UNIT - III

CENTRE OF GRAVITY

AND

MOMENT OF INERTIA OF AREAS

UNIT – III

CENTRE OF GRAVITY & CENTROID.

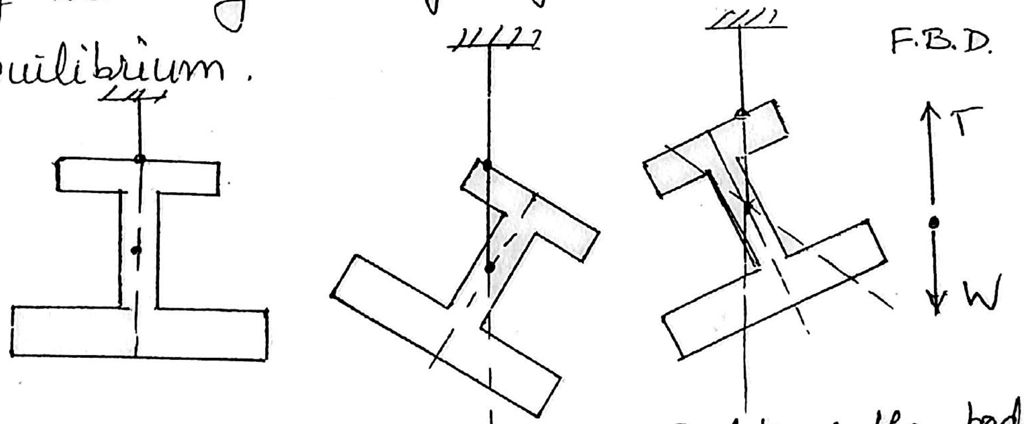
Centre of Gravity:

Centre of gravity is a very useful concept in statics as well as dynamics.

We tacitly assume in problems pertaining to statics that the weight of a solid acts through a point always, no matter what its orientation and call this point as the centre of gravity. The point may not always lie within the matter of the body.

What is C.G.?

If we hang a body by a thread, it takes a positi[on] of equilibrium.



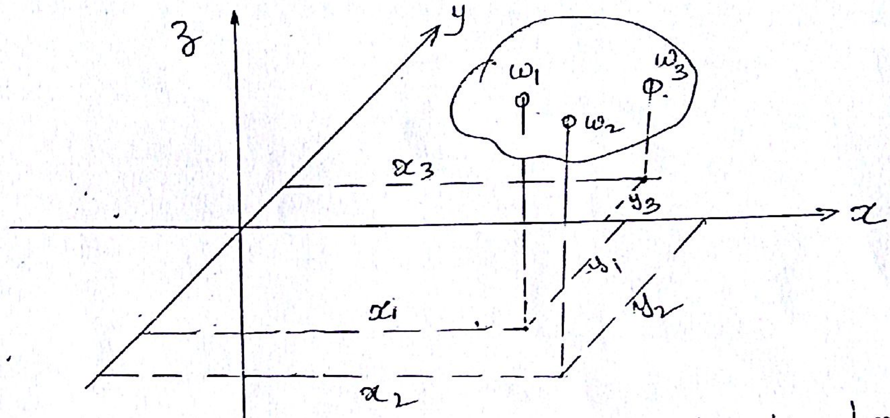
The resultant of the forces of weight of the body must lie along and coincide with the vertical line of the threa[d] for in equilibrium, no net moments or forces exist. If we change the point of suspension, the body will lie in a different position of equilibrium. It is observed tha[t] no matter how we suspend the body the vertical lin[e] extended from the thread of suspension always passes through a particular point in the body.

The resultant force of the weight of the body is equivalent to a single force always passing through th[e] point. This is called the centre of gravity. Thus, the existance of the C.G. can be verified experimentally.

Theory:

A solid can be thought of as made up of a large number of small particles. Each particle is attracted by the force of gravity towards the centre of the earth. The resultant of this set of parallel forces, whose resultant must be an ~~single force~~ equivalent single force parallel to this set of forces, and its magnitude is equal to the sum of these forces.

$$\text{i.e. } W = \sum w_i = \int \rho \, dV$$



Let us visualize a set of X-Y axes in the horizontal plane and a body of weight W considered w.r.to this axis set:

If (x_1, y_1), (x_2, y_2), (x_3, y_3) .... (x_n, y_n) are the x and y coordinate sets of particles w_1, w_2, w_3 .... w_n located at $(x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3)$ ... (x_n, y_n, z_n), then, if W is the sum of the weight of all these particles $(W = \sum w_i)$, using Varignon's theorem, we have

$$\sum x_i w_i = w_1 x_1 + w_2 x_2 + \dots w_n x_n = W \cdot \bar{x} \qquad --(1)$$

where $\bar{x}$ is the distance of the resultant force from the 'y' axis. Hence, $\bar{x} = \dfrac{x_1 w_1 + w_2 x_2 + \dots w_n x_n}{w_1 + w_2 + \dots w_n} = \dfrac{\sum w_i x_i}{\sum w_i} = \dfrac{\sum x_i w_i}{W}$

Similarly, $\bar{y} = \dfrac{\sum w_i y_i}{\sum w_i}$ $\qquad --(2)$

Now if we tilt the body by rotating it along with the axes set about the x axis such that the y axis becomes the z axis; the weight would act along the

Hence, by extension, we have

$$\bar{z} = \frac{\Sigma' w_i z_i}{\Sigma' w_i} \quad \ldots\ldots (3)$$

The point defined by these coordinates $(\bar{x}, \bar{y}, \bar{z})$ is unique in the solid and is called the 'Centre of Gravity'

Different types of Solids & their C.G :-

We divide the objects that we find around us in the following categories for convenience.

1. Objects where one dimension is constant & much less than other dimensions - Laminas.
2. Wire like objects in which one dimension is predominant and other dimensions are uniform but very small compared to the length.
3. Surfaces formed by thin sheets which can be develop into plane areas.
4. Proper Solids of regular shapes.
5. Solids which are composed of several regular shapes.
6. Axi-symmetric solids.

