UNIT-5

LAPLACE TRANSFORMS AND Z-TRANSFORMS

LAPLACE TRANSFORMS:

28/10 1, 2, 6, 10, 11, 12, 14, 20, 21, 23, 25, 26, 28, 33, 35, 38, 41, 42, 48, 50, 55,

→ Laplace transform represents continuous time signals in terms of complex exponentials i.e e^{-st}. It is used to analyze the signals or functions which are not absolutely integrable.

→ More effectively continuous time signals can be analyzed using Laplace transform.

→ Laplace transform provides broader characterization compared to Fourier Transform

DEFINITION:

→ To transform a time domain signal x(t) to S-domain, multiply the signal by e^{-st} and then integrate from $-\infty$ to ∞.

→ The transformed signal is represented as X(S) and transformation is denoted by letter $\mathcal{L}$

Laplace transform is given as for continuous time signal x(t)

i.e

$$\boxed{X(S) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt} \rightarrow ①$$

where 'S' is complex in nature and given as $S = \sigma + j\omega$.

realpart / attenuation constant

imaginary / complex frequency

→ If x(t) is defined for $t \geq 0$ (i.e x(t) is causal) then

$$\boxed{\mathcal{L}\{x(t)\} = X(S) = \int_{0}^{\infty} x(t)\, e^{-st}\, dt} \rightarrow ②$$

TYPES OF LAPLACE TRANSFORM:

(i) Bilateral or two sided Laplace transform: If the integration is taken from $-\infty$ to ∞ as shown in eq(1) then it is called Bilateral L.T

(ii) Unilateral or one sided Laplace transform: If the integration is taken from 0 to ∞ as shown in eq(2) then it is called Unilateral LT

→ Useful in analysis of networks and solving differential equations

INVERSE LAPLACE TRANSFORM:

→ The s-domain signal X(s) can be transformed to time domain signal x(t) by using inverse laplace transform and is defined as

$$\boxed{L^{-1}[X(s)] = x(t) = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} X(s)e^{st}\,ds}$$

→ The signal x(t) and X(s) are called Laplace transform pair

$$x(t) \underset{L^{-1}}{\overset{L}{\rightleftarrows}} X(s)$$

RELATION BETWEEN FOURIER TRANSFORM AND LAPLACE TRANSFORM:

Fourier transform is given as

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)\,e^{-j\omega t}\,dt \rightarrow (1)$$

→ FT can be calculated only if x(t) is absolutely integrable i.e

$$\int_{-\infty}^{\infty} |x(t)|\,dt < \infty \longrightarrow (2)$$

→ Laplace transform is written as

$$X(s) = \int_{-\infty}^{\infty} x(t)\,e^{-(\sigma+j\omega)t}\,dt \quad \text{putting } s=\sigma+j\omega$$

$$= \int_{-\infty}^{\infty} x(t)\,e^{-\sigma t}\cdot e^{-j\omega t}\,dt$$

$$= \left\{\int_{-\infty}^{\infty} x(t)\,e^{-\sigma t}\right\} e^{-j\omega t}\,dt \rightarrow (3)$$

comparing eq(3) with eq(1), laplace transform of x(t) is basically fourier transform of $x(t)e^{-\sigma t}$.

→ If $\sigma=0$, then above equation i.e $s=j\omega$ the above eqn

$$X(s) = \int_{-\infty}^{\infty} x(t)\,e^{-j\omega t}\,dt = X(j\omega) \text{ when } s=j\omega$$

→ It is basically fourier transform on imaginary (jω) axis in s-plane.

CONVERGENCE / REGION OF CONVERGENCE (ROC):

→ From eqn $\left\{ \int_{-\infty}^{\infty} (x(t) e^{-\sigma t}) e^{-j\omega t} dt \right\}$ we know that Laplace transform is basically the fourier transform of $x(t) e^{-\sigma t}$. If fourier transform of $x(t) e^{-\sigma t}$ exists then laplace transform of $x(t)$ exists.

→ $\int_{-\infty}^{\infty} |x(t) e^{-\sigma t}| dt < \infty$ must be absolutely integrable for fourier transform to exists.

→ Laplace transform of $x(t)$ will exist, if above condition is satisfied.

→ The range of values of 'σ' for which laplace transform converges is called ROC or region of convergence.

