2.003J/1.053J Dynamics and Control I, Spring 2007
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2/26/2007

Lecture 6

Collisions

Impulses

Large forces acting over a short period of time

Impulsive forces result in an instantaneous change of velocity (linear momentum)

Newton's 2nd Law Application:

$$\int_{t_1}^{t_2} \underline{f} dt = \underline{p}_2 - \underline{p}_1 = m\underline{v}_2 - m\underline{v}_1$$

For an impulsive force:

$$\underline{f} = \Delta \underline{p} \delta(t)$$

$$\int_{0^-}^{0^+} \delta(t) dt = 1$$

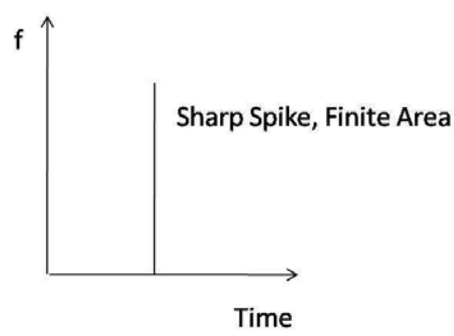

Figure 1: Impulsive force. In a time versus force graph, this is shown by a sharp spike with finite area. Figure by MIT OCW.

Impulse occurring between $t = 0^-$ and $t = 0^+$:

$$\int_{t=0^-}^{t=0^+} \Delta \underline{p} \delta(t) dt = \Delta \underline{p} = m[\underline{v}(0^+) - \underline{v}(0^-)] \Rightarrow \underline{v}(0^+) - \underline{v}(0^-) = \frac{\Delta \underline{p}}{m}$$

Therefore, impulse response is just the natural free response for the initial condition:

$$\underline{v}(0) = \frac{\Delta \underline{p}}{m}$$

Key Points

All other forces (e.g. gravity, dashpot damping) are considered negligible during impact.

As the time interval shrinks, the effect of these finite forces becomes negligible.

By direct analogy, for rotational systems one can have torques of an impulsive nature. The impulsive torque changes the angular momentum.

$$\underline{\tau} = \Delta \underline{H} \delta(t)$$

$\int_{t=0^-}^{t=0^+} \Delta \underline{H} \delta(t) dt = \Delta \underline{H}$ where $\Delta \underline{H}$ is the change in angular momentum of the system.

Collisions in a 1-D System

Before:

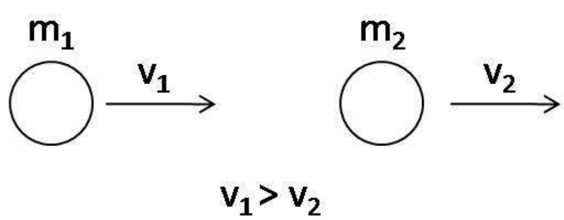

Figure 2: Two balls traveling with $v_1 > v_2$. Figure by MIT OCW.

After:

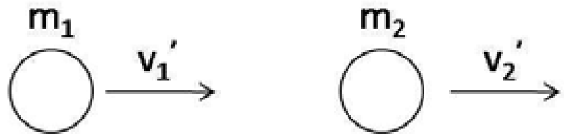

Figure 3: Two balls post-collision. The balls now travel with new velocities, $v_1^{'}$ and $v_2^{'}$. Figure by MIT OCW.

Forces between the colliding spheres are impulsive (all other forces make no contribution).

Impulse on m_1 (acts to the left): $m_1(v_1 - v_1^{'})$.
Impulse on m_2 (acts to the right): $m_2(v_2 - v_2^{'})$.

By Newton's Third Law (Impulse given to m_1 is equal and opposite to the impulse on m_2. Action and reaction are equal and opposite.)

$$m_1(v_1 - v_1^{'}) = m_2(v_2^{'} - v_2)$$

$$\boxed{m_1 v_1 + m_2 v_2 = m_2 v_1^{'} + m_2 v_2^{'}} \tag{1}$$

Linear momentum is conserved.

Note that this is not enough to solve for the result of a collision (Two unknowns $v_1^{'}$, $v_2^{'}$; but only one equation).

We need more information. Obtained from experiment.

Is energy conserved? $\rightarrow$ Yes, energy is conserved in universe!

Is mechanical energy conserved? $\rightarrow$ Not in general.

Typically $KE_{after} < KE_{before}$: Where does it go?
Large stresses and strains developed during impact (deformation) $\Rightarrow$ Energy lost to inelastic straining.
Sound waves

Measure energy loss by *coefficient of restitution*.

Coefficient of Restitution

$$e = \frac{\text{Relative Velocity After Collision}}{\text{Relative Velocity Before Collision}} = \frac{v_2^{'} - v_1^{'}}{v_1 - v_2}$$

This provides us with the second equation:

$$v_2^{'} - v_1^{'} = e(v_1 - v_2) \tag{2}$$

e must be measured for collisions betweeen any two materials:
- will be different based on the materials
- can be different based on how strong the collision is

Discussion

Combine (1) and (2)

$$v_1^{'} = \frac{m_1 - em_2}{m_1 + m_2} v_1 + \frac{(1+e)m_2}{m_1 + m_2} v_2$$

$$v_2^{'} = \frac{(1+e)m_1}{m_1 + m_2} v_1 + \frac{m_2 - em_1}{m_1 + m_2} v_2$$

If $e = 0$:

$$v_1^{'} = \frac{m_1 v_1}{m_1 + m_2} + \frac{m_2 v_2}{m_1 + m_2}$$

$$v_2^{'} = \frac{m_1 v_1}{m_1 + m_2} + \frac{m_2 v_2}{m_1 + m_2}$$

The two masses stick together → inelastic (calculate K.E., kinetic energy is lost)

If $e = 1$:

$$v_2^{'} - v_1^{'} = v_1 - v_2 \text{ Perfectly Elastic Case}$$

Can show that K.E. is conserved:

$$\frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 = \frac{1}{2} m_1 {v_1^{'}}^2 + \frac{1}{2} m_2 {v_2^{'}}^2$$

Collisions. See

Bedford, A. and Wallace L. Fowler.
Engineering Mechanics: Dynamics. 2nd Ed.
Menlo Park, Ca: Addison-Wesley Publishing, Inc, 1998. ISBN: 9780201180718.