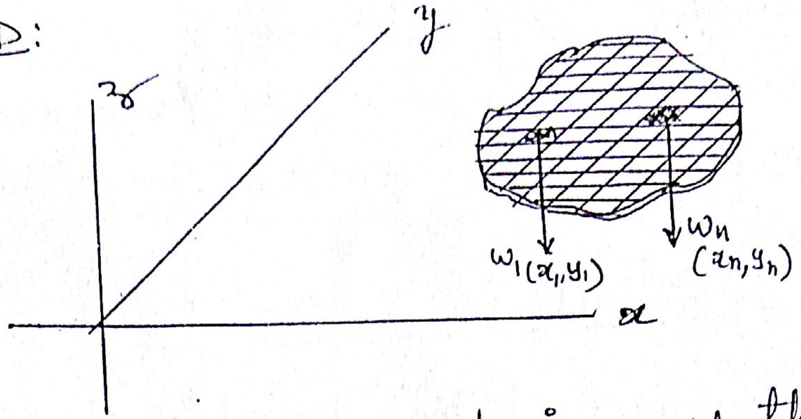
In this unit, we will learn how to find to find the C.G. of all the types mentioned above.

Why determination of C.G. is important ?

i. To ensure that the body support is such that it will not topple.
ii To minimise stresses in lifting tackles by locat lifting points closest to the C.G.
iii) To perform static & dynamic force analysis

If a body is suspended with its C.G. not in the lin vertically down the line of suspension a moment caus by the weight and the tension in the string will th the body till it aligns its C.G. vertically down the poi of suspension.

CENTROID:



Let us consider a thin lamina of thickness 't' and uniform density 'ρ' with its plane ⊥ to vertical direction. If a small weight all has its edges ∥ to z-axis, then,

$$w_1 = A_1 \rho . t . g$$
$$w_2 = A_2 \rho t . g$$
$$w_i = A_i \rho . t . g .$$

Hence, $\sum w_i = \sum (A_1 \rho t g + A_2 \rho t g + A_3 \rho t g + \cdots A_i \rho t g + \cdots$

$$= \sum A_i (\rho . t g) \quad \text{------ (3)}$$

and, $\sum w_i x_i = A_1 x_1 \rho t g + A_2 x_2 \rho t g + \cdots A_i x_i \rho t g + \cdots$

$$= \sum A_i x_i (\rho . t . g) \quad \text{---- (4)}$$

Hence, $$\bar{x} = \frac{\sum w_i x_i}{\sum w_i} = \frac{\sum A_i x_i (\rho t g)}{\sum A_i (\rho t g)}$$

$$= \frac{\sum A_i x_i}{\sum A_i} \quad \text{---- (5)}$$

Similarly, $$\bar{y} = \frac{\sum A_i y_i}{\sum A_i} \quad \text{--- (6)}$$

Thus, the coordinates are independent of the density, gravity constant or the thickness of the lamina.

If the thickness tends to zero, the centre of gravity tends to a point in the plane of the lamina.

The centre of gravity can then ~~th~~ be thought of as an attribute of the area.

Defined for an area, thus the C.G. is called 'Centroid'.

Centroid of an area has no direct physical significance but it is a concept extremely useful in the subject of 'strength of materials'.

Another interpretation can be seen. If we apply uniform pressure on an area, the resultant force will have a line of action passing through the 'Centroid'.

CENTROID OF A LENGTH:

Anologous to the definition of centroid of areas we can define centroid for wirelike long objects thus.

$$\bar{x} = \frac{\Sigma' x_i L_i}{\Sigma' L_i}, \quad \bar{y} = \frac{\Sigma' x_i L_i}{\Sigma' L_i}, \quad \bar{z}_i = \frac{\Sigma' z_i L_i}{\Sigma' L_i}.$$

Centroid / Centre of Gravity by Integration:

If the size of the particles into which the body is assumed to be divided becomes smaller and smaller, the number of partic comprising the solid becomes larger and larger.

In the limit, we have

$$\bar{x} = \frac{\Sigma w_i x_i}{\Sigma w_i} = \frac{\int_V x_i \, dw}{\int_V dw} = \frac{\int_V x \, dw}{W}$$

and similar expressions for $\bar{y}$ and $\bar{z}$.

Similarly, for areas,

$$\bar{x} = \frac{\Sigma A_i x_i}{\Sigma' A_i} = \frac{\int_A x \, dA}{\int_A dA}, \quad \bar{y} = \frac{\int_A y \, dA}{\int_A dA}$$

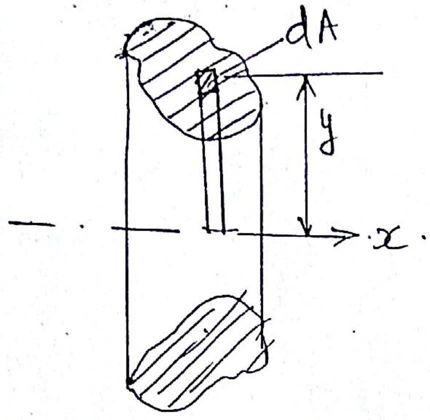
Thus, $\bar{x} W = \int x \, dw$, $\bar{x} A = \int x \, dA$ etc.

The quantities $\bar{x}W$, $\bar{x}A$ etc. are called 'First Moment of the weight (or area).

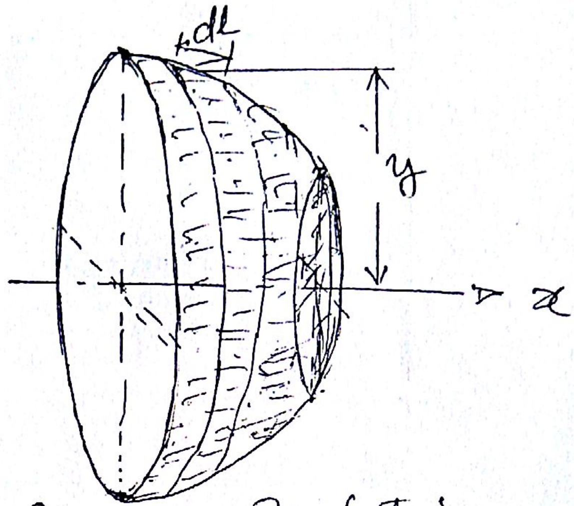
For volumes, the C.G. is similarly determined by

$$\bar{x} = \frac{\int_V x \, dv}{\int dv}$$ etc, if the density is constant

Centre of Gravity of Axi-symmetric Objects, that is, Solids or Surfaces of revolution



Solid of Revolution (volume).



Surface of Revolution

An axi-~~symmetric~~ solid is the volume obtained by the rotation of a plane area about an axis.

An axi-symmetric ~~or~~ surface is the surface obtained by the rotation of a curved line about an axis.

