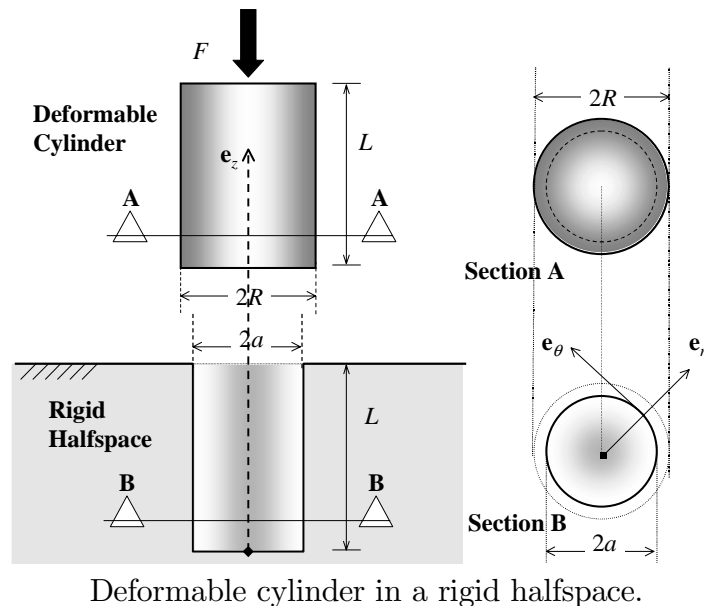


1.033/1.57: H#4: “Champagne Method” (3D-ElastoPlasticity)

Due: Part 1-3 (H#4): December 1, 2003
Part 4 (Q#3): December 5, 2003

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We consider a rigid infinite half-space with an empty hole of diameter $2a$ and length L (see figure below). In this hole we want to force a deformable cylinder sample of same length L but of a greater diameter $2R > 2a$ than the bore hole. The cylinder is initially stress free, and it is entirely forced into the hole. During this process, the cylinder preserves its length. The contact between the cylinder walls and the bore hole walls is frictionless. We want to study the stress fields and the required force F to maintain the cylinder in the bore hole. Throughout this exercise we assume isothermal and quasi-static evolutions, and body forces are disregarded.



1. **Deformation and Strain:** We consider a displacement field of the form:

$$\boldsymbol{\xi} = u(r)\mathbf{e}_r + \mathbf{D}$$

with $u(r)$ the displacement in the radial direction \mathbf{e}_r ; it is a pure radial displacement field. \mathbf{D} is a rigid body displacement field which ensures during the process that the axis of the cylinder sample coincides with the axis of the bore hole.

- (a) Specify the condition which the displacement field needs to satisfy to be kinematically admissible.
 - (b) Once the cylinder sample is in the hole, we want to deal with the problem within the hypothesis of small perturbations. For the problem in hand, specify the restriction on the considered displacement field.
 - (c) Determine the linearized strain tensor.
2. **Elasticity:** We want to determine the stress field in the sample at the end of the process, and the required force to enforce the sample in the hole. In a first approximation, we suppose a linear elastic isotropic behavior of the cylinder sample (Lamé constants λ, μ).
- (a) Determine the elastic stress field $\boldsymbol{\sigma}$ in the cylinder sample.
 - (b) Which equation needs to satisfy the radial displacement $u(r)$ for the elastic stress field $\boldsymbol{\sigma}$ to be statically admissible? By solving this equation, determine function $u(r)$ and the linear elastic stress solution.
 - (c) Determine the force intensity F that is required to enforce the sample in the bore hole. Determine the pressure p which the surrounding rigid medium exerts on the sample.
 - (d) Determine the elastic energy that is stored in the cylinder once the sample is in the bore hole.
3. **Elastic Strength Limit:** We want to determine the maximum difference in radius $R - a$ between the sample and the bore hole, when the stress reaches the elastic limit.
- (a) Plot the Mohr Circle of the elastic solution determined previously.
 - (b) The sample is assumed to be composed of a Tresca Material, characterized by a uniaxial compressive strength f'_c . Determine the admissible radius difference $R - a$. Where and on which surfaces occurs the maximum shear?
 - (c) What is the maximum admissible radius difference $R - a$ for a Mohr-Coulomb Material, defined by the cohesion C and the friction angle φ .
4. **Plasticity:** More realistically, we assume now that the cylinder is composed of an isotropic ideal elastoplastic Von-Mises material (shear strength k). We want to determine the required force F to enforce the sample in the hole for a given radius difference $R - a_1 > R - a_0$. $R - a_0$ corresponds to the limit radius difference when the sample starts yielding.

