# UNIT-IV

# Fourier Series

Suppose that a given function $f(x)$ defined in $[-\pi,\pi]$ (or) $[0,2\pi]$ (or) in any other interval can be expressed as

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos nx+b_n\sin nx\right)$$

The above series is known as the Fourier series for $f(x)$ and the constants $a_0,a_n,b_n\left(n=1,2,3------\right)$ are called Fourier coefficients of $f(x)$

**Periodic Functions:-**

A function $f(x)$ is said to be periodic with period $T>0$ if for all $x$ $f(x+T)=f(x)$ and T is the least of such values

Example:- $\sin x=\sin(x+2\pi)=\sin(x+4\pi)=----$ the function $\sin x$ is periodic with period $2\pi$ there is no positive value T, $0<T<2\pi$ such that $\sin(x+T)=\sin x\,\forall\, x$

**Euler's Formula:-**

The Fourier series for the function $f(x)$ in the interval $c\le x\le c+2\pi$ is given by

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos nx+b_n\sin nx\right)$$

Where $a_0=\frac{1}{\pi}\int_c^{c+2\pi}f(x)\,dx$

$$a_n=\frac{1}{\pi}\int_c^{c+2\pi}f(x)\cos nx.dx$$ and

$$b_n=\frac{1}{\pi}\int_c^{c+2\pi}f(x)\sin nx.dx$$

These values of $a_0,a_n,b_n$ are known as Euler's formula

**Corollary:-** if $f(x)$ is to be expanded as a Fourier series in the interval $0\le x\le 2\pi$, put $c=0$ then the formulae (1) reduces to

$$a_0=\frac{1}{\pi}\int_0^{2\pi}f(x)\,dx$$

$$a_n=\frac{1}{\pi}\int_0^{2\pi}f(x)\cos nx.dx$$

$$b_n=\frac{1}{\pi}\int_0^{2\pi}f(x)\sin nx.dx$$

**Corollay 2:-** if $f(x)$ is to expanded as a fourier series in $[-\pi,\pi]$ put $c=-\pi$, the interval becomes $-\pi \le x \le \pi$ and the formula (1) reduces to

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx.dx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx.dx$$

**Conditions For Fourier Expansion:-**

Dirichlet has formulated certain conditions known as Dirichlet conditions under which certain functions posses valid Fourier Expansions.

A given function $f(x)$ has a valid Fourier series expansion of the form $\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$

Where $a_0, a_n, b_n$ are constants, provided

(i) $f(x)$ is well defined and single – valued except possibly at a finite number of points in the interval of definition

(ii) $f(x)$ has a finite number of discontinuities in the interval of definition

(iii) $f(x)$ has al most a finite number of maxima and minima in the interval of definition

**Note:-** The above conditions are sufficient but not necessary

**Functions Having Points of Discontinuity :-**

In Euler's formulae for $a_0, a_n, b_n$ it was assumed that $f(x)$ is continuous. Instead a function may have a finite number of discontinuities. Even then such a function is expressible as a Fouries series

Let $f(x)$ be defined by

$$f(x) = \phi(x) \quad c < x < x_0$$
$$= \phi(x) \quad x_0 < x < c+2\pi$$

Where $x_0$ is the point of discontinuity in $(c, c+2\pi)$ in such cases also we obtain the Fourier series for $f(x)$ in the usual way. The values of $a_0, a_n, b_n$ are given by

$$a_0 = \frac{1}{\pi}\left[\int_{c}^{x_0} \phi(x)\,dx + \int_{x_0}^{c+2\pi} \phi(x)\,dx\right]$$

$$a_n = \frac{1}{\pi}\left[\int_{c}^{x_0} \phi(x)\cos nx.dx + \int_{x_0}^{c+2\pi} \phi(x)\cos nx.dx\right]$$

$$b_n = \frac{1}{\pi}\left[\int_{c}^{x_0} \phi(x)\sin nx.dx + \int_{x_0}^{c+2\pi} \phi(x)\sin nx.dx\right]$$

**Note :-**

$$(i)\int_{-\pi}^{\pi}\cos mx\,\cos nx\,dx=\begin{cases}0 \;\; for\;\; m\neq n\\ \pi,\;\; for\;\; m=n>0\\ 2\pi,\;\; for\;\; m=n=0\end{cases}$$

$$(ii)(i)\int_{-\pi}^{\pi}\sin mx\,\sin nx\,dx=\begin{cases}0 \;\; for\;\; m\neq n \;\; and \;\; m=n=0\\ \pi,\;\; for\;\; m=n>0\end{cases}$$

**Examples:-**

**1. Express $f(x)=x-\pi$ as Fourier series in the interval $-\pi<x<\pi$**

Sol Let the function $x-\pi$ be represented by the Fourier series

$$x-\pi=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx+\sum_{n=1}^{\infty}b_n\sin nx\rightarrow(1)$$

Then

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)dx=\frac{1}{\pi}\int_{-\pi}^{\pi}(x-\pi)dx$$

$$=\frac{1}{\pi}\left[\int_{-\pi}^{\pi}xdx-\pi\int_{-\pi}^{\pi}dx\right]$$

$$=\frac{1}{\pi}\left[0-\pi.2\int_{0}^{\pi}dx\right]\quad(\because x\ is\ odd\ \ function)$$

$$=\frac{1}{\pi}\left[-2\pi(x)_0^{\pi}\right]$$

$$=-2(\pi-0)=-2\pi\ and$$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nx.dx$$

$$=\frac{1}{\pi}\int_{-\pi}^{\pi}(x-\pi)\cos nx.dx$$

$$=\frac{1}{x}\left[\int_{-\pi}^{\pi}x\cos nx.dx-\pi\int_{-\pi}^{\pi}\cos nx.dx\right]$$

$$=\frac{1}{\pi}\left[0-2\pi\int_{0}^{\pi}\cos nx.dx\right]$$

$$\therefore a_n = -2\int_0^{\pi} \cos nx.dx$$
$$= -2\left(\frac{\sin nx}{n}\right)_0^{\pi}$$
$$= \frac{-2}{n}(\sin n\pi - \sin 0)$$
$$= \frac{-2}{n}(0-0) = 0 \quad for \ n = 1, 2, 3..........$$
$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx.dx$$
$$= \frac{1}{\pi}\int_{-\pi}^{\pi}(x-\pi)\sin nx.dx$$
$$= \frac{1}{\pi}\left[\int_{-\pi}^{\pi} x\sin nx - \pi\int_{-\pi}^{\pi}\sin nx.dx\right]$$
$$= \frac{1}{\pi}\left[2\int_0^{\pi} x\sin nx.dx - \pi(0)\right]$$

($\because x\cos nx$ is odd function and $\cos nx$ is even function)
$$= \frac{2}{\pi}\left[x\left(\frac{-\cos nx}{n}\right) - 1\left(\frac{-\sin nx}{n^2}\right)\right]_0^{\pi}$$
$$= \frac{2}{\pi}\left[\left(\frac{-\pi\cos n\pi}{n} + 0\right) - (0+0)\right]$$

$$= \frac{-2}{\pi}\cos n\pi = \frac{-2}{n}(-1)^n$$
$$= \frac{2}{n}(-1)^{n+1} \ \forall n = 1, 2, 3.....$$

Substituting the values of $a_0, a_n, b_n$ in (1),

We get
$$x - \pi = -\pi + \sum_{n=1}^{\infty}(-1)^{n+1}\frac{2}{\pi}\sin nx$$
$$= -\pi + 2\left[\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \frac{1}{4}\sin 4x + ......\right]$$

**2. Find the Fourier series to represent the function $e^{-ax}$ $from \ x = -\pi \ to \ \pi$. Deduce from this that**
$$\frac{\pi}{\sinh \pi} = 2\left[\frac{1}{2^2+1} - \frac{1}{3^2+1} + \frac{1}{4^2+1} - - - - -\right]$$

Sol. Let the function $e^{-ax}$ be represented by the Fourier series
$$e^{-ax} = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) \rightarrow (1)$$