Theorem of Pappus and Gladinus

Theorem 1: It states that 'The area of a surface of revolution is the product of the length of the geometric curve generating the surface, and the distance travelled by the centroid of the curve in generating the same.' provided the generating curve does not cross the axis of revolution.

Theorem 2: 'The volume of a solid of revolution' is the product of the area of the generating section surface multiplied by the distance travelled by the centroid of the area in generating the volume', provided the generating area does not cross the axis of revolution.

Proof: (Theorem 1):

We know that the surface area (dS) of a small length of the curve 'dL' is $dS = 2\pi y\, dL$

Hence, $S = \int dS = \int 2\pi y . dL = 2\pi \int y\, dL$

But, by defination, $\int y\, dL = \bar{y} L$

Hence, $S = 2\pi \bar{y} . L$ (proved)

Theorem 2: We know that

$$dV = 2\pi y \, dA$$

Hence, $V = \int dV = \int 2\pi y \, dA$

$$= 2\pi \int y \, dA$$

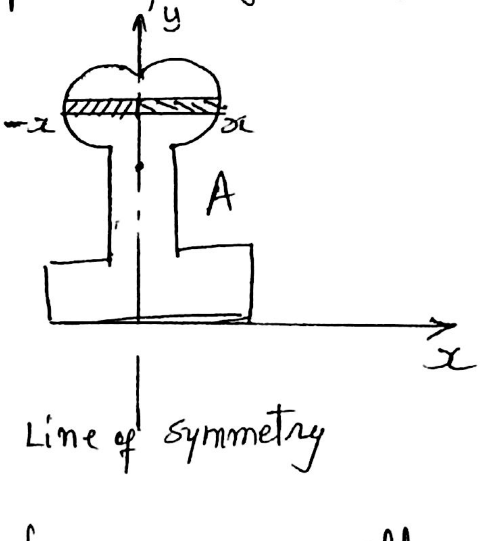
$$= 2\pi \bar{y} . A \quad \text{(by definition)}.$$

Hence, $V = 2\pi \bar{y} A$. proved

Centre of Gravity / Centroid of Symmetric Objects.

Symmetry: If there exists a line for an area such that the area is divided into two by the line in two exactly similar parts, one being the mirror image of the other, the line is said to be the line of symmetry.

Similarly, if a plane divides a volume in two halves such that one is the mirror image of the other, the plane being the mirror, the plane is said to be a plane of symmetry.



We can divide the area A into strips ⊥ to the line of symmetry, that is thin strips parallel to X axis.

For every such strip,

$$\bar{x} = \int_{-x}^{x} x \, dA = \int_{-x}^{x} x . y \, dx .$$

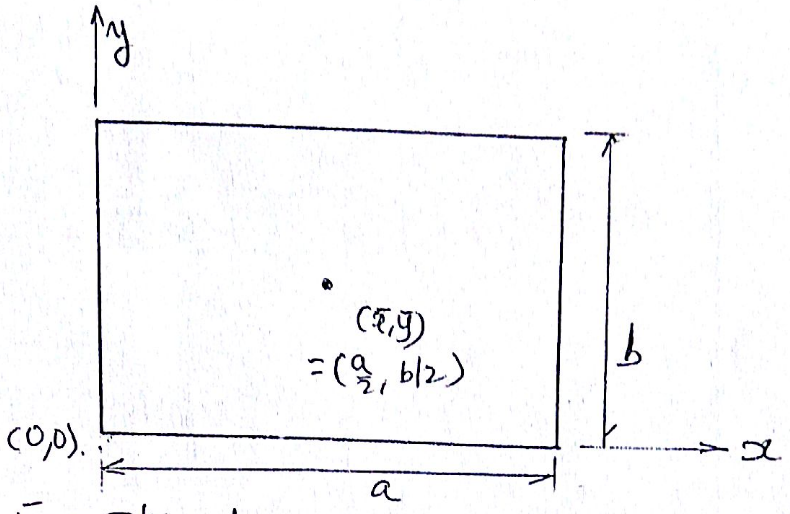
$\int x.y$ being an odd integral $\int_{-x}^{x} xy \, dx = 0$.

Thus, the Centroid for the strip, and by induction, of the whole area lies on Y axis. ($\bar{X} = 0$).

If an area has two axes of symmetry, the centroid lies on the point of intersection of the two lines. ...

CENTROIDS / CENTRES OF GRAVITY OF SIMPLE SHAPES

I) Rectangle :

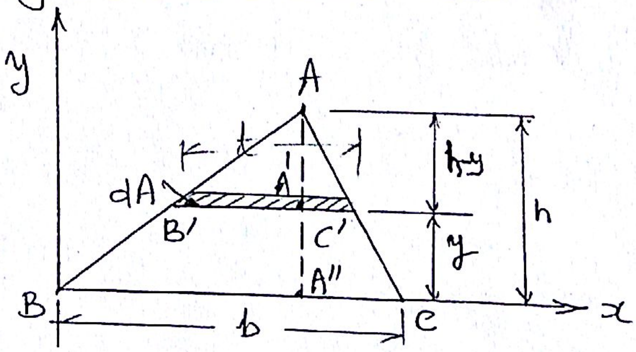


By Symmetry, $\bar{x},\bar{y}$ works out as $(a/2, b/2)$.

II) Circle / Annular Area.

By symmetry, the geometrical centre is the centre of gravity.

III) Triangle w.r. to its base.



For convenience, x axis is chosen as the base of the triangle.
We assume the triangle to comprise of infinitesimal area strips parallel to x axis.

