1.225J (ESD 205) Transportation Flow Systems

Lecture 6

Introduction to Optimization

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Lecture 6 Outline

- Mathematical programs (MPs)
- Formulation of shortest path problems as MPs
- Formulation of U.O. traffic assignment as an MP
- Relationship between U.O. and S.O. traffic assignment
- Solving S.O. traffic assignment by hand
- Lecture summary

Optimization: Mathematical Programs

- General formulation (n variables, m constraints):

$$\min \; z(x_1, x_2, ..., x_n) \Rightarrow \text{Objective function}$$

$$\text{Subject to (s.t.): } \left.\begin{aligned} g_1(x_1, x_2, ..., x_n) &\geq b_1 \\ g_2(x_1, x_2, ..., x_n) &\geq b_2 \\ &\vdots \\ g_m(x_1, x_2, ..., x_n) &\geq b_m \end{aligned}\right\} \Rightarrow \text{Feasible set}$$

- $(x_1, x_2, ..., x_n)$: decision variables
- $g_j(x_1, x_2, ..., x_n) \geq b_j$: A constraint
- Notes:

$$\text{Max } f(x_1, x_2, ..., x_n) = Min \text{ z(x)} = -f(x_1, x_2, ..., x_n)$$

$$g(x_1, x_2, ..., x_n) \leq b \Leftrightarrow -g(x_1, x_2, ..., x_n) \geq -b$$

$$g(x_1, x_2, ..., x_n) = b \Leftrightarrow g(x_1, x_2, ..., x_n) \geq b \text{ and } -g(x_1, x_2, ..., x_n) \geq -b$$

Types of Mathematical Programs (MPs)

- **Linear programs (LPs)**: objective function is linear, and constraints are linear
- **Non-linear programs (NLPs)**: objective function is linear. (constraints are usually linear. Otherwise, there might be more than one optimal solution (finding such a solution can be a very time consuming task))

- If decision variables are further constrained to take integer values, a linear program is an **integer program**

- If decision variables are constrained to take 0/1 values: an integer program is an **0/1 integer program**

- If some, but not all, variables are constrained to take integer values: a linear program is called a **mixed integer program**

MPs Are Tools for Transportation Analysts

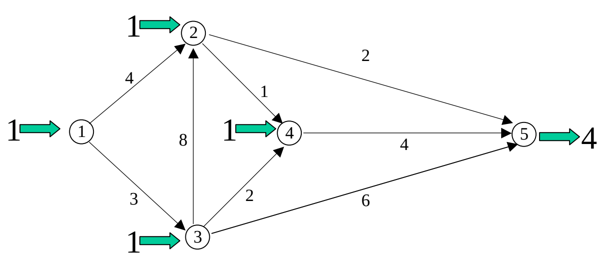
- Various models and quantitative analysis questions in transportation can be formulated as minimization (or maximization) problems:
  - Shortest path problems
  - Traffic assignment models in congested networks
  - Signal setting problems
  - Ramp-metering optimization
- Examples of questions related to modeling:
  - Formulate a model as a mathematical program (MP) (there might be more than one model for the same modeling question)
  - Study the properties of the model (i.e. does the model possess one or multiple solutions? Is it easy to find a solution?
  - Find a "solution" to the model (One may settle for an approximate reasonable solution, as it is not always possible (desirable) to find an optimal solution in a reasonable amount of time)

How to Solve Mathematical Programs?

- Graphically: This gives you a feel of what is happening
- By hand using a systematic analytical method
- Use your software such as Xpress-MP, GAMES, LINDO, CPLEX, Excel:
  - Software tools are computer implementations of systematic methods
- There are also specialized software for some transportation applications
  - Examples: TRANSCAD and EMME/2 for static traffic assignment (We have licenses of these software systems in the CTS Computing lab)

Shortest Path Problems As An LP: Example

- We want to find shortest paths from all nodes to Node 5
- Decision variables: $x_{ij} = \begin{cases} 1, & if\ arc(i,j)\ is\ used \\ 0, & otherwise \end{cases}$
- $c_{ij},\ (i,j) \in A$ is the cost of each arc



All-to-One Shortest Path Problem As An LP

- Formulation:

$$\begin{aligned} \min \quad & 4x_{12} + 3x_{13} + x_{24} + 2x_{25} + \\ & 8x_{32} + 2x_{34} + 6x_{35} + 4x_{45} \\ s.t. \quad & x_{12} + x_{13} = 1 \\ & x_{24} + x_{25} - x_{12} - x_{32} = 1 \\ & x_{32} + x_{34} + x_{35} - x_{13} = 1 \\ & x_{45} - x_{24} - x_{34} = 1 \\ & -x_{25} - x_{35} - x_{45} = -4 \\ & x_{ij} \geq 0, \quad (i,j) \in A \end{aligned}$$

- Note: constraints "the decision variables must be integers" should have been added. However, it is known theoretically that the above LP possesses an integer solution, and tools exist to it.