Then

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} e^{-ax}dx = \frac{1}{\pi}\left(\frac{e^{-ax}}{-a}\right)_{-\pi}^{\pi}$$

$$= \frac{-1}{a\pi}\left(e^{-a\pi} - e^{a\pi}\right) = \frac{e^{a\pi} - e^{-a\pi}}{a\pi}$$

$$\therefore \frac{a_0}{2} = \left[\frac{e^{a\pi} - e^{-a\pi}}{2}\right]\frac{1}{a\pi} = \frac{\sinh a\pi}{a\pi}$$

And

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} e^{-ax}\cos nx.dx$$

$$= \frac{1}{\pi}\left[\frac{e^{-ax}}{a^2+n^2}\left(-a\cos nx + n\sin nx\right)\right]_{-\pi}^{\pi}$$

$$\int e^{ax}\cos bx.dx = \frac{e^{ax}}{a^2+b^2}\left(a\cos bx + b\sin bx\right)$$

$$\therefore a_n = \frac{1}{\pi}\left\{\frac{e^{-ax}}{a^2+n^2}\left(-a\cos n\pi + 0\right) - \frac{e^{a\pi}}{a^2+n^2}\left(-a\cos n\pi + 0\right)\right\}$$

$$= \frac{a}{\pi\left(a^2+n^2\right)}\left(e^{a\pi} - e^{-a\pi}\right)\cos n\pi$$

$$= \frac{2a\cos n\pi \sinh a\pi}{\pi\left(a^2+n^2\right)}$$

$$= \frac{(-1)^n 2a\sinh a\pi}{\pi\left(a^2+n^2\right)}\left(\because \cos n\pi = (-1)^n\right)$$

Finally $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} e^{-ax}\sin nx.dx$

$$= \frac{1}{\pi}\left[\frac{e^{-ax}}{a^2+n^2}\left(-a\sin nx - n\cos nx\right)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{e^{-a\pi}}{a^2+n^2}\left(0 - n\cos n\pi\right) - \frac{e^{a\pi}}{a^2+n^2}\left(0 - n\cos n\pi\right)\right]$$

$$= \frac{n\cos n\pi e^{a\pi} - e^{-a\pi}}{\pi\left(a^2+n^2\right)} = \frac{(-1)^n 2n\sinh a\pi}{\pi\left(a^2+n^2\right)}$$

Substituting the values of $\frac{a_0}{2}, a_n$ $and$ $b_n$ in (1) we get

$$e^{-ax} = \frac{\sinh a\pi}{a\pi} + \sum_{n=1}^{\infty}\left[\frac{(-1)^n 2a\sinh a\pi}{\pi\left(a^2+n^2\right)}\cos nx + (-1)^n 2n\frac{\sinh a\pi}{\pi\left(a^2+n^2\right)}\sin nx\right]$$

$$= \frac{2\sinh a\pi}{\pi}\left\{\left(\frac{1}{2a} - \frac{a\cos x}{1^2+a^2} + \frac{a\cos 2x}{2^2+a^2} - \frac{a\cos 3x}{3^2+a^2} + -----\right) - \left(\frac{\sin x}{1^2+a^2} - \frac{2\sin 2x}{2^2+a^2} + \frac{3\sin 3x}{3^2+a^2} ---\right)\right\}$$

**Deduction:-**

Putting $x=0$ and $a=1$ in (2), we get

$$1=\frac{2\sinh\pi}{\pi}\left[\frac{1}{2}-\frac{1}{2}+\frac{1}{2^2+1}-\frac{1}{3^2+1}+\frac{1}{4^2+1}---\right]$$

$$\frac{\pi}{\sinh\pi}=2\left(\frac{1}{2^2+1}-\frac{1}{3^2+1}+\frac{1}{4^2+1}----\right)$$

**3. Find the Fourier series of the periodic function defined as** $f\left(x\right)=\begin{bmatrix}-\pi & \pi<x<0\\ x & 0<x<\pi\end{bmatrix}$

Hence deduce that $\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+------=\frac{\pi^2}{8}$

Sol. Let $f\left(x\right)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx+\sum_{n=1}^{\infty}b_n\sin nx\rightarrow(1)$

Then

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f\left(x\right)dx$$

$$=\frac{1}{\pi}\left[\int_{-\pi}^{o}\left(-\pi\right)dx+\int_0^{\pi}xdx\right]$$

$$=\frac{1}{\pi}\left[-\pi\left(x\right)_{-\pi}^{0}+\left(\frac{x^2}{2}\right)_0^{\pi}\right]$$

$$=\frac{1}{\pi}\left[-\pi^2+\frac{\pi^2}{2}\right]=\frac{1}{\pi}\left[\frac{-\pi^2}{2}\right]=\frac{-\pi}{2}$$

$$=\frac{1}{\pi n^2}\left(\cos n\pi-1\right)=\frac{1}{\pi n^2}\left[\left(-1\right)^{n-1}\right]$$

$$a_1=\frac{-2}{1^2.\pi},a_2=0,a_3=\frac{-2}{3^2.\pi},a_4=0,a_5=\frac{-2}{5^2.\pi},---$$

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f\left(x\right)\sin nx.dx$$

$$=\frac{1}{\pi}\left[\int_{\pi}^{0}\left(-\pi\right)\sin nx.dx+\int_0^{\pi}x\sin nx.dx\right]$$

$$=\frac{1}{\pi}\left[\pi\left(\frac{\cos nx}{n}\right)_{-\pi}^{0}+\left(-x\frac{\cos nx}{n}+\frac{\sin nx}{n^2}\right)_0^{\pi}\right]$$

$$=\frac{1}{\pi}\left[\frac{\pi}{n}\left(1-\cos n\pi\right)-\frac{\pi}{n}\cos n\pi\right]$$

$$=\frac{1}{n}\left(1-2\cos n\pi\right)$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx.dx$$
$$= \frac{1}{\pi}\left[\int_{-\pi}^{o}(-\pi)\cos nx.dx + \int_{0}^{\pi} x\cos nx.dx\right]$$
$$= \frac{1}{\pi}\left[-\pi\left(\frac{\sin nx}{n}\right)_{-\pi}^{0} + \left(x\frac{\sin nx}{n} + \frac{\cos nx}{n^2}\right)_{0}^{\pi}\right]$$
$$= \frac{1}{\pi}\left[0 + \frac{1}{n^2}\cos n\pi - \frac{1}{\pi n^2}\right]$$

$b_1 = 3, b_2 = \frac{-1}{2}, b_3 = 1, b_4 = \frac{-1}{4}$ and so --- on substituting the values of $a_0, a_n$ *and* $b_n$ in (1), we get

$$f(x) = \frac{-\pi}{4} - \frac{2}{\pi}\left(\cos x + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + --- \right) + \left(3\sin x - \frac{\sin 2x}{2} + \frac{3\sin 3x}{3} - \frac{\sin 4x}{4} + --\right)$$

**Deduction:-**

Putting $x = 0$ in (2), we obtain

$$f(0) = \frac{-\pi}{4} - \frac{2}{4}\left(1 + \frac{1}{3^2} + \frac{1}{5^2} + ---\right)$$

Now $f(x)$ is discontinuous at $x = 0$

$$f(0-0) = -\pi \text{ and } f(0+0) = 0$$

$$f(0) = \frac{1}{2}\left[f(0-0) + f(0+0)\right] = \frac{-\pi}{2}$$

Now (3) becomes

$$\frac{-\pi}{2} = \frac{-\pi}{4} - \frac{2}{\pi}\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + ----\right)$$
$$= \frac{\pi^2}{8}$$

**Even and Odd Functions:-**

A function $f(x)$ is said to be even *if* $f(-x) = f(x)$ and odd if $f(-x) = -f(x)$

**Example:-** $x^2, x^4 + x^2 + 1, e^x + e^{-x}$ are even functions

$x^3, x, \sin x, \cos ecx$ are odd functions

**Note:-**

1. Product of two even (or) two odd functions will be an even function
2. Product of an even function and an odd function will be an odd function

**Note 2:-** $\int_{-a}^{a} f(x)dx = 0$ when $f(x)$ is an odd function

$= 2\int_{0}^{a} f(x)dx$ when $f(x)$ is even function

Fourier series for even and odd functions

We know that a function $f(x)$ defined in $(-\pi,\pi)$ can be represented by the Fourier series

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx+\sum_{n=1}^{\infty}b_n\sin nx$$

Where $a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)dx$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nx.dx$$