If dA is area of an strip at 'y' from the base,

$$A = \int dA = \int x\,dy \quad (\text{or} \int t \cdot dy)$$

and,

$$\bar{y} = \frac{\int y\,dA}{\int dA} = \frac{\int x_y \cdot y \cdot dy}{\int dA} \quad \cdots\cdots (1)$$

But, since Δ's ABC and $AB'C'$ are similar,

$$\frac{h-y}{h} = \frac{x}{b} \quad \text{.......(2)}$$

or, $x_h = \frac{b}{h}(h-y)$

Hence, $A\bar{y} = \int_0^h y \cdot x \cdot dy = \int_0^h \frac{b}{h}(h-y)\,y\,dy$

$$= \frac{b}{h}\int_0^h (hy - y^2)\,dy$$

or, $A\bar{y} = \frac{b}{h}\left[\frac{hy^2}{2} - \frac{y^3}{3}\right]_0^h = \frac{b}{h}\left[\frac{h^3}{2} - \frac{h^3}{3}\right]$

$$= \frac{1}{6}bh^2 \quad \text{....(3)}$$

and, $A = \int_0^h \frac{b}{h}(h-y)\,dy = \frac{b}{h}\left[hy - \frac{y^2}{2}\right]_0^h$

$$= \frac{b}{h}\left[h^2 - \frac{h^2}{2}\right] = \frac{1}{2}bh \quad \text{...(4)}$$

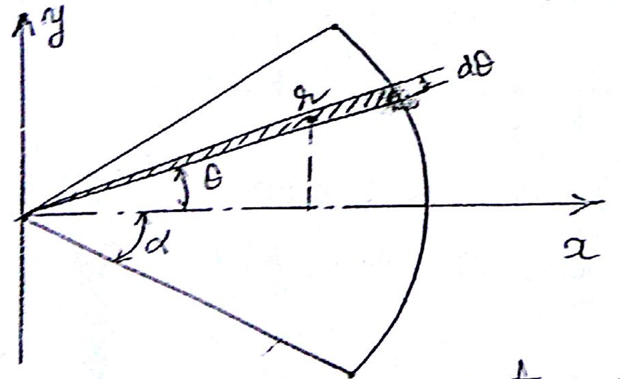
Hence, substituting, in (3)

$$\frac{1}{2}bh.(\bar{y}) = \frac{1}{6}bh^2 \quad \text{.......(4)}$$

or, $\bar{y} = \underline{\frac{1}{3}h}$.

Thus, the centroid of a triangle is $(1/3)^{rd}$ of the perp. heig[ht] from every side (and is hence the median bisector).

IV) Sector of a Circle:



Let us consider a sector of included angle 2α. If we draw the bisector line, due to symmetry, the centroi[d] will lie on this line.

Let us consider an elemental area 'dA' formed by t[he] lines at θ and $\theta + d\theta$ and between the centre and the c[ircle]

The area $dA = \frac{1}{2} r\,d\theta \times r = \frac{r^2}{2}\,d\theta$ ----- (1)

Centroid of this area lies at $2/3^{rd}$ of radius $= 2r/3$.

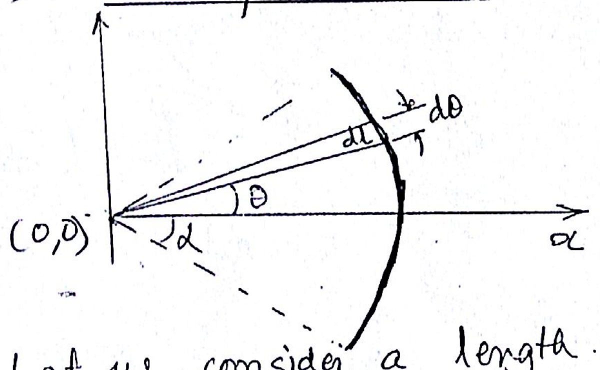
$\therefore\ x = \frac{2}{3} r\cos\theta$, and $\bar{y} = 0$.

$$\text{Hence}\quad \bar{x} = \frac{\int x\,dA}{\int dA} = \frac{\int_{-\alpha}^{\alpha} \frac{2}{3} r\cos\theta\,\frac{1}{2} r^2\,d\theta}{\int_{-\alpha}^{\alpha} \frac{1}{2} r^2\,d\theta}$$

$$\text{or,}\quad \bar{x} = \frac{\frac{1}{3} r^3 \int_{-\alpha}^{\alpha} \cos\theta\,d\theta}{\frac{1}{2} r^2 \int_{-\alpha}^{\alpha} d\theta} = \frac{\frac{1}{3} r^3 [\sin\theta]_{-\alpha}^{\alpha}}{\frac{r^2}{2}[\theta]_{-\alpha}^{\alpha}}$$

$$= \left(\frac{2}{3}\right)\frac{r\sin\alpha}{\alpha} \quad \text{----- (3).}$$

V). Arc of a Circle:



Again, due to symmetry

$\bar{y} = 0$.

Let us consider a length element of the arc at 'θ' and of length dl. ($dl = r\,d\theta$)

The c.g. of this element lies at the centre of the elem

Hence, $x_{cg} = r\cos\theta$

$$\therefore\ \bar{x} = \frac{\int x\,dl}{\int dl} = \frac{\int_{-\alpha}^{\alpha} r\cos\theta \cdot r\,d\theta}{\int_{-\alpha}^{\alpha} r\,d\theta} = \frac{r^2 \int_{-\alpha}^{\alpha} \cos\theta\,d\theta}{r \int_{-\alpha}^{\alpha} d\theta}$$

$$\text{or,}\quad \bar{x} = \frac{r^2 [\sin\theta]_{-\alpha}^{\alpha}}{r[\theta]_{-\alpha}^{\alpha}} = \frac{2r^2 \sin\alpha}{2r\alpha} = \frac{r\sin\alpha}{\alpha} \quad \text{--(4)}$$

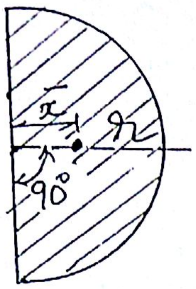
Thus, $\bar{x} = \left(\frac{r\sin\alpha}{\alpha}\right)$.

Important cases of C.G. / Centroid (to be remembered)

CENTROIDS OF CIRCULAR AREAS :

$\alpha = 90^\circ$

$(\pi/2)$



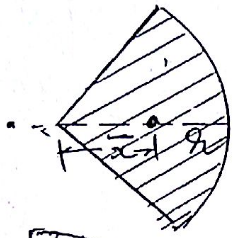
$$\bar{x} = \frac{2}{3}\frac{r\sin\alpha}{\alpha}$$

$$= \frac{2}{3} r \times \frac{1}{\pi/2} = \left(\frac{4r}{3\pi}\right) \quad \text{--- (1)}$$

$$\bar{y} = 0.$$

$\alpha = 45^\circ$

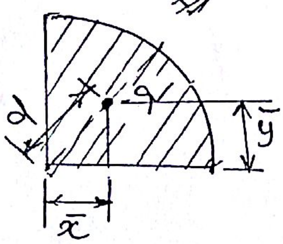
$(\pi/4)$



$$\bar{x} = \frac{2}{3} r\sin\alpha/\alpha$$

$$= \frac{2}{3} r \frac{1/\sqrt{2}}{\pi/4} = \left(\frac{4\sqrt{2}}{3\pi} r\right) \text{--- (2)}$$