One-to-One Shortest Path Problem As An LP

- Formulation:

$$\begin{aligned}
\min \quad & 4x_{12} + 3x_{13} + x_{24} + 2x_{25} + \\
& 8x_{32} + 2x_{34} + 6x_{35} + 4x_{45} \\
s.t. \quad & x_{12} + x_{13} = 1 \\
& x_{24} + x_{25} - x_{12} - x_{32} = 0 \\
& x_{32} + x_{34} + x_{35} - x_{13} = 0 \\
& x_{45} - x_{24} - x_{34} = 0 \\
& -x_{25} - x_{35} - x_{45} = -1 \\
& x_{ij} \geq 0, \quad (i, j) \in A
\end{aligned}$$

- Note: we could have added the fact that the decision variables are either 0 or 1. However, it is known theoretically that the above LP possesses a 0-1 solution, and tools exist that provide such solution

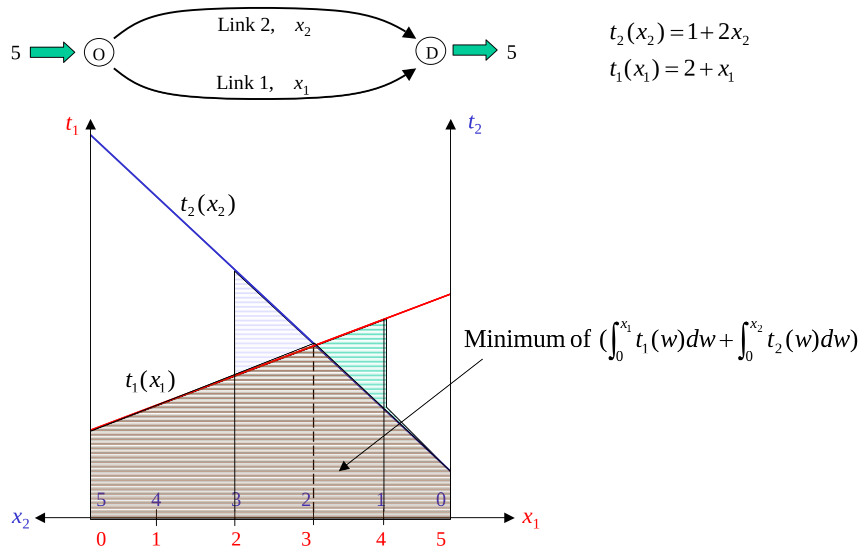
SO Static Traffic Assignment as an MP: Example

$$\begin{aligned}
\min \quad & x_1 t_1(x_1) + x_2 t_2(x_2) + x_3 t_3(x_3) & \\
s.t. \quad & f_1^{ac} = q_{ac} & \text{Demand} \\
& f_1^{bc} + f_2^{bc} = q_{bc} & \\
& f_1^{ac} \geq 0,\ f_1^{bc} \geq 0,\ f_2^{bc} \geq 0 & \text{Non-negativity} \\
& x_1 = f_1^{ac} + f_2^{bc} & \\
& x_2 = f_1^{bc} & \text{Definition of link flows} \\
& x_3 = f_2^{bc} &
\end{aligned}$$

- $\min\ x_1 t_1(x_1) + x_2 t_2(x_2) + x_3 t_3(x_3) = \min \int_0^{x_1} m_1(x_1) dx_1 + \int_0^{x_2} m_2(x_2) dx_2 + \int_0^{x_3} m_3(x_3) dx_3$

$$where\ \ m_1(x_1) = \frac{d(x_1 t_1(x_1))}{dx_1},\ \ m_2(x_2) = \frac{d(x_2 t_2(x_2))}{dx_2},\ \ m_3(x_3) = \frac{d(x_3 t_3(x_3))}{dx_3}$$