And $b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin nx.dx$

**Case (i):-** when $f(x)$ is even function

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)dx=\frac{2}{\pi}\int_{0}^{\pi}f(x)dx$$

Since $\cos nx$ is an even function, $f(x)\cos nx$ is also an even function

Hence

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nx.dx$$

$$=\frac{2}{\pi}\int_{0}^{\pi}f(x)\cos nx.dx$$

Since $\sin nx$ is an odd function, $f(x)\sin nx$ is an odd function

$$\therefore b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin nx.dx=0$$

$\therefore$ If a function $f(x)$ is even in $(-\pi,\pi)$, its Fourier series expansion contains only cosine terms

$$\therefore f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx$$

Where $a_n=\frac{2}{\pi}\int_{0}^{\pi}f(x)\cos nx.dx, n=0,1,2,-----$

**Case 2:-** when $f(x)$ is an odd function in $(-\pi,\pi)$

$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)dx=0$ since $f(x)$ is odd

Since $\cos nx$ is an even function, $f(x)\cos nx$ is an odd function and hence

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nx.dx=0$$

Since $\sin nx$ is odd function ; $f(x)\sin nx$ is an even function

$$\therefore b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin nx.dx$$

$$=\frac{2}{\pi}\int_{0}^{\pi}f(x)\sin nx.dx$$

$$\therefore f(x)=\sum_{n=1}^{\infty}b_n\sin nx$$

Where $b_n=\frac{2}{\pi}\int_{0}^{\pi}f(x)\sin nx.dx$

Thus, if a function $f(x)$ defined in $(-\pi,\pi)$ is odd, its Fourier expansion contains only sine terms

**Examples:-**

**1. Expand the function $f(x)=x^2$ as a Fourier series in $(-\pi,\pi)$, hence deduce that**

$$\textbf{(i)}\ \frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+----=\frac{\pi^2}{12}$$

Sol. Since $f(-x)=(-x)^2=x^2=f(x)$

Hence in its Fourier series expansion, the sine terms are absent

$$\therefore x^2=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx$$

Where

$$a_0=\frac{2}{\pi}\int_0^{\pi}x^2dx$$

$$=\frac{2}{\pi}\left(\frac{x^3}{3}\right)_0^{\pi}=\frac{2\pi^2}{3}$$

$$a_n=\frac{2}{\pi}\int_0^{\pi}f(x)\cos nx.dx$$

$$=\frac{2}{\pi}\int_0^{\pi}x^2\cos nx.dx$$

$$=\frac{2}{\pi}\left[x^2\left(\frac{\sin nx}{n}\right)-2x\left(\frac{-\cos nx}{n^2}\right)+2\left(\frac{-\sin nx}{n^3}\right)\right]_0^{\pi}$$

$$=\frac{2}{\pi}\left[0+2\pi\frac{\cos nx}{n^2}+2.0\right]$$

$$=\frac{4\cos n\pi}{n^2}=\frac{4}{n^2}(-1)^n$$

Substituting the values of $a_0$ *and* $a_n$ from (2) and (3) in (1) we get

$$x^2=\frac{\pi^2}{3}+\sum_{n=1}^{\infty}\frac{4}{n^2}(-1)^n\cos nx$$

$$=\frac{\pi^2}{3}-4\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}\cos nx$$

$$=\frac{\pi^2}{3}-4\left(\cos x-\frac{\cos 2x}{2^2}+\frac{\cos 3x}{3^2}-\frac{\cos 4x}{4^2}+---\right)\to(4)$$

**Deductions:-**

Putting $x=0$ in (4), we get

$$0=\frac{\pi^2}{3}-4\left(1-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+---\right)$$

$$\Rightarrow 1-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+---=\frac{\pi^2}{12}$$

**2. Find the Fourier series to represent the function** $f\left(x\right)=\left|\sin x\right|, -\pi < x < \pi$

**Sol Since** $\left|\sin x\right|$ **is an even function,**

$b_n = 0 \quad for\ all\ n$

Let $f\left(x\right)=\left|\sin x\right|=\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n \cos nx \rightarrow (1)$

Where

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi}\left|\sin x\right| dx$$

$$=\frac{2}{\pi}\int_{0}^{\pi}\sin x\, dx$$

$$=\frac{2}{\pi}\left(-\cos x\right)_0^{\pi}$$

$$=\frac{-2}{\pi}\left(-1-1\right)=\frac{4}{\pi} \quad and$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f\left(x\right)\cos nx.dx = \frac{2}{\pi}\int_{0}^{\pi}\sin x.\cos nx\, dx$$

$$=\frac{1}{\pi}\int_{0}^{\pi}\left[\sin\left(1+n\right)x+\sin\left(1-n\right)x\right]dx$$

$$=\frac{1}{\pi}\left[-\frac{\cos\left(1+n\right)x}{1+n}-\frac{\cos\left(1-n\right)x}{1-n}\right]_0^{\pi}, n \neq 1$$

$$=-\frac{1}{\pi}\left[\frac{\cos\left(1+n\right)\pi}{1+n}+\frac{\cos\left(1-n\right)\pi}{1-n}-\frac{1}{1+n}-\frac{1}{1-n}\right]_0^{\pi} \quad n+1$$

$$=\frac{-1}{\pi}\left[\frac{\left(-1\right)^{n+1}-1}{1+n}+\frac{\left(-1\right)^{n+1}-1}{1-n}\right]$$

$$=\frac{-1}{\pi}\left[\left(-1\right)^{n+1}\left\{\frac{1}{1+n}+\frac{1}{1-n}\right\}-\left\{\frac{1}{1+n}+\frac{1}{1-n}\right\}\right]$$

$$= \frac{-1}{\pi}\left[(-1)^{n+1}\frac{2}{1-n^2} - \frac{2}{1-n^2}\right]$$

$$= \frac{2}{\pi(n^2-1)}\left[(-1)^{n+1}-1\right]$$

$$= \frac{-2}{\pi(n^2-1)}\left[1+(-1)^n\right]$$

$$\therefore a_n = \begin{cases} 0 \quad if \;\; n \; is \; odd \\ \dfrac{-4}{\pi(n^2-1)} \;\; if \;\; n \;\; is \;\; even \end{cases}$$

$$for \;\; n=1, a_1 = \frac{2}{\pi}\int_0^\pi \sin x.\cos x \; dx$$

$$= \frac{1}{\pi}\int_0^\pi \sin 2x \; dx$$

$$= \frac{1}{\pi}\left(\frac{-\cos 2x}{2}\right)_0^\pi$$

$$= \frac{-1}{\pi}(\cos 2\pi - 1) = 0$$

Substituting the values of $a_0, a_1 \; and \; a_n$ in (1)

We get $|\sin x| = \dfrac{2}{\pi} + \displaystyle\sum_{n=2,4,--}^{\infty} \frac{-4}{\pi(n^2-1)}\cos nx$

$$= \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=2,4---}^{\infty}\frac{\cos nx}{n^2-1}$$

$$= \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\cos 2nx}{4n^2-1}$$

( replace n by 2n)

Hence $|\sin x| = \dfrac{2}{\pi} - \dfrac{4}{\pi}\left(\dfrac{\cos 2x}{3} + \dfrac{\cos 4x}{15} + -----\right)$

**Half –Range Fourier Series:-**

**1) The sine series:-**

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$where \; b_n = \frac{2}{\pi}\int_0^\pi f(x)\sin nx.dx$$

**2) The cosine series:-**

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

$$where \;\; a_0 = \frac{2}{\pi}\int_0^\pi f(x)dx \; and$$

$$a_n = \frac{2}{\pi}\int_0^\pi f(x)\cos nx.dx$$

**Note:-**

1) Suppose $f(x) = x \ in \ [0, \pi]$, it can have Fourier cosine series expansion as well as Fourier sine series expansion in $[0, \pi]$
2) If $f(x) = x^2 \ in \ [0, \pi]$, can have Fourier cosine series as well as sine series

**Examples:-**

1. Find the half range sine series for $f(x) = x(\pi - x) \ in \ 0 < x < \pi$. Deduce that $\frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + ---- = \frac{\pi^3}{32}$