$\alpha = 45^\circ$



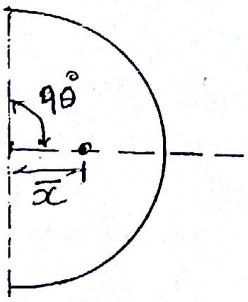
$$d = \frac{2}{3}\frac{r\sin\alpha}{\alpha} = \left(\frac{4\sqrt{2}}{3\pi} r\right).$$

$$\bar{x} = d\sin 45^\circ = \frac{d}{\sqrt{2}} = \left(\frac{4r}{3\pi}\right) \quad \text{--- (3)}$$

$$\bar{y} = \left(\frac{4r}{3\pi}\right) \quad \text{--- (4)}$$

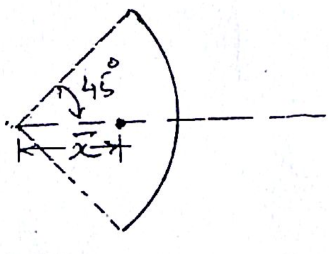
CENROIDS OF ARCS

$\alpha = 90^\circ$



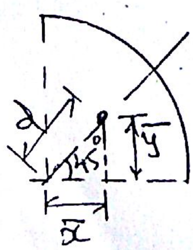
$$\bar{x} = \frac{r\sin\alpha}{\alpha} = \frac{r\cdot 1}{\pi/\pi} = \left(\frac{2r}{\pi}\right) \quad \text{--- (5)}$$

$\alpha = 45^\circ$,



$$\bar{x} = \frac{r\sin\alpha}{\alpha} = \frac{r\cdot 1/\sqrt{2}}{\pi/4} = \left(\frac{2\sqrt{2}}{\pi} r\right) \quad \text{--- (6)}$$

$\alpha = 45^\circ$

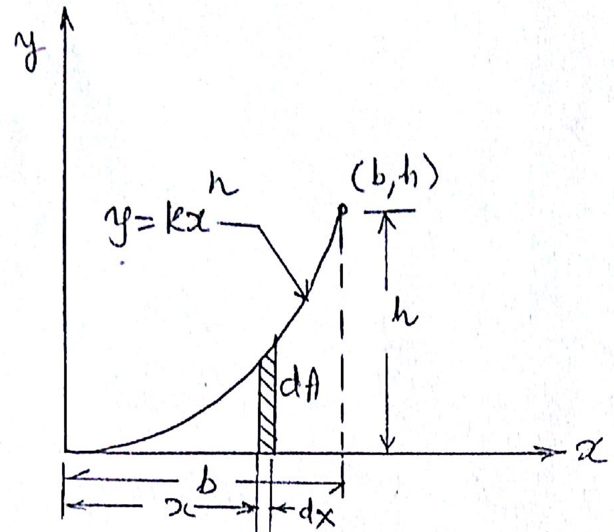


$$d = \frac{r\sin\alpha}{\alpha} = \frac{r\sin\alpha}{\alpha} = \frac{2\sqrt{2}}{\pi} r$$

$$\bar{x} = \frac{d}{\sin 45^\circ} = \left(\frac{2r}{\pi}\right) \quad \text{--- (7)}$$

$$\bar{y} = \bar{x} = \left(\frac{2r}{\pi}\right) \quad \text{--- (8)}$$

VI n^{th} order parabola: (17-A)



Let the equation of the n^{th} order parabola be

$$y = kx^n \quad \text{------}(1)$$

We assume the parabola to be divided into infinitesimal strips parallel to the y axis.

Then, Area of the parabola $= A = \int_0^b y\,dx$

and, $\bar{x} = \dfrac{\int_0^b x \cdot y\,dx}{\int_0^b y\,dx}$, and, $\bar{y} = \dfrac{\int_0^b (\frac{1}{2}y)\cdot y\,dx}{\int_0^b y\,dx}$ --- (2)

Now, $A = \int_0^b y\,dx = \int_0^b kx^n \cdot dx = k\left[\dfrac{x^{n+1}}{n+1}\right]_0^b = \dfrac{kb^{n+1}}{n+1} = \dfrac{k.b^n.b}{n+1}$

$$\therefore A = \frac{b.h.}{n+1} \quad \text{----}(3)$$

and, $\int_0^b x \cdot y\,dx = \int_0^b x.k.x^n dx = k\left[\dfrac{x^{n+2}}{n+2}\right]_0^b$

$= \dfrac{k\,b^{n+2}}{n+2} = \dfrac{b^2 h}{(n+2)}$ -- (4)

and $\int_0^b (\frac{1}{2}y).y.dx = \int_0^b \frac{1}{2}.kx^n.kx^n.dx$

$= \dfrac{k^2}{2}\int x^{2n} dx = \dfrac{k^2}{2}\left[\dfrac{x^{2n+1}}{2n+1}\right]_0^b = \dfrac{k^2\,b.b^{2n}}{2(2n+1)} = \dfrac{h^2 b}{(4n+2)}$ (5)

Hence, substituting (3) and (4) in (1) --- (6)

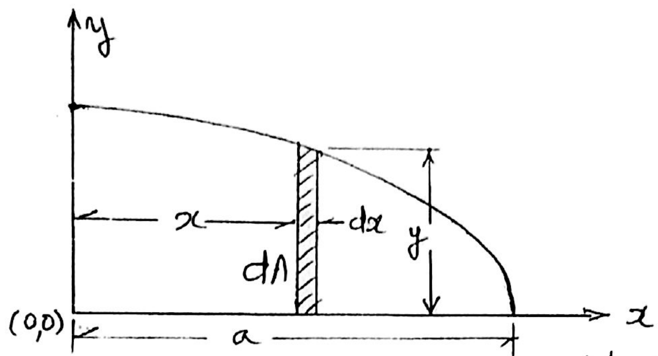
$\bar{x} = \dfrac{b^2.h.}{(n+2)} \times \dfrac{n+1}{(b h)} = \left(\dfrac{n+1}{n+2}\right).b$

$\text{-----}(7)$

and, $\bar{y} = \dfrac{h^2 b}{(4n+2)} \times \dfrac{n+1}{b.h} = \dfrac{n+1}{(4n+2)}.h$ -- (8)

Thus, the coordinates of C.G. are $\left(\dfrac{n+1}{n+2}.b, \dfrac{n+1}{4n+2}.h\right)$

VII) Quarter of an ellipse:



The ellipse of semi-axes 'a' and 'b' is shown.