- (a) Determine the radius difference $R - a_0$ which corresponds to the elastic limit state of the Von-Mises material sample.
- (b) We want to determine the elastoplastic stress field σ prevailing in the cylinder once the cylinder is in the hole.
 - i. For the problem at hand, write the constitutive equations required to solve the problem.
 - ii. Show that the consistency condition $df = 0$ leads to:

$$dp - \frac{dF}{\pi R^2} = 0$$

where dp is the pressure increment applied from the surrounding medium on the sample; and dF the force applied on the cylinder head (see figure). From the previous result, determine the plastic multiplier.

- iii. Determine the stress tensor σ at the end of the elastoplastic loading process.
- (c) Illustrate the elastoplastic process in the Mohr-Stress-Plane. How is the stress vector on the critical shear plane affected by the elastoplastic process?
- (d) Determine the force intensity F during the elastoplastic process. Compare the result with the elastic case (i.e., Question 2c). Comment!
- (e) Plot the pressure p exerted by the surrounding rigid half-space versus the radius variation for this loading process.
- (f) Determine the amount of energy that is dissipated into heat during the process.
- (g) Last, we want to “uncork” the cylinder sample from the rigid half-space by means of a “corkscrew” which preserves the length of the cylinder sample during uncorking. Determine the required force F to do so.

1 Deformation and Strain

1.1 (a) K.A. Displacement Field

The displacement field must satisfy the displacement boundary condition on $\partial\Omega_{\xi^d}$. This is here the prescribed radial displacement:

$$r = R : u^d = a - R$$

This displacement field is such that any point at the cylinder wall, defined by position vector \mathbf{X} in the undeformed configuration, becomes after deformation $\mathbf{x} = \mathbf{X} + u^d \mathbf{e}_r + \mathbf{D}$ (*i.e.*, in the r -direction, $x_r = a$, $X_r = R$; therefore $u^d = x_r - X_r$).

1.2 (b) Small Displacement Hypothesis

$$|a - R| \ll R$$

The consequence of the small perturbation hypothesis is:

$$u(r = R) = u(r = a)$$

1.3 (c) Linearized Strain Tensor

Rigid body motion does not provoke strain; $\varepsilon_{zz} = 0$ since same length in bore hole as original length. Due to radial symmetry, the shear strains are zero. It follows:

$$\varepsilon_{rr} = \frac{\partial u(r)}{\partial r}; \quad \varepsilon_{\theta\theta} = \frac{u(r)}{r}; \quad \varepsilon_{zz} = \frac{\partial u_z}{\partial z} = 0; \quad \text{other } \varepsilon_{ij} = 0$$

2 Elasticity

2.1 (a) Elastic Stress Field in Cylinder

$$\boldsymbol{\sigma} = 2\mu\boldsymbol{\varepsilon} + \lambda(\text{tr}\boldsymbol{\varepsilon})\mathbf{1};$$

Thus,

$$\begin{aligned} \sigma_{rr} &= (2\mu + \lambda) \frac{\partial u(r)}{\partial r} + \lambda \frac{u(r)}{r} \\ \sigma_{\theta\theta} &= (2\mu + \lambda) \frac{u(r)}{r} + \lambda \frac{\partial u(r)}{\partial r} \\ \sigma_{zz} &= \lambda \left(\frac{\partial u(r)}{\partial r} + \frac{u(r)}{r} \right) \end{aligned}$$

2.2 (b) SA Stress Field

Equilibrium equation:

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r}(\sigma_{rr} - \sigma_{\theta\theta}) = 0$$

Thus, with the stress components previously defined:

$$u'' + u'/r - u/r^2 = 0 \rightarrow u(r) = Ar + \frac{B}{r}$$

Since $u(r=0) = 0 \rightarrow B = 0$. The displacement boundary condition at $r = R$ gives:

$$u(r=R) = a - R = AR \rightarrow A = \frac{a - R}{R}$$

Whence the elastic solution:

$$\begin{aligned} \sigma_{rr} = \sigma_{\theta\theta} &= 2(\mu + \lambda) \left[\frac{a - R}{R} \right] \\ \sigma_{zz} &= 2\lambda \left[\frac{a - R}{R} \right] = \nu(\sigma_{rr} + \sigma_{\theta\theta}) \end{aligned}$$