Ans. The Fourier sine series expansion of $f(x)$ in $(0, \pi)$ is

$$f(x) = x(\pi - x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$where \ b_n = \frac{2}{\pi}\int_0^{\pi} f(x) \sin nx.dx$$

$$hence \ b_n = \frac{2}{\pi}\int_0^{\pi} x(\pi - x) \sin nx.dx$$

$$= \frac{2}{\pi}\int_0^{\pi} (\pi x - x^2) \sin nx.dx$$

$$= \frac{2}{\pi}\left[(\pi x - x^2)\left(\frac{-\cos nx}{n}\right) - (\pi - 2x)\left(\frac{-\sin nx}{n^2}\right) + (-2)\frac{\cos nx}{n^3}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{2}{n^3}(1 - \cos n\pi)\right]$$

$$= \frac{4}{n\pi^3}\left(1 - (-1)^n\right)$$

$$b_n = \begin{cases} 0, \ when \ n \ is \ even \\ \frac{8}{\pi n^3}, \ when \ n \ is \ odd \end{cases}$$

Hence

$$x(\pi - x) = \sum_{n=1,3,5...} \frac{8}{\pi n^3} \sin nx \ (or)$$

$$x(\pi - x) = \frac{8}{\pi}\left(\sin x + \frac{\sin 3x}{3^3} + \frac{\sin 5x}{5^3} + ----\right) \to (1)$$

**Deduction:-**

Putting $x = \frac{\pi}{2}$ in (1), we get

$$\frac{\pi}{2}\left(x - \frac{\pi}{2}\right) = \frac{8}{\pi}\left(\sin\frac{\pi}{2} + \frac{1}{3^3}\sin\frac{3\pi}{2} + \frac{1}{5^3} sin\frac{5\pi}{2} + --\right)$$

$$\frac{\pi^2}{4} = \frac{8}{\pi}\left[1 + \frac{1}{3^3}\sin\left(\pi + \frac{\pi}{2}\right) + \frac{1}{5^3}\sin\left(2\pi + \frac{\pi}{2}\right) + \frac{1}{7^3}\sin\left(3\pi + \frac{\pi}{2}\right) + ---\right]$$

$$(or) \frac{\pi^2}{32} = 1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + ---$$

**3) Find the half- range sine series for the function** $f(x)=\frac{e^{ax}-e^{-ax}}{e^{a\pi}-e^{-a\pi}}in\ (0,\pi)$

Ans. Let $f(x)=\sum_{n=1}^{\infty}b_n\sin nx$

$$then\ b_n=\frac{2}{\pi}\int_0^{\pi}f(x)\sin nx.dx$$

$$=\frac{2}{\pi}\int_0^{\pi}\frac{e^{ax}-e^{-ax}}{e^{a\pi}-e^{-a\pi}}.\sin nx.dx$$

$$=\frac{2}{\pi\left(e^{a\pi}-e^{-a\pi}\right)}\left[\int_0^{\pi}e^{ax}\sin nx.dx-\int_0^{\pi}e^{-ax}\sin nx.dx\right]$$

$$=\frac{2}{\pi\left(e^{a\pi}-e^{-a\pi}\right)}\left[\left[\frac{e^{ax}}{a^2+n^2}(a\sin nx-n\cos nx)\right]_0^{\pi}-\left[\frac{e^{-ax}}{a^2+b^2}(-a\sin nx-n\cos nx)\right]_0^{\pi}\right]$$

$$=\frac{2}{\pi\left(e^{a\pi}-e^{-a\pi}\right)}\left[\frac{-e^{a\pi}}{a^2+n^2}n(-1)^n+\frac{n}{a^2+b^2}+\frac{e^{-a\pi}}{a^2+b^2}n(-1)^n-\frac{n}{a^2+b^2}\right]$$

$$=\frac{2n(-1)^n}{\pi\left(e^{a\pi}-e^{-a\pi}\right)}\left[\frac{e^{-a\pi}-e^{a\pi}}{n^2+a^2}\right]$$

$$=\frac{2n(-1)^{n+1}}{\pi\left(n^2+a^2\right)}$$

Substituting (2) in (1), we get

$$f(x)=\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{n(-1)^{n+1}}{a^2+n^2}\sin nx$$

$$=\frac{2}{\pi}\left[\frac{\sin nx}{a^2+1^2}-\frac{2\sin 2x}{a^2+2^2}+\frac{3\sin 3x}{a^2+3^2}-----\right]$$

**Fourier series of** $f(x)$ **defined in** $[c_1c+2l]$

It can be seen that role played by the functions

$1,\cos x,\cos 2x,\cos 3x,.....\sin x,\sin 2x.........$

In expanding a function $f(x)$ defined in $[c_1c+2\pi]$ as a Fourier series, will be played by

$$1,\cos\left(\frac{\pi x}{e}\right),\cos\left(\frac{2\pi x}{e}\right),\cos\left(\frac{3\pi x}{e}\right),.....$$

$$\sin\left(\frac{\pi x}{e}\right),\sin\left(\frac{2\pi x}{e}\right),\sin\left(\frac{3\pi x}{e}\right),.....$$

In expanding a function $f(x)$ defined in $[c,c+2I]$ it can be verified directly that, when m, n are integers

$$\int_c^{c+2l} \sin\left(\frac{m\pi x}{l}\right).\cos\left(\frac{2\pi x}{l}\right) dx = 0$$

$$\int_c^{c+2l} \sin\left(\frac{m\pi x}{l}\right).\sin\left(\frac{n\pi x}{l}\right) dx = \begin{cases} 0 & if \ \ m \neq n \\ l \ if & m = n \neq 0 \\ 2l \ if & m = 2n = 0 \end{cases}$$

$$\int_c^{c+2l} \cos\left(\frac{m\pi x}{l}\right).\cos\left(\frac{n\pi x}{l}\right) dx = \begin{cases} 0 & if \ \ m \neq n \\ l \ if & m = n \neq 0 \\ 2l \ if & m = n = 0 \end{cases}$$

**Fourier series of $f(x)$ defined in $[0, 2l]$:-**

Let $f(x)$ be defined in $[0, 2l]$ and be periodic with period $2l$. Its Fourier series expansion is defined as

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left[a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right] \rightarrow (1)$$

$$where \ a_n = \frac{1}{l}\int_0^{2l} f(x)\cos\frac{n\pi x}{l} dx \ and \rightarrow (2)$$

$$b_n = \frac{1}{l}\int_0^{2l} f(x)\sin\frac{n\pi x}{l} dx \rightarrow (3)$$

**Fourier series of $f(x)$ defined in $[-l, l]$:-**

Let $f(x)$ be defined in $[-l, l]$ and be periodic with period $2l$. Its Fourier series expansion is defined as

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right)$$

$$where \ a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l} dx$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l} dx$$

**Fourier series for even and odd functions in $[-l, l]$:-**

Let $f(x)$ be defined in $[-l, l]$. If $f(x)$ is even $f(x)\cos\frac{n\pi x}{l}$ is also even

$$\therefore a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l} dx$$

$$= \frac{2}{l}\int_0^{l} f(x)\cos\frac{n\pi x}{l} dx \qquad and \quad f(x)\sin\frac{n\pi x}{l} is \ odd$$

$$\therefore b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l} dx = 0 \forall n$$

Hence if $f(x)$ is defined in $[-l,l]$ and is even its Fourier series expansion is given by

$$f(x)=\frac{1}{2}a_0+\sum_{n=1}^{\infty}a_n\cos\frac{n\pi x}{l}$$

$$\textit{where } a_n=\frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}dx$$

If $f(x)$ is defined in $[-l,l]$and its odd its Fourier series expansion is given by

$$f(x)=\sum_{n=1}^{\infty}b_n\sin\frac{n\pi x}{l}\quad \textit{where } b_n=\frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}dx$$

**Note:-** In the above discussion if we put $2l=2\pi, l=\pi$ we get the discussion regarding the intervals $[0,2\pi]$ and $[-\pi,\pi]$ as special cases