We have, $(x^2/a^2) + (y^2/b^2) = 1$ as the governing equation.

An area element 'dA' at 'x' along y axis is given by

$$dA = y.dx. \qquad \because y^2 = b^2\left(1 - \frac{x^2}{a^2}\right)$$

Hence, by definition, $\Rightarrow y = \frac{b}{a}\sqrt{a^2-x^2}$

$$\bar{x} = \int_A x_i\, dA \Big/ \int_A dA, \quad \bar{y} = \int_A y_i\, dA \Big/ \int_A dA.$$

Here, $x_i = x$, and, $y_i = y/2$

Hence, $\int_A x_i\, dA = \int x.y\, dx = \int_0^a x.\left(\frac{b}{a}\right)\sqrt{a^2-x^2}.\, dx.$ --- (i)

$\int_A y_i\, dA = \frac{1}{2}\int_0^a \left(\frac{b}{a}\right)\sqrt{a^2-x^2}\left(\frac{b}{a}\right)\sqrt{a^2-x^2}.\, dx$

$= \frac{1}{2}\left(\frac{b}{a}\right)^2 \int_0^a (a^2-x^2)dx$ ------ (ii)

and, $A = \int_A y\, dx = \int_0^a \left(\frac{b}{a}\right)\sqrt{a^2-x^2}.\, dx = \frac{b}{a}\int_0^a \sqrt{a^2-x^2}\, dx$ --- (iii)

Now, $\left(\frac{b}{a}\right)\int_0^a \sqrt{a^2-x^2}\, dx$ $\because$ Let $x = a\cos\theta$, $dx = a\sin\theta\, d\theta$.

$\therefore I_A = \left(\frac{b}{a}\right)\int_{\pi/2} a\sin\theta.(-a\sin\theta)d\theta = +\frac{ba^2}{a}\int_0^{\pi/2} \sin^2\theta.\, d\theta$

$= \left(\frac{b}{a}\right)a^2\int_0^{\pi/2}\left(\frac{1-\cos 2\theta}{2}\right)d\theta = \frac{ba^2}{2a}\left[\theta - \frac{\sin 2\theta}{2}\right]_0^{\pi/2}$

$= \frac{ba^2}{2a}.\left[\left(\frac{\pi}{2} - 0\right) - (0 - 0)\right]$

$= \frac{\pi}{4}ab$ -------- (4)

$$I_{Ax} = \frac{1}{2}\left(\frac{b}{a}\right)^2 \int_0^a (a^2-x^2)\,dx = \frac{b^2}{2a^2}\left[a^2x - \frac{x^3}{3}\right]_0^a$$

$$= \frac{b^2}{2a^2} \cdot \frac{2}{3}a^3 = \frac{ab^2}{3} \quad \cdots\cdots (5)$$

and, $$I_{Ax} = \frac{b}{a}\int_0^a \sqrt{a^2-x^2}\,.\,x\,dx.$$

Let $x = a\cos\theta$, $\Rightarrow$ $dx = -a\sin\theta\,d\theta$

$$\therefore I_{Ax} = \frac{b}{a}\int_{\pi/2}^{0} a\sin\theta \,.\, a\cos\theta \,.\,(-a\sin\theta\,d\theta)$$

$$= -a^2b\int_{\pi/2}^{0} \sin^2\theta\,.\cos\theta\,d\theta$$

$$= a^2b\int_0^{\pi/2} \sin^2\theta\,.\cos\theta\,.\,d\theta$$

$$= a^2b\,.\,\frac{\Gamma\frac{2+1}{2}\,.\,\Gamma\frac{1+1}{2}}{2\,\Gamma\frac{2+1+2}{2}} = a^2b\,.\,\frac{\Gamma\frac{3}{2}\;\;\Gamma 1}{2\,\Gamma\frac{5}{2}}$$

$$= a^2b\,\frac{\frac{1}{2}\,.\sqrt{\pi}\,.\,1}{2\,.\,\frac{3}{2}\,.\,\frac{1}{2}\,.\,\sqrt{\pi}} = \frac{a^2b}{3} \quad \cdots (6)$$

Hence, $\bar{y} = \frac{ab^2}{3} \div \frac{\pi}{4}ab = \frac{4}{3\pi}b,$

and $\bar{x} = \frac{a^2b}{3} \div \frac{\pi}{4}ab = \frac{4}{3\pi}a$ } Ans.

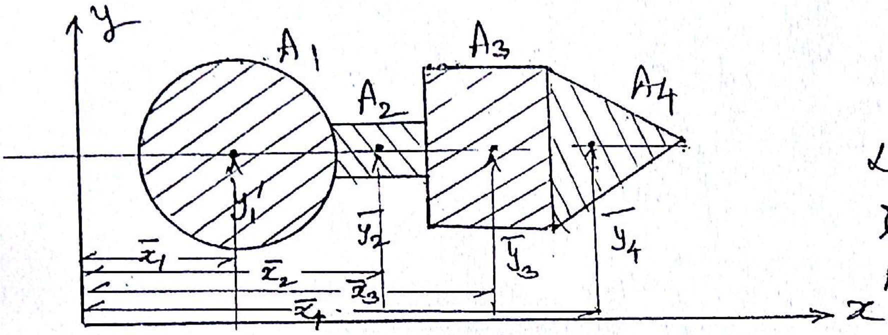
Important:-

$$\int_0^{\pi/2} \cos^n x\,.\sin^m x\,.\,dx = \frac{\Gamma\frac{m+1}{2}\;\;\Gamma\frac{n+1}{2}}{2\,\Gamma\frac{m+n+2}{2}}$$

$\Gamma 1 = 1$, $\Gamma\frac{1}{2} = (\sqrt{\pi})$, $\Gamma n = (n-1)\,\Gamma(n-1)$

C.G. / CENTROID OF COMPOSITE PARTS:

When it is required to find out the C.G. of a solid / centroid of an area or a line, it is done by breaking the entity in simple parts whose centroid / C.G. can be easily found. From that, the composite C.G. / centroid can be obtained as follows.



I) Centroid of a composite Area A

Let the area be divided into simple parts A_1, A_2, A_3.. who centroids are known.