2.3 (c) Force Intensity

Application of Reduction Formula delivers:

$$-F\mathbf{e}_z = \int_{A=\pi a^2 \simeq \pi R^2} \boldsymbol{\sigma} \cdot \mathbf{e}_z da \simeq 2\lambda \left[\frac{a - R}{R} \right] \pi R^2 \mathbf{e}_z \rightarrow F = 2\pi\lambda [R - a] R$$

The pressure p exerted from the surrounding rigid medium is:

$$\text{on } r = R : \mathbf{T}(\mathbf{n} = \mathbf{e}_r) = \boldsymbol{\sigma} \cdot \mathbf{e}_r \equiv -p\mathbf{e}_r \rightarrow p = -\sigma_{rr} = 2(\mu + \lambda) \left[\frac{R - a}{R} \right]$$

or equivalently:

$$p = \frac{F}{2\nu\pi R^2}$$

2.4 (d) Stored Energy

Since $\boldsymbol{\xi}$ and $\boldsymbol{\sigma}$ are solution of the linear elastic problem, Clapeyron's formula applies:

$$W(\boldsymbol{\varepsilon}) = \int_{\Omega} \psi(\boldsymbol{\varepsilon}) d\Omega = \frac{1}{2} [\Phi(\boldsymbol{\xi}) + \Phi^*(\boldsymbol{\sigma})]$$

Noting

$$\Phi(\boldsymbol{\xi}) = \int_{\partial\Omega_{\mathbf{T}^d}} \mathbf{T}^d \cdot (\boldsymbol{\xi} - \mathbf{D}) da = 0$$

we obtain:

$$\begin{aligned} W(\boldsymbol{\varepsilon}) &= \frac{1}{2} \Phi^*(\boldsymbol{\sigma}) = \frac{1}{2} \int_{\partial\Omega_{\boldsymbol{\xi}^d}} \mathbf{T}(\mathbf{e}_r) \cdot (\boldsymbol{\xi}^d - \mathbf{D}^d) da = \frac{1}{2} \int_{\partial\Omega_{\boldsymbol{\xi}^d} = 2\pi RL} [-pu(r=R)] da \\ &= p(R - a)L\pi R = 2\pi(\mu + \lambda)(R - a)^2 L \end{aligned}$$

3 Elastic Strength Limit

3.1 (a) Mohr Circle

Here, $\sigma_I = \sigma_{zz} \geq \sigma_{II} = \sigma_{III} = \sigma_{rr} = \sigma_{\theta\theta}$. Radius and center of the Mohr Circle are:

$$|\tau| = \frac{1}{2}(\sigma_I - \sigma_{III}) = \mu \left[\frac{R - a}{R} \right]$$

$$\sigma_c = \frac{1}{2}(\sigma_I + \sigma_{III}) = -(2\lambda + \mu) \left[\frac{R - a}{R} \right]$$

The Mohr Circle is displayed in figure 1.

3.2 (b) Tresca Criterion

Note that $\sigma_0 = f'_c$. The Tresca criterion for the problem delivers:

$$f(\boldsymbol{\sigma}) = |\tau_{crit}| - f'_c/2 = 0 \rightarrow \max(R - a) = R \frac{f'_c}{2\mu}$$

Maximum shear occurs on a cone of which the surfaces are inclined with regard to \mathbf{e}_z -direction by (see figure 1):

$$\vartheta(\mathbf{u}_I = \mathbf{e}_z, \mathbf{n}) = -\pi/4$$

3.3 (c) Mohr-Coulomb Criterion

$$f(\boldsymbol{\sigma}) = 0 : |\tau_{crit}| + \sigma_{crit} \tan \varphi - c = 0$$

If we note $|\tau_{crit}| = |\tau| \cos \varphi$, $\sigma_{crit} = \sigma_c + |\tau| \sin \varphi$, we obtain for the problem at hand:

$$\max(R - a) = R \frac{c \cos \varphi}{\mu - \sin \varphi [\mu + 2\lambda]}$$

Maximum shear occurs on a cone inclined with regard to \mathbf{e}_z -direction by (see figure 1):

$$\vartheta(\mathbf{u}_I = \mathbf{e}_z, \mathbf{n}) = \varphi/2 - \pi/4$$

4 Plasticity

4.1 (a) Von-Mises Criterion

$$f(\boldsymbol{\sigma}) = \sqrt{J_2} - k \leq 0$$