**Examples:-**

**1. Express $f(x)=x^2$ as a Fourier series in $[-l,l]$**

Sol $f(-x)=f(-x)^2=x^2=f(x)$

Therefore $f(x)$ is an even function

Hence the Fourier series of $f(x)$ in $[-l,l]$ is given by

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos\frac{n\pi x}{l}$$

$$\textit{where } a_n=\frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}dx$$

$$\textit{hence } a_0=\frac{2}{l}\int_0^l x^2dx=\frac{2}{l}\left(\frac{x^3}{3}\right)_0^l=\frac{2l}{3}$$

$$\textit{also } a_n=\frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}dx$$

$$=\frac{2}{l}\int_0^l x^2\cos\frac{n\pi x}{l}dx$$

$$=\frac{2}{l}\left[x^2\left[\frac{\sin\left(\frac{n\pi x}{l}\right)}{\frac{n\pi}{l}}\right]-2x\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right)+2\left(\frac{-\sin\frac{n\pi x}{l}}{\frac{n^3\pi^3}{l^3}}\right)\right]_0^l$$

$$=\frac{2}{l}\left[2x\frac{\cos\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right]_0^l$$

Since the first and last terms vanish at both upper and lower limits

$$\therefore a_n = \frac{2}{l}\left[2l\frac{\cos n\pi}{n^2\pi^2/l^2}\right] = \frac{4l^2\cos n\pi}{n^2\pi^2}$$

$$= \frac{(-1)^n 4l^2}{n^2\pi^2}$$

Substituting these values in (1), we get

$$x^2 = \frac{l^2}{3} + \sum_{n=1}^{\infty}\frac{(-1)^n 4l^2}{n^2\pi^2}\cos\frac{n\pi x}{l}$$

$$= \frac{l^2}{3} - \frac{4l^2}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}\cos\frac{n\pi x}{l}$$

$$= \frac{l^2}{3} - \frac{4l^2}{\pi^2}\left[\frac{\cos(\pi x/l)}{1^2} - \frac{\cos(2\pi x/l)}{2^2} + \frac{\cos(3\pi x/l)}{3^2} - - - - -\right]$$

**2. Find a Fourier series with period 3 to represent** $f(x) = x + x^2 \ in \ (0,3)$

Sol. Let $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{l} + b_n\sin\frac{n\pi x}{l}\right) \rightarrow (1)$

Here $2l = 3$, $l = 3/2$

Hence (1) becomes

$$f(x) = x + x^2$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\frac{2n\pi x}{3} + b_n\sin\frac{2n\pi x}{3}\right) \rightarrow (2)$$

$$Where \ a_0 = \frac{1}{l}\int_0^{2l} f(x)\,dx$$

$$= \frac{2}{3}\int_0^3 (x + x^2)\,dx = \frac{2}{3}\left[\frac{x^2}{2} + \frac{x^3}{3}\right]_0^3 = 9$$

$$and \ a_n = \frac{1}{l}\int_0^2 f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{2}{3}\int_0^3 (x + x^2)\cos\left(\frac{2n\pi x}{3}\right)dx$$

Integrating by parts, we obtain

$$a_n = \frac{2}{3}\left[\frac{3}{4n^2\pi^2 - 4n^2\pi^2}\right] = \frac{2}{3}\left(\frac{54}{9n^2\pi^2}\right) = \frac{9}{n^2\pi^2}$$

Finally $b_n = \frac{1}{l}\int_0^{2l} f(x)\sin\frac{n\pi x}{l}\,dx$

$$= \frac{2}{3}\int_0^3 \left(x + x^2\right)\sin\left(\frac{2n\pi x}{3}\right)dx$$

$$= \frac{-12}{n\pi}$$

Substituting the values of a's and b's in (2) we get

$$x + x^2 = \frac{9}{2} + \frac{9}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\cos\left(\frac{2n\pi x}{3}\right) - \frac{12}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\left(\frac{2n\pi x}{3}\right)$$

**Half- Range Expansion of $f(x)\ in\ [0,l]$:-**

1. The half range sine series expansion of $f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$ in $(0,2)$ is given by

   Where $b_n = \frac{2}{l}\int_0^1 f(x)\sin\frac{n\pi x}{l}dx$

2. The half range cosine series expansion of $f(x)\ in\ [0,l]$ is given by

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}$$

$$where\ b_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}dx$$

**<u>Examples:-</u>**

1. Find the half- range sine series of $f(x) = 1\ in\ [0,l]$

Ans. The Fourier sine series of $f(x)\ in\ [0,l]$ is given by $f(x) = 1 = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$

$$here\ b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}dx$$

$$= \frac{2}{l}\int_0^l 1.\sin\frac{n\pi x}{l}dx$$

$$= \frac{2}{l}\left(\frac{-\cos\frac{n\pi x}{l}}{n\pi / l}\right)_0^l$$

$$= \frac{2}{n\pi}\left[-\cos\frac{n\pi x}{l}\right]_0^l$$

$$= \frac{2}{n\pi}\left(-\cos n\pi + 1\right)$$

$$= \frac{2}{n\pi}\left[(-1)^{n+1} + 1\right]$$

$\therefore b_n = 0$ when n is even

$$= \frac{4}{n\pi}, \text{ when n is odd}$$

Hence the required Fourier series is $f\left(x\right) = \sum_{n=1,3,5---}^{\infty} \frac{4}{n\pi} \sin\frac{n\pi x}{l}$

2. Find the half – range cosine series expansion of $f\left(x\right) = \sin\left(\frac{\pi x}{l}\right)$ in the range $0 < x < l$

Sol

$$f\left(xa_0\right) = \sin\left(\frac{\pi x}{l}\right) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}$$

$$where\ a_0 = \frac{2}{l}\int_0^l f\left(x\right)dx = \frac{2}{l}\int_0^l \sin\frac{\pi x}{l}dx$$

$$= \frac{2}{l}\left[\frac{-\cos \pi x/l}{\pi/l}\right]_0^l$$

$$= \frac{2}{l}\left(\cos \pi - 1\right) = \frac{4}{\pi}\ and$$

$$a_n = \frac{2}{l}\int_0^l f\left(x\right)\cos\frac{n\pi x}{l}dx$$

$$= \frac{2}{l}\int_0^l \sin\left(\frac{\pi x}{l}\right)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{1}{l}\int_0^l \left[\frac{\sin\left(n+1\right)\pi x}{l} - \frac{\sin\left(n-1\right)\pi x}{l}\right]dx$$

$$= \frac{1}{l}\left[-\frac{-\dfrac{\cos\left(n+1\right)\pi x}{l}}{\left(n+1\right)\pi/l} + \frac{\cos\left(n-1\right)\pi x/l}{\left(n-1\right)\pi/l}\right]_0^l$$

$$= \frac{1}{\pi}\left[-\frac{\left(-1\right)^{n+1}}{n+1} + \frac{\left(-1\right)^{n-1}}{n-1} + \frac{1}{n+1} - \frac{1}{n-1}\right]$$

When n is odd

$$a_n = \frac{1}{\pi}\left[\frac{-1}{n+1} + \frac{1}{n-1} + \frac{1}{n+1} - \frac{1}{n-1}\right] = 0$$

When n is even

$$a_n = \frac{1}{\pi}\left[\frac{1}{n+1} - \frac{1}{n-1} + \frac{1}{n+1} - \frac{1}{n-1}\right]$$

$$= \frac{-4}{\pi\left(n+1\right)\left(n-1\right)}$$

$$\therefore \sin\left(\frac{\pi x}{l}\right) = \frac{2}{\pi} - \frac{4}{\pi}\left[\frac{\cos\left(2\pi x/l\right)}{1.3} + \frac{\cos\left(4\pi x/l\right)}{3.5} + ----\right]$$

# Fourier Transforms

Fourier Transforms are widely used to solve Partial Differential Equations and in various boundary value problems of Engineering such as Vibration of Strings, Conduction of heat, Oscillation of an elastic beam, Transmission lines etc.

**Integral Transforms:**

The Integral transform of a function f(x) is defined as

$I\{f(x)\} = \bar{f}(s) = \int_{x=x_1}^{x_2} f(x)K(s,x)dx$

Where K(s,x) is a known function of s & x, called the 'Kernel' of the transform.