Objective Function For U.O. Traffic Assignment



UO & SO Traffic Assignment As MPs: R7 Example

- $\min \int_0^{x_1} t_1(x_1)dx_1 + \int_0^{x_2} t_2(x_2)dx_2 + \int_0^{x_3} t_3(x_3)dx_3$

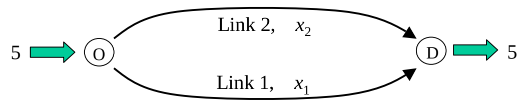
$$\text{s.t.} \quad f_1^{ac} = q_{ac}$$
$$f_1^{bc} + f_2^{bc} = q_{bc}$$
$$f_1^{ac} \geq 0,\ f_1^{bc} \geq 0,\ f_2^{bc} \geq 0$$
$$x_1 = f_1^{ac} + f_2^{bc}$$
$$x_2 = f_1^{bc}$$
$$x_3 = f_2^{bc}$$

- $\min \int_0^{x_1} m_1(x_1)dx_1 + \int_0^{x_2} m_2(x_2)dx_2 + \int_0^{x_3} m_3(x_3)dx_3$

$$\text{s.t.} \quad f_1^{ac} = q_{ac}$$
$$f_1^{bc} + f_2^{bc} = q_{bc}$$
$$f_1^{ac} \geq 0,\ f_1^{bc} \geq 0,\ f_2^{bc} \geq 0$$
$$x_1 = f_1^{ac} + f_2^{bc}$$
$$x_2 = f_1^{bc}$$
$$x_3 = f_2^{bc}$$

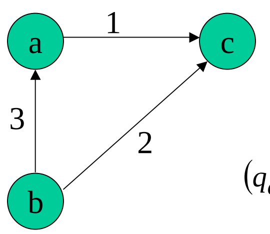
- *U.O.: All used paths have, between any O-D pair, equal and minimum travel time*
- *S.O.: All used paths have, between any O-D pair, equal and minimum marginal travel times*

Solving SO Static Traffic Assignment: Examples



$t_2(x_2) = 1 + 2x_2$

$t_1(x_1) = 2 + x_1$



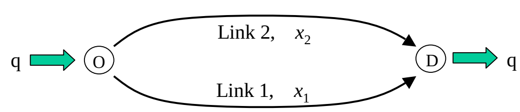
$t_1(x_1) = 10 + x_1$

$t_2(x_2) = 90 + x_2$

$t_3(x_3) = 0$

$(q_{ac}, q_{bc}) = (80,10)$

Solving SO Static Traffic Assignment: Examples



$t_2(x_2) = 1 + 2x_2$

$t_1(x_1) = 2 + x_1$

- UO solutions for three values of q:
  - q=1/8: x1=0, x2=q
  - q=1/2: x1=0, x2=q
  - q=5: x1=3, x2=2
- SO solutions for three values of q:
  - m1(x1)=2+2x1; m2(x2)=1+4x2
  - q=1/8: x1=0, x2=q
  - q=1/4: x1=0, x2=q
  - q=5: x1=(-1+4q)/6, x2=(1+2q)/6

SO Static Traffic Assignment: Example

$$\min \quad x_1 t_1(x_1) + x_2 t_2(x_2) + x_3 t_3(x_3)$$

$$s.t. \quad f_1^{ac} = q_{ac} \qquad \text{Demand}$$

$$f_1^{bc} + f_2^{bc} = q_{bc}$$

$$f_1^{ac} \geq 0,\ f_1^{bc} \geq 0,\ f_2^{bc} \geq 0 \qquad \text{Non-negativity}$$

$$x_1 = f_1^{ac} + f_2^{bc}$$
$$x_2 = f_1^{bc} \qquad \text{Definition of link flows}$$
$$x_3 = f_2^{bc}$$

- S.O. solution: $(x_1^*, x_2^*, x_3^*) = (80,10,0)$
- $\min\ x_1 t_1(x_1) + x_2 t_2(x_2) + x_3 t_3(x_3) = \min \int_0^{x_1} m_1(x_1)dx_1 + \int_0^{x_2} m_2(x_2)dx_2 + \int_0^{x_3} m_3(x_3)dx_3$

$$where\ \ m_1(x_1) = \frac{d(x_1 t_1(x_1))}{dx_1},\ \ m_2(x_2) = \frac{d(x_2 t_2(x_2))}{dx_2},\ \ m_3(x_3) = \frac{d(x_3 t_3(x_3))}{dx_3}$$

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