_X(x(y)) = 1 - e^{-\lambda ln(y)} = 1 - y^{-\lambda}$. This distribution is known as the (strict) Pareto Distribution.

- **Power Functions**

Suppose $Y = X^{1/\alpha}, \alpha > 0, \Rightarrow x = ln(y), y \geq 0$. This is a monotonic increasing function, and $F_Y(y) = F_X(x(y)) = 1 - e^{-\lambda y^{\alpha}}$. This distribution is known as the Weibull (Extreme Type III) Distribution.

- **Log Functions**

Suppose $Y = -ln(X), \Rightarrow x = e^{-y}, -\infty \leq y \leq \infty$. This is a monotonic decreasing function, and $F_Y(y) = 1 - F_X(x(y)) = e^{-\lambda e^{-y}}$. This distribution is known as the Gumbell (Extreme Type I) Distribution.

(b) Functions of Two or More Random Variables

- **Problem**

Given the JCDF of the random vector $\begin{bmatrix} X \\ Y \end{bmatrix}$, $F_{X,Y}(x,y)$, and a deterministic function $Z = g(x,y)$, find the (derived) distribution of the random variable Z.

- **General Solution**

Let $\Omega_Z = \{x, y : g(x,y) \le z\}$. Then:

$F_Z(z) = P[Z \le z] = P[(x,y) \in \Omega_Z] = \iint_{\Omega_Z} f_{X,Y}(x,y)dxdy$

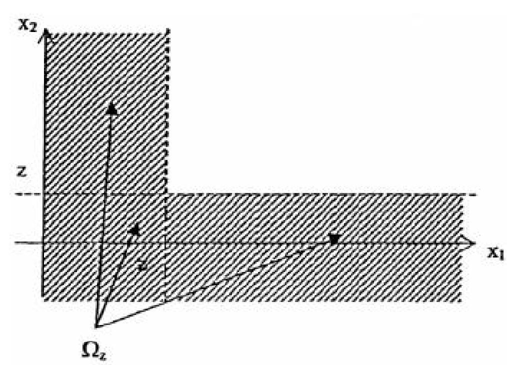
- **Special Cases**

  - Minimum/maximum functions

    i.e. $Z = Min[X_1, X_2, \ldots, X_n]$ (eg. minimum strength)
    or $Z = Max[X_1, X_2, \ldots, X_n]$ (eg. maximum load)

    - $Z = Min[X_1, X_2, \ldots, X_n]$. For $n = 2$,

$$F_Z(z) = P[Z \le z] = \iint_{\Omega_Z} f_{X_1,X_2}(x_1,x_2)dx_1dx_2, \quad \text{with } \Omega_Z \text{ shown in figure}$$
$$= 1 - \int_z^\infty dx_1 \int_z^\infty f_{X_1,X_2}(x_1,x_2)dx_2$$



If X_1 and X_2 are independent:

$\int_z^\infty dx_1 \int_z^\infty f_{X_1,X_2}(x_1,x_2)dx_2 = [1-F_{X_1}(z)][1-F_{X_2}(z)]$

Therefore,

$F_Z(z) = 1 - [1-F_{X_1}(z)][1-F_{X_2}(z)]$

For n iid variables:

$$\begin{aligned} F_Z(z) = P[Z \leq z] &= 1 - P[(X_1 > z) \cap \ldots \cap (X_n > z)] \\ &= 1 - [1 - F_X(z)]^n \end{aligned}$$

or, with $G_X(x) = 1 - F_X(x)$,

$$\begin{aligned} G_Z(z) &= P[Z > z] = [G_X(z)]^n \\ f_Z(z) &= \frac{d}{dz}F_Z(z) = -\frac{d}{dz}G_Z(z) = n[G_X(z)]^{n-1}f_X(z) \end{aligned}$$