The function f(x) is called the Inverse transform of $\bar{f}(s)$

1.**Laplace Transform**: When K(s,x) = $e^{-sx}$

$L\{f(x)\} = \bar{f}(s) = \int_0^\infty f(x)e^{-sx}dx$

2.**Fourier Transform**: When K(s,x) = $e^{isx}$

$F\{f(x)\} = \bar{f}(s) = \frac{1}{\sqrt{2\pi}}\int_0^\infty f(x)e^{isx}dx$

3.**Fourier Sine Transform**: When K(s,x)=Sinsx

$F_s\{f(x)\} = \bar{f}(s) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)sinsx\ dx$

4. **Fourier Cosine Transform**: When K(s,x)=Cossx

$F_c\{f(x)\} = \bar{f}(s) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)cossx\ dx$

5.**Mellin Transform**: When K(s,x)= $x^{s-1}$

$M(s) = \int_0^\infty f(x)x^{s-1}dx$

6.**Hankel Transform**: When K(s,x) = $xJ_n(sx)$

$H(s) = \bar{f}(s) = \int_0^\infty f(x)xJ_n(sx)dx$

Where $J_n(sx)$ is a Bessel function.

**Fourier Integral Theorem**:- If f(x) satisfies Dirichlet's conditions for expansion of Fourier series in (-c,c) and $\int_{-\infty}^{\infty}|f(x)|$ converges, then

$$f(x) = \frac{1}{\pi}\int_0^\infty \int_{-\infty}^{\infty} f(t)cos\lambda(t-x)dt\ d\lambda$$

Which is known as Fourier Integral of f(x)

**Proof:** Since f(x) satisfies Dirichlet's conditions in (-c,c), Fourier series of f(x) is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n cos\frac{n\pi x}{c} + b_n sin\frac{n\pi x}{c}\right) \quad \ldots\ldots\ldots\ldots\ldots..(1)$$

Where $a_0 = \frac{1}{c}\int_{-c}^{c} f(t)dt$ , $a_n = \frac{1}{c}\int_{-c}^{c} f(t)cos\frac{n\pi t}{c}dt$ , $b_n = \frac{1}{c}\int_{-c}^{c} f(t)sin\frac{n\pi t}{c}dt$

Substitute the values of $a_0$, $a_n$ and $b_n$ in (1) , we get

$$f(x) = \frac{1}{2c}\int_{-c}^{c} f(t)dt + \frac{1}{c}\sum_{n=1}^{\infty}\int_{-c}^{c} f(t)\cos\frac{n\pi(t-x)}{c}\, dt \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

Since $\int_{-\infty}^{\infty}|f(x)|\, dx$ converges i.e., f(x) is absolutely integrable on x-axis,

The first term on R.H.S of (2) approaches to '0' as $c \to \infty$

Since $\left|\frac{1}{2c}\int_{-\infty}^{\infty} f(t)dt\right| \le \frac{1}{2c}\int_{-\infty}^{\infty}|f(t)|\, dt$

The second term on R.H.S of (2) tends to

$$Lt_{c\to\infty}\frac{1}{c}\sum_{n=1}^{\infty}\int_{-\infty}^{\infty} f(t)\cos\frac{n\pi(t-x)}{c}\, dt = Lt_{\frac{\pi}{c}\to 0}\frac{1}{\pi}\sum_{n=1}^{\infty}\int_{-\infty}^{\infty} f(t)\cos\frac{n\pi(t-x)}{c}\, dt$$

Let $\frac{\pi}{c} = \delta\lambda$ $so\ that\ \delta\lambda \to 0\ as\ c \to \infty$

$$f(x) = Lt_{\delta\lambda\to 0}\frac{1}{\pi}\sum_{n=1}^{\infty}\int_{-\infty}^{\infty} f(t)\cos n(t-x)\delta\lambda\, dt \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

This is of the form $Lt_{\delta\lambda\to 0}\sum_{n=1}^{\infty}F(n\delta\lambda)$ $i.e.,$ $\int_0^{\infty}F(\lambda)\, d\lambda$

Thus $as\ c \to \infty$, (3) becomes

$$f(x) = \frac{1}{\pi}\int_0^{\infty}\int_{-\infty}^{\infty} f(t)\cos\lambda(t-x)dt\, d\lambda$$

Which is known as Fourier Integral of f(x)

**Fourier Sine & Cosine Integrals:-**

From Fourier Integral theorem

$$f(x) = \frac{1}{\pi}\int_0^{\infty}\int_{-\infty}^{\infty} f(t)\cos\lambda(t-x)dt\, d\lambda \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

w.k.t $\cos\lambda(t-x) = \cos\lambda t\cos\lambda x + \sin\lambda t\ \sin\lambda x$

Sub. This value in eq(1), we get

$$f(x) = \frac{1}{\pi}\int_0^{\infty}\cos\lambda x\int_{-\infty}^{\infty} f(t)\cos\lambda t\, dt\, d\lambda + \frac{1}{\pi}\int_0^{\infty}\sin\lambda x\int_{-\infty}^{\infty} f(t)\sin\lambda t\, dt\, d\lambda \ \ldots\ldots\ldots(2)$$

when f(t) is odd function, then $f(t)\cos\lambda t$ is an odd function while $f(t)\sin\lambda t$ is an even function.
then eq(2) becomes

$$f(x) = \frac{2}{\pi}\int_0^{\infty}\sin\lambda x\int_0^{\infty} f(t)\sin\lambda t\, dt\, d\lambda$$

This is called "Fourier sine Integral"

when f(t) is even function then f(t) cosλt is an even function, while f(t)sinλt is an odd function then eq(2) becomes

$$f(x) = \frac{2}{\pi}\int_0^{\infty} cos\lambda x \int_0^{\infty} f(t)cos\lambda t\ dt\ d\lambda$$

This is called "Fourier cosine Integral"

**Complex form of Fourier Integral:-**

From Fourier Integral theorem

$$f(x) = \frac{1}{\pi}\int_0^{\infty}\int_{-\infty}^{\infty} f(t)cos\lambda(t-x)dt\ d\lambda \quad \ldots\ldots\ldots\ldots\ldots(1)$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)dt\ \{\int_0^{\infty} cos\lambda(t-x)d\lambda\}$$

since cos λ(t-x) is an even function

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)cos\lambda(t-x)dt\ d\lambda \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

w.k.t sinλ(t-x) is an odd function,

$$\int_{-\infty}^{\infty} sin\lambda(t-x)\ d\lambda = 0$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)sin\lambda(t-x)dt\ d\lambda = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

multiply (3) by i and add it to (2), then

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)[cos\lambda(t-x) + isin\lambda(t-x)]dt\ d\lambda$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)\ e^{i\lambda(t-x)}dt\ d\lambda$$

which is known as 'Complex form of Fourier Integral'.

**Problems:**

1. Express the function $f(x) = \begin{cases} 1; & |x| \le 1 \\ 0; & |x| > 1 \end{cases}$ as a Fourier integral and hence evaluate

$$\int_0^{\infty} \frac{sin\lambda cos\lambda x}{\lambda}\ d\lambda$$

**sol:** The Fourier Integral of f(x) is given by

$$f(x) = \frac{1}{\pi}\int_{\lambda=0}^{\infty} d\lambda \int_{t=-\infty}^{\infty} f(t)cos\lambda(t-x)dt \quad \ldots\ldots\ldots\ldots\ldots(1)$$

given that $f(t) = \begin{cases} 0; & -\infty < t < -1 \\ 1; & -1 < t < 1 \\ 0; & 1 < t < \infty \end{cases}$

$$f(x) = \frac{1}{\pi}\int_{\lambda=0}^{\infty} d\lambda \int_{t=-1}^{1} cos\lambda(t-x)dt$$

$$= \frac{1}{\pi}\int_{\lambda=0}^{\infty}\left(\frac{sin\lambda(t-x)}{\lambda}\right)_{t=-1}^{1} d\lambda$$

$$= \frac{1}{\pi}\int_{\lambda=0}^{\infty} \frac{1}{\lambda}\ (\ sin\lambda(1-x) -\ sin\lambda(-1-x))d\lambda$$

$$= \frac{1}{\pi}\int_{\lambda=0}^{\infty} \frac{1}{\lambda}\ (\ sin\lambda(1-x) +\ sin\lambda(1+x))d\lambda$$

$= \frac{2}{\pi}\int_{\lambda=0}^{\infty}\frac{\sin\lambda\cos\lambda x}{\lambda}\,d\lambda$ ……………………(2)

which is fourier integral of f(x)

from (2), $\frac{2}{\pi}\int_{\lambda=0}^{\infty}\frac{\sin\lambda\cos\lambda x}{\lambda}\,d\lambda = f(x)$