The C.G. of the composite is given by

$$\bar{x}A = \int_A x \cdot dA = \int_{A_1} x\,dA + \int_{A_2} x\,dA + \cdots \int_{A_n} x\,dA \text{ , and}$$

$$\bar{y}A = \int_A y \cdot dA = \int_{A_1} y\,dA + \int_{A_2} y\,dA + \cdots \int_{A_n} y\,dA .$$

But, we know, $\bar{x}_1 = \dfrac{\int_{A_1} x\,dA}{A_1}$, $\Rightarrow \int_{A_1} x\,dA = \bar{x}_1 A_1$

and $\bar{x}_i A_i = \int_{A_i} x\,dA$.

Similarly, $\bar{y}_i A_i = \int_{A_i} y\,dA$.

$\therefore$ Substituting in — for $\int_{A_1} x\,dA$, $\int_{A_2} x\,dA$ .. etc.

$$\bar{x}A = A_1\bar{x}_1 + A_2\bar{x}_2 + \cdots A_n\bar{x}_n$$

and $\bar{y}A = A_1\bar{y}_1 + A_2\bar{y}_2 + \cdots A_n\bar{y}_n$

$$\rightarrow \bar{x} = (A_1\bar{x}_1 + A_2\bar{x}_2 + \cdots A_n\bar{x}_n)/A = \frac{\Sigma(A_i x_i)}{A}$$

and $\bar{y} = (A_1\bar{y}_1 + A_2\bar{y}_2 + \cdots A_n\bar{y}_n)/A = \dfrac{\Sigma_i(A_i y_i)}{A}$

The same principle can be extended to the evaluation of C.G./ centroid of lengths, volumes.

For lengths $L_1, L_2, \ldots L_n$ forming the total length L,

$$\bar{x} = \sum_{i=1}^{n} (L_i x_i)/L$$

$$\bar{y} = \sum_{i=1}^{n} (L_i y_i)/L$$

C.G. of Composite Solids.

If density is assumed constant, for a solid

$$\bar{x} = \frac{\int x\,dv}{V}, \quad \bar{y} = \frac{\int y\,dv}{V}, \quad \bar{z} = \frac{\int z\,dv}{V}$$

If a solid is composed of simple parts, V_1, V_2, ... etc.

$$\bar{x} = \sum (V_i x_i)/V,$$

$$\bar{y} = \sum (V_i y_i)/V, \text{ and}$$

$$\bar{z} = \sum (V_i z_i)/V$$

If the density is not constant and varies,

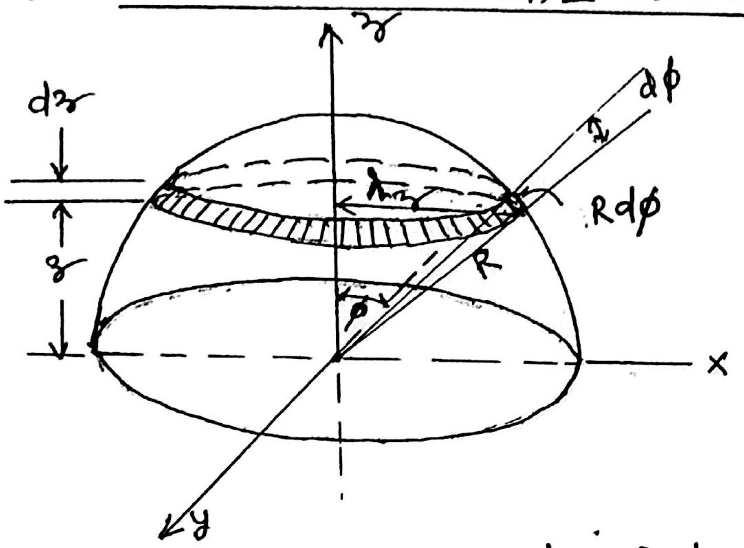
$$\bar{x} = \sum (V_i x_i \rho_i)/W \quad \text{etc}$$

$$\bar{y} = \sum (V_i y_i \rho_i)/W, \text{ and}$$

$$\bar{z} = \sum (V_i z_i \rho_i)/W$$

CENTRE OF GRAVITY OF SOLIDS:

I) A HEMESPHERICAL SOLID.



Let us consider a hemispherical solid with its flat base coinciding with the X-Y plane with origin at its centre.

By symmetry, the X and Y coordinates of the hemisphere are zero ($\bar{X} = \bar{Y} = 0$). Only '$\bar{Z}$' needs evaluation.

Let the hemisphere be considered to be a set of a large number of disks of varying radius, stacked one on top of the other.

Considering one such disk between heights 'z' & 'z+dz',

The volume of the disk element $= \pi r_z^2 dz$

and the z coordinate of its c.g $= z + \frac{dz}{2} = $ 'z', neglecting 'dz'.

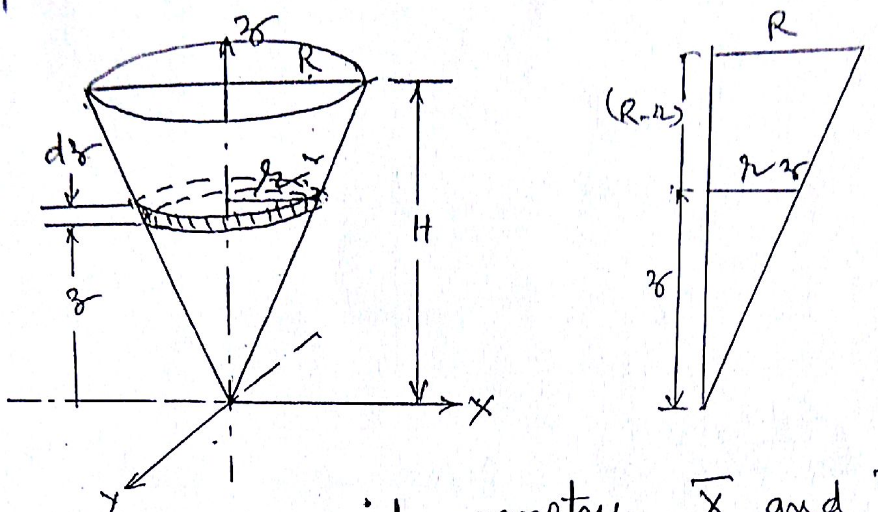
$$\therefore \bar{z} = \frac{\int z\,dV}{\int dV} = \frac{\int \pi r_z^2 . dz . 'z'}{\frac{2}{3}\pi R^3} \quad \text{--- (1)}$$

But, from trigonometry, $R^2 - z^2 = r_z^2$

$$\therefore \bar{z} = \frac{\int_0^R \pi (R^2 - z^2) z\,dz}{\frac{2}{3}\pi R^3} = \frac{\pi \left[\frac{R^2 z^2}{2} - \frac{z^4}{4}\right]_0^R}{\frac{2}{3}\pi R^3} = \frac{3}{8}R. \quad \text{--- (2)}$$

iii) A RIGHT CIRCULAR CONE :

Let the 'z' axis coincide with the geometrical axis of the solid cone and its apex the origin of coordinates



With the aforesaid geometry, $\bar{X}$ and $\bar{Y}$ are zero by symmetry considerations and only $\bar{Z}$ needs determination. The cone is assumed to comprise of infinitesimally small disks stacked one over the other.