- $Z = Max[X_1, X_2, \ldots, X_n]$

$$F_Z(z) = P\left[\bigcap_i (X_i \leq z)\right] = F_{\underline{X}}\begin{bmatrix} z \\ \vdots \\ z \end{bmatrix}$$

$$= \prod_i F_{X_i}(z) \quad \text{(if } X_i\text{'s are independent)}$$

$$= [F_X(z)]^n \quad \text{and} \quad f_Z(z) = n[F_x(z)]^{n-1}f_X(z) \quad \text{(if } X_i\text{'s are iid)}$$

- Linear transformations

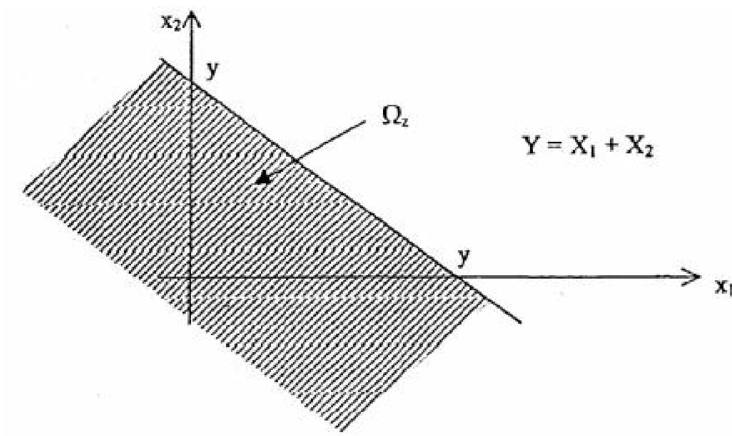
$Y = \sum_i a_i x_i$

- **Simplest case:** $Y = X_1 + X_2$

$$F_Y(y) = P[Y \leq y] = P[x_1 + x_2 \leq y] = \iint\limits_{x_1+x_2 \leq y} f_{X_1,X_2}(x_1,x_2)dx_1dx_2$$

$$= \int_{-\infty}^{\infty} dx_2 \int_{-\infty}^{y-x_2} f_{X_1,X_2}(x_1,x_2)dx_1$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X_1,X_2}(y-x_2,x_2)dx_2$$



If X_1 and X_2 are independent, then:

$f_Y(y) = \int_{-\infty}^{\infty} f_{X_1}(y-x_2)f_{X_2}(x_2)dx_2$ (convolution)

- **Example: Derivation of Gamma distribution**

Consider $Y = X_1 + X_2$, where X_1 and X_2 are iid exponential , with density:

$$f_{X_i} = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

Then,

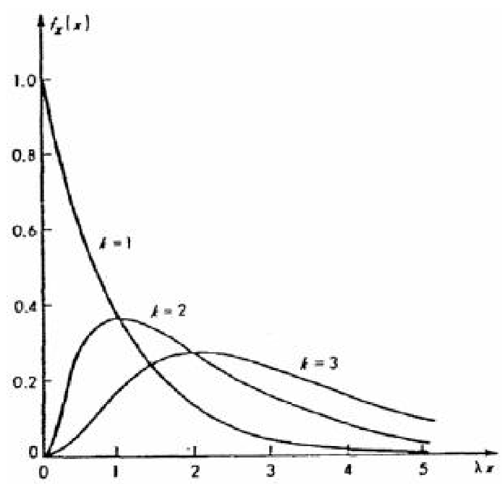
$$f_Y(y) = \int_0^\infty f_X(y - x_1) f_X(x_1) dx_1$$
$$= \lambda^2 y e^{-\lambda y} \quad \text{(Rayleigh or Gamma (2) distribution)}$$

In general, for any n, the probability density of $Y = X_1 + X_2 + \ldots + X_n$, where X_i are iid exponential, is:

$$f_Y(y) = \frac{\lambda(\lambda y)^{n-1} e^{-\lambda y}}{\Gamma(n)}, \quad y \geq 0, \text{ where } \Gamma(n) = (n-1)!$$

(Gamma(n) distribution)

Note: For n = 1, the Gamma distribution reduces to the exponential distribution.



Gamma distributions $G(k, \lambda)$.