$\int_{\lambda=0}^{\infty}\frac{\sin\lambda\cos\lambda x}{\lambda}\,d\lambda = \frac{\pi}{2}f(x)$

given f(x) $= \begin{cases} 1; & |x| \le 1 \\ 0; & |x| > 1 \end{cases}$

$\int_{\lambda=0}^{\infty}\frac{\sin\lambda\cos\lambda x}{\lambda}\,d\lambda = \begin{cases} \frac{\pi}{2}; & |x| < 1 \\ 0; & |x| > 1 \end{cases}$

at $|x| = 1$ i.e., when x=±1

f(x) is discontinuous & the integral has the value $\frac{1}{2}\left(\frac{\pi}{2}+0\right) = \frac{\pi}{4}$

$\int_{\lambda=0}^{\infty}\frac{\sin\lambda\cos\lambda x}{\lambda}\,d\lambda = \frac{\pi}{4}$ $at\ |x| = 1$

**2**.Find Fourier Sine Integral representation of f(x) $= \begin{cases} 0, \infty < x < 1 \\ x, -1 < x < 0 \\ 0, \quad x > 0 \end{cases}$

**sol**: Fourier Sine integral of f(x) is given by

f(x) $= \frac{2}{\pi}\int_0^{\infty}\sin\lambda x \int_0^{\infty} f(t)\sin\lambda t\, dt\, d\lambda$ ……………..(1)

given that f(t) $= \begin{cases} t, & -1 < t < 0 \\ 0, & else\ where \end{cases}$

f(x) $= \frac{2}{\pi}\int_0^{\infty}\sin\lambda x \left\{\int_{-1}^{0} t\sin\lambda t\, dt\right\} d\lambda$

$= \frac{2}{\pi}\int_0^{\infty}\sin\lambda x \left(\frac{-t\cos\lambda t}{\lambda}+\frac{\sin\lambda t}{\lambda^2}\right)_{t=-1}^{0} d\lambda$

f(x) $= \frac{2}{\pi}\int_0^{\infty}\sin\lambda x \left(\frac{\sin\lambda}{\lambda^2}-\frac{\cos\lambda}{\lambda}\right) d\lambda$

**Fourier Transforms:-**

Complex form of Fourier Integral of f(x) is

f(x) $= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)\, e^{i\lambda(t-x)}\, dt\, d\lambda$

replace λ by s

f(x) $= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-isx}\, ds \,.\, \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{-ist}\, dt$

If we define F(s) $= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{-ist}\, dt$

then f(x) $= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(s)e^{-isx}\, ds$

F(s) is called Fourier Transform (F.T) of f(x) and f(x) is called inverse Fourier transform of F(s)

**Fourier Sine & Cosine transforms:-**

The Fourier sine integral of f(x) is defined as

f(x) $= \frac{2}{\pi}\int_0^{\infty}\sin sx \int_0^{\infty} f(x)\sin sx\, dx\, ds$

$$f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty sinsx\ ds.\ \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)sinsx\ dx$$

If we define $F_s(s) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)sinsx\ dx$

then $f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty F_s(s)sinsx\ ds$

here $\boldsymbol{F_s(s)}$ is called **Fourier sine transform** of **f(x)** and **f(x)** is called **Inverse Fourier sine transform** of $\boldsymbol{F_s(s)}$

similarly, Fourier cosine integral of f(x) is

$$f(x) = \frac{2}{\pi}\int_0^\infty cossx\int_0^\infty f(x)cossx\ dx\ ds$$

if we define $F_c(s) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)cossx\ dx$

then $f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty F_c(s)cossx\ ds$

here $\boldsymbol{F_c(s)}$ is called **Fourier cosine transform** of **f(x)** and **f(x)** is called **Inverse Fourier cosine transform** of $\boldsymbol{F_c(s)}$

**NOTE**: 1. Some authors define F.T as follows

i) $F(s) = \int_{-\infty}^{\infty} f(x)e^{isx}dx$ ii) $f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(s)e^{-isx}ds$

iii) $F(s) = \sqrt{\frac{2}{\pi}}\int_{-\infty}^{\infty} f(x)e^{-isx}dx$ iv) $f(x) = \sqrt{\frac{2}{\pi}}\int_{-\infty}^{\infty} F(s)e^{isx}ds$

2.Some authors define Fourier sine & cosine transforms as follows

i) $F_s(s) = \int_0^\infty f(x)sinsx\ dx$ ii) $f(x) = \frac{2}{\pi}\int_0^\infty F_s(s)sinsx\ ds$

iii) $F_c(s) = \int_0^\infty f(x)cossx\ dx$ iv) $f(x) = \frac{2}{\pi}\int_0^\infty F_c(s)cossx\ ds$

**Properties of Fourier Transforms**:-

1.**Linearity Property**:- If $F_1(s)$ and $F_2(s)$ be the Fourier transforms of $f_1(x)$ and $f_2(x)$ respectively then $f\{af_1(x) + b\,f_2(x)\} = a\ F_1(s) + b\,F_2(s)$, where $a$ & $b$ are constants

**proof**:- by definition of Fourier transform,

$$F\{af_1(x) + b\,f_2(x)\} = \int_{-\infty}^{\infty} e^{isx}(af_1(x) + b\,f_2(x))\,dx$$

$$= a\int_{-\infty}^{\infty} e^{isx} f_1(x)\,dx + b\int_{-\infty}^{\infty} e^{isx} f_2(x)\,dx$$

$$= a\ F_1(s) + b\,F_2(s)$$

2.**Change of Scale Property**:- If F{f(x)} = F(s) then F{f(ax)} $= \frac{1}{a} F(\frac{s}{a})$

**proof**:- By definition of F.T,

F{f(x)} = F(s) = $\int_{-\infty}^{\infty} e^{isx} f(x)\, dx$ …….(1)

F{f(ax)} = $\int_{-\infty}^{\infty} e^{isx} f(ax)\, dx$

put ax = t
then a dx = dt

F{f(ax)} = $\int_{-\infty}^{\infty} e^{is(\frac{t}{a})} f(t)\, dt/a$

$= \frac{1}{a} \int_{-\infty}^{\infty} e^{i(\frac{s}{a})t} f(t)\, dt$

$= \frac{1}{a} F(\frac{s}{a})$ by(1)

3.**Shifting Property**:- If F{f(x)} = F(s) then F{f(x-a)} = $e^{isa} F(s)$

**Proof**:- By definition of F.T,

F{f(x)} = F(s) = $\int_{-\infty}^{\infty} e^{isx} f(x)\, dx$ …….(1)

F{f(x-a)} = $\int_{-\infty}^{\infty} e^{isx} f(x-a)\, dx$

put x-a = t
then dx = dt

F{f(x-a)} = $\int_{-\infty}^{\infty} e^{is(t+a)} f(t)\, dt$

$= e^{isa} \int_{-\infty}^{\infty} e^{ist} f(t)\, dt$

$= e^{isa} F(s)$ by(1)

4.**Modulation Property**:- If F{f(x)} = F(s) then F{f(x)cosax} = ½ {F(s+a)+F(s-a)}

**Proof**:- By definition of F.T,

F{f(x)} = F(s) = $\int_{-\infty}^{\infty} e^{isx} f(x)\, dx$ …….(1)
F{f(x)cosax} = $\int_{-\infty}^{\infty} e^{isx} f(x) cosax\, dx$

$= \int_{-\infty}^{\infty} e^{isx} f(x) (\frac{e^{iax} + e^{-iax}}{2})\, dx$

$=\frac{1}{2}\{\int_{-\infty}^{\infty} e^{i(s+a)} f(x)dx + \int_{-\infty}^{\infty} e^{i(s-a)} f(x)dx\}$

$=\frac{1}{2}$ {F(s+a) + F(s-a)}

5.**Convolution Property**:- The convolution of two functions f(t) and g(t) in (-∞,∞) is defined as f(t) * g(t) = $\int_{-\infty}^{\infty} f(t).g(x-t)\ dt$