Alternative Viewpoint of Coefficient of Restitution

We have defined e for a pair of colliding particles:
Before:

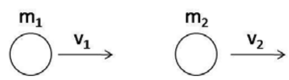

Figure 4: Two moving balls prior to collision. Figure by MIT OCW.

After:

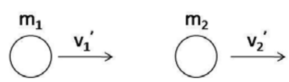

Figure 5: Two moving balls post-collision moving at new velocities. Figure by MIT OCW.

$$v_2^{'} - v_1^{'} = e(v_1 - v_2)$$

Alternatively:

Before:

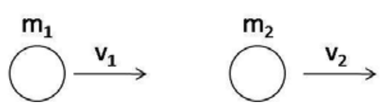

Figure 6: Two moving balls prior to collision. Figure by MIT OCW.

During:

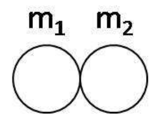

Figure 7: Two balls during collision. Figure by MIT OCW.

After:

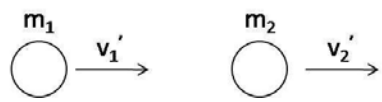

Figure 8: Two balls post-collision now moving at new velocities. Figure by MIT OCW.

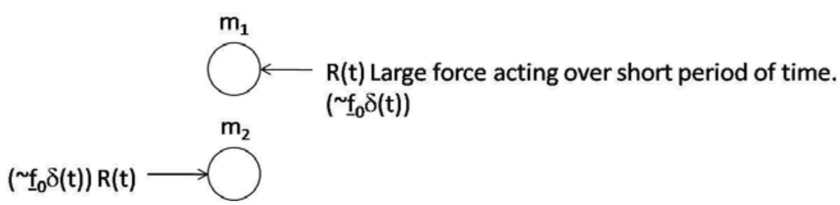

Figure 9: Diagram of large forces acting on balls during collision. Figure by MIT OCW.

Let t_A be the time that they first impact. Then they will deform slightly and at time t_B their centers of mass will be in closest proximity.

At time t_B the relative velocity of the centers of mass is zero; they both have velocity v_B.

The objects move apart at time t_C.

For particle 1:

$$\int_{t_A}^{t_B} -R dt = m_1 v_B - m_1 v_1 \text{ (a)}$$

$$\int_{t_B}^{t_C} -Rdt = m_1 v_1^{'} - m_1 v_B \text{ (b)}$$

For particle 2:

$$\int_{t_A}^{t_B} Rdt = m_2 v_B - m_2 v_2 \text{ (c)}$$

$$\int_{t_B}^{t_C} Rdt = m_2 v_2^{'} - m_2 v_B \text{ (d)}$$

The impuse imparted in the second half of the collision ($t_B \rightarrow t_C$) is less than in first half.

$\Rightarrow$ The ratio of these impulses is the coefficient of restitution.

$$e = \frac{\int_{t_B}^{t_C} Rdt}{\int_{t_A}^{t_B} Rdt}$$

$\int_{t_B}^{t_C} Rdt$: Impulse in second half of collision
$\int_{t_A}^{t_B} Rdt$: Impulse in first half of collision

Newton's Third Law has action and reaction, equal and opposite so therefore putting:

$$\frac{\text{(b)}}{\text{(a)}} = e = \frac{\text{(d)}}{\text{(c)}}$$

$$e = \frac{v_2^{'} - v_1^{'}}{v_1 - v_2}$$

Collision in the Reference Frame of Center of Mass

It is interesting to view the collision in the reference frame of center of mass.
1-D Discussion:

$$v_c = \frac{P}{M} = \frac{m_1 v_1 + m_2 v_2}{m_1 + m_2}$$

Velocity of center of mass is the same; does not change during collision.

In the center of mass frame $p = 0$.

What do we see in the center of mass frame?

Before:

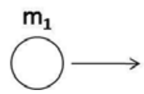

Figure 10: Ball m_1 moves to the right. Figure by MIT OCW.

$$\frac{m_2(v_2 - v_1)}{m_1 + m_2}$$

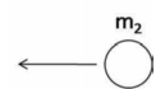

Figure 11: Ball m_2 moves to the left. Figure by MIT OCW.

$$\frac{m_1(v_2 - v_1)}{m_1 + m_2}$$

After:

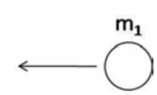

Figure 12: Ball m_1 after collision. Figure by MIT OCW.

$$\frac{-em_2(v_1 - v_2)}{m_1 + m_2}$$

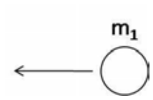

Figure 13: Ball m_2 after collision. Figure by MIT OCW.

$$\frac{-em_1(v_2 - v_1)}{m_1 + m_2}$$

As a result of the collision, masses change direction and velocities reduce by a factor of e. For $e = 0$, the masses stick together and remain stationary in center of mass frame.

$$v_{icm} = v_1 - \frac{m_1 v_1 + m_2 v_2}{m_1 + m_2} = \frac{m_1 v_1 + m_2 v_1 - m_1 v_1 - m_2 v_2}{m_1 + m_2} = \frac{m_2(v_1 - v_2)}{m_1 + m_2}$$