Considering a disk at height z,

vol. of disk $= \pi r_z^2 \, dz$ --- (1)

$$\Rightarrow \bar{z} = \frac{\int z\,dv}{\int dv} = \frac{\int \pi r_z^2 \cdot dz \cdot z}{\left(\frac{1}{3}\pi R^2 H\right)} \quad \text{--- (2)}$$

But, from trigonometry of the cone geometry,

$$\frac{r_z}{R} = \frac{z}{H} \rightarrow r_z^2 = z^2\left(\frac{R^2}{H^2}\right),$$

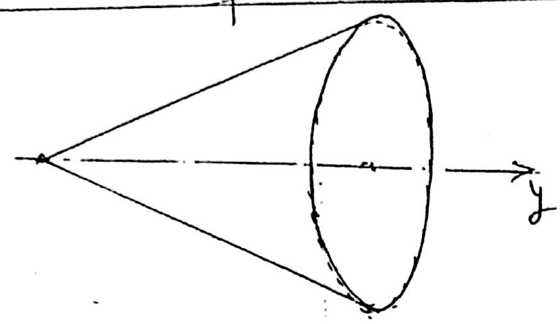
Hence, $\bar{z} = \int_0 \pi z^2\left(\frac{R^2}{H^2}\right) z\,dz \Big/ \frac{1}{3}\pi R^2 H$

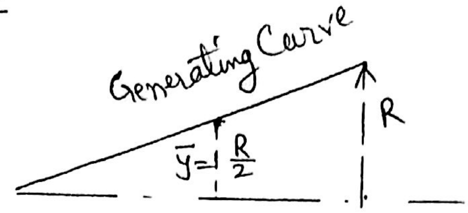
$$= \pi \cdot \frac{R^2}{H^2}\left\{\int_0^R z^3 dz\right\} \Big/ \frac{1}{3}\pi R^2 H$$

$$= \frac{3}{H^3}\left[\frac{z^4}{4}\right]_0^R = \frac{3H}{4} \quad \text{--- (3)}$$

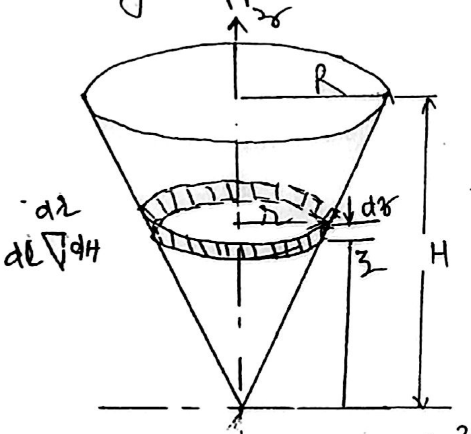
Thus, the C.G. is $\frac{3}{4}H$ from the apex and $\frac{1}{4}H$ from the base of the cone.

Centroid of a Conical Shell:





Using Pappus's Theorem. $y = \frac{R}{2}$, $S = 2\pi \cdot \left(\frac{R}{2}\right) \cdot L = \pi R L$



For a ring at height z,

$\frac{r}{R} = \frac{z}{H}$, $\Rightarrow z = \left(\frac{H}{R}\right) r$

$dz = \left(\frac{H}{R}\right) dr$

Now, $z = \dfrac{\int z\, ds}{\int ds}$

and $ds = 2\pi r\, dl$

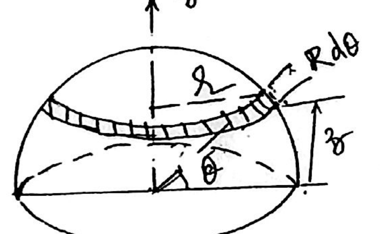
But $dl^2 = dr^2 + dz^2$, $\Rightarrow dl^2 = dr^2 \left\{ 1 + \left(\frac{H}{R}\right)^2 \right\}$

or, $dl = \sqrt{1 + \left(\frac{H}{R}\right)^2}\, dr = K\, dr$ where $K = \sqrt{1 + \frac{H^2}{R^2}}$

$$\therefore \bar{z} = \frac{\int z\, ds}{\int ds} = \frac{\int_0^R 2\pi r\, K\, dr \left(\frac{H}{R}\right) r}{\int_0^R 2\pi r \cdot K\, dr} = \frac{\frac{2\pi K H}{R} \int_0^R r^2\, dr}{2\pi K \int_0^R r\, dr}$$

$$\text{or, } \bar{z} = \frac{\frac{2\pi K H}{R}\left(R^3/3\right)}{2\pi K \left(R^2/2\right)} = \underline{\frac{2}{3} H}$$

Centroid of a Spherical Shell:



For a spherical shell. $z = R \sin\theta$.

$\bar{z} = \dfrac{\int z \cdot ds}{\int ds}$, but $ds = 2\pi r \cdot R\, d\theta$
$= 2\pi R \cos\theta \cdot R\, d\theta$

$$\therefore \bar{z} = \frac{\int_0^{\pi/2} 2\pi R^2 \cos\theta \cdot (R \sin\theta)\, d\theta}{\int_0^{\pi/2} 2\pi R^2 \cos\theta\, d\theta} = \frac{\pi R^3 \int_0^{\pi/2} \sin 2\theta\, d\theta}{2\pi R^2 \int_0^{\pi/2} \cos\theta\, d\theta} = \frac{R}{2} \frac{\left[-\frac{\cos 2\theta}{2}\right]_0^{\pi/2}}{\left[\sin\theta\right]_0^{\pi/2}}$$

$= \frac{R}{2}$ <u>Ans</u>