**Theorem:**- If F{f(t)}=$F_1(s)$ and F{g(t)}= $F_2(s)$ then F{f(t)*g(t)} = $F_1(s).F_2(s)$

**Proof**:- By definition of F.T we have

F{f(t)*g(t)} $= \int_{-\infty}^{\infty} \big(f(t) * g(t)\big)e^{ist}dt$

$= \int_{-\infty}^{\infty} \{\int_{-\infty}^{\infty} f(u)g(t-u)du\}\, e^{ist}dt$

$= \int_{-\infty}^{\infty} f(u)e^{isu}\{\int_{-\infty}^{\infty} g(t-u)\,e^{is(t-u)}\,d(t-u)\}\,du$

on changing the order of integration,

$= \int_{-\infty}^{\infty} f(u)e^{isu}.F_2(s)du$

$= \{\int_{-\infty}^{\infty} f(u)e^{isu}du\}.F_2(s)$

$=F_1(s).F_2(s)$

F{f(t)*g(t)} $= F_1(s).F_2(s)$

6.If F{f(x)} = F(s) then F{f(-x)}= F(-s)

**Proof**: By definition, F{f(x)} $= \int_{-\infty}^{\infty} f(x)e^{isx}dx$ ………….(1)

F{f(-x)} $= \int_{-\infty}^{\infty} f(-x)e^{isx}dx$

put –x = t then dx = -dt

as x→∞, t→-∞ and as x→-∞ , t→∞

F{f(x)} $= \int_{\infty}^{-\infty} f(t)e^{-ist}(-dt)$

$= \int_{-\infty}^{\infty} f(t)e^{-ist}dt$

$= \int_{-\infty}^{\infty} f(t)e^{i(-s)t}dt$

= F(-s) (by (1))

7.$\overline{F\{f(x)\}} = \overline{F(-s)}$

**Proof**: By definition, F{f(x)} = F(s) = $\int_{-\infty}^{\infty} f(x)e^{isx}dx$ ………….(1)

F(-s) = $\int_{-\infty}^{\infty} f(x)e^{-isx}dx$

taking complex conjugate on both sides

$\overline{F(-s)} = \int_{-\infty}^{\infty} \overline{f(x)}e^{isx}dx$

$$= \overline{F\{f(x)\}}$$

8. $\overline{F\{f(-x)\}} = \overline{F(s)}$

**Proof**: By definition, F{f(x)} = F(s) = $\int_{-\infty}^{\infty} f(x) e^{isx} dx$

take complex conjugate on both sides

$$\overline{F(s)} = \int_{-\infty}^{\infty} \overline{f(x)} e^{-isx} dx$$

put x=-z then dx = -dz

$$\overline{F(s)} = \int_{\infty}^{-\infty} \overline{f(-z)} e^{isz} (-dz)$$

$$= \int_{-\infty}^{\infty} \overline{f(-z)} e^{isz} dz$$

$$= F\{\overline{f(-x)}\}$$

$$\overline{F\{f(-x)\}} = \overline{F(s)}$$

9. $F_c\{xf(x)\} = \frac{d}{ds} F_s\{f(x)\}$

**Proof**: By definition of Fourier sine transform

$$F_s\{f(x)\} = \int_0^{\infty} f(x) sinsx\, dx$$

$$\frac{d}{ds} F_s\{f(x)\} = \frac{d}{ds} \{ \int_0^{\infty} f(x) sinsx\, dx \}$$

$$= \int_0^{\infty} \frac{d}{ds} \{ f(x) sinsx\} dx$$

$$= \int_0^{\infty} f(x). xcossx\, dx$$

$$= \int_0^{\infty} \{xf(x)\} cossx\, dx$$

$$= F_c\{xf(x)\}$$

Note: $F_s\{xf(x)\} = -\frac{d}{ds} F_c\{f(x)\}$

**Problems:**

**1. Find the F.T of f(x) =** $e^{-|x|}$

**sol**: Given f(x) = $e^{-|x|}$

$$= \begin{cases} e^{x}; x < 0 \\ e^{-x}; x > 0 \end{cases}$$

by definition, F{f(x)} $= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx$

$$= \frac{1}{\sqrt{2\pi}} \{ \int_{-\infty}^{0} f(x) e^{isx} dx + \int_0^{\infty} f(x) e^{isx} dx \}$$

$$= \frac{1}{\sqrt{2\pi}}\{\int_{-\infty}^{0} e^{(1+is)x}dx + \int_{0}^{\infty} e^{(-1+is)x}dx\}$$

$$= \frac{1}{\sqrt{2\pi}}\ \{ \left(\frac{e^{(1+is)x}}{1+is}\right)_{-\infty}^{0} + \left(\frac{-e^{-(1-is)x}}{1-is}\right)_{0}^{\infty}\ \}$$

$$= \frac{1}{\sqrt{2\pi}}\left(\frac{1}{1+is} + \frac{1}{1-is}\right)$$

$$= \sqrt{\frac{2}{\pi}}.\frac{1}{1+s^2}$$

**2. S.T the Fourier Sine transform of f(x) =** $\begin{cases} x, 0 < x \le 1 \\ 2-x, 1 < x < 2 \\ 0, x > 2 \end{cases}$ **is** $\frac{2sins(1-coss)}{s^2}$

**sol:** By definition,

$$F_s\{f(x)\} = \int_0^\infty f(x) sinsx\ dx$$

$$= \int_0^1 f(x)sinsx\ dx + \int_1^2 f(x)sinsx\ dx + \int_2^\infty f(x)sinsx\ dx$$

$$= \int_0^1 xsinsx\ dx + \int_1^2 (2-x)sinsx\ dx$$

$$= [x.\left(\frac{-cossx}{s}\right) - \left(\frac{-sinsx}{s^2}\right)]_0^1 + [(2-x).\left(\frac{-cossx}{s}\right) - (-1)\left(\frac{-sinsx}{s^2}\right)]_1^2$$

$$= \frac{-coss}{s} + \frac{sins}{s^2} - \frac{sin2s}{s^2} + \frac{coss}{s} + \frac{sins}{s^2}$$

$$= \frac{2sins(1-coss)}{s^2}$$

## <u>FINITE FOURIER TRANSFORMS:-</u>

If f(x) is a function defined in the interval (0,c) then, the Finite Fourier sine transform of f(x) in 0<x<c is defined as

$F_s(n) = \int_0^c f(x) sin\frac{n\pi x}{c}\ dx$ , where n is an integer.

The Inverse finite Fourier sine transform of $F_s(n)$ is f(x) and is given by

$$f(x) = \frac{2}{c}\sum_{n=1}^{\infty} F_s(n)\ sin\frac{n\pi x}{c}$$

The Finite Fourier cosine transform of f(x) in 0<x<c is defined as

$F_c(n) = \int_0^c f(x) cos\frac{n\pi x}{c}\ dx$ , where n is an integer

The Inverse finite Fourier cosine transform of $F_c(n)$ is f(x) and is given by

$$f(x) = \frac{1}{c}F_c(0) + \frac{2}{c}\sum_{n=1}^{\infty} F_c(n)\ cos\frac{n\pi x}{c}$$

**<u>Problems:-</u>**

**1. Find the Finite Fourier sine and cosine transforms of f(x)=1 in (0,c)**

**sol**: By definition,

$$F_s(n) = \int_0^c f(x) sin\frac{n\pi x}{c}\ dx$$

$$= \int_0^c sin\frac{n\pi x}{c}\ dx$$

$$= [\left(\frac{-c}{n\pi}\right)\cos\left(\frac{n\pi x}{c}\right)]_0^c$$

$$= \frac{-c}{n\pi}\ (cosn\pi - 1)$$

$$= \frac{c}{n\pi}(1-(-1)^n)$$

$$F_s(n) = \{\begin{matrix} 0, if\ \ n\ is\ odd \\ \frac{2c}{n\pi}, if\ \ n\ is\ even \end{matrix}$$

Now, $F_c(n) = \int_0^c f(x) cos\frac{n\pi x}{c}\ dx$

$$= \int_0^c cos\frac{n\pi x}{c}\ dx$$

$$= [\left(\frac{c}{n\pi}\right)\sin\left(\frac{n\pi x}{c}\right)]_0^c$$

$$= \frac{c}{n\pi}\ sinn\prod\ = 0$$