(Or)

→ The Laplace transform of a signal given by $\int_{-\infty}^{\infty} x(t) e^{-st} dt$. The values of '$s$' for which the integral $\int_{-\infty}^{\infty} x(t) e^{-st} dt$ converges is called ROC.

PROBLEMS:

(1) Calculate the Laplace transform and ROC for $x(t) = e^{-at} u(t)$ (Right sided causal sgl)

Sol

$x(t) = e^{-at} u(t)$ where $a > 0$

$= e^{-at}$ for $t \geq 0$

$$L[x(t)] = X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt = \int_{-\infty}^{\infty} e^{-at} \cdot u(t) e^{-st} dt$$

$$= \int_{0}^{\infty} e^{-at} e^{-st} dt = \int_{0}^{\infty} e^{-(s+a)t} dt$$

$$= \frac{1}{-(s+a)} \left[e^{-(s+a)t} \right]_0^{\infty} = \frac{-1}{s+a} [0 - 1]$$

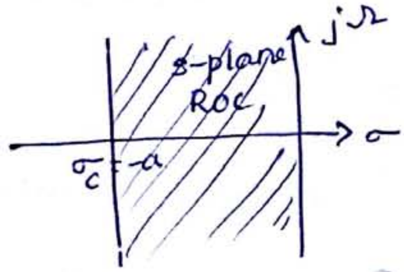
$$= \frac{1}{s+a}$$

ROC: $s > -a$

$$= -\left[\frac{e^{-(s+a)t}}{s+a}\right]_0^\infty$$

$$= -\left\{\lim_{t\to\infty}\left[\frac{e^{-(s+a)t}}{s+a}\right] - \lim_{t\to 0}\left[\frac{e^{-(s+a)t}}{s+a}\right]\right\}$$

$$X(s) = +\left[\frac{e^{-(s+a)\infty}}{-(s+a)} - \frac{e^{-(s+a)0}}{-(s+a)}\right].$$



$S = j\omega + \sigma$

$$L\{x(t)\} = \frac{e^{-K\times\infty}e^{-j\Omega\times\infty}}{s+a} + \frac{1}{s+a}$$

where $K = \sigma + a = \sigma - (-a)$.

when $\sigma > a$, $k = \sigma - (-a)$ = positive $\therefore e^{-K\infty} = e^{-\infty} = 0$

When $\sigma < -a$, $k = \sigma - (-a)$ = Negative $\therefore e^{-K\infty} = e^{+\infty} = \infty$

$\therefore$ $X(s)$ converges when $\sigma > -a$ and does not converge for $\sigma < -a$.

$\therefore$ When $\sigma > -a$, the $X(s)$ is given by

$$L\{x(t)\} = X(s) = \frac{-e^{-K\times\infty}e^{-j\Omega\times\infty}}{s+a} + \frac{1}{s+a} = \frac{-0\times e^{-j\Omega\times\infty}}{s+a} + \frac{1}{s+a} = \frac{1}{s+a}$$

2)

Properties of L.T :

(1) Amplitude Scaling

If $L[x(t)] = X(s)$ then $L[A\,x(t)] = A\,X(s)$

Proof:

$$X(s) = L\{x(t)\} = \int_{-\infty}^{\infty} x(t)\cdot e^{-st}dt$$

$$L\{A\,x(t)\} = \int_{-\infty}^{\infty} A\cdot x(t)e^{-st}dt = A\int_{-\infty}^{\infty} x(t)e^{-st}dt$$

$$= A\,X(s).$$

(2) Linearity:

If $L\{x_1(t)\} = X_1(s)$ and $L\{x_2(t)\} = X_2(s)$ then $L\{a_1x_1(t) + a_2x_2(t)\} = a_1X(s) + a_2X_2(s)$

proof

$$X_1(s) = L\{x_1(t)\} = \int_{-\infty}^{\infty} x_1(t)e^{-st}dt$$

$$X_2(s) = L\{x_2(t)\} = \int_{-\infty}^{\infty} x_2(t)e^{-st}dt$$

$$L\{a_1x_1(t) + a_2x_2(t)\} = \int_{-\infty}^{\infty} [a_1x_1(t) + a_2x_2(t)]e^{-st}dt$$

$$= a_1\int_{-\infty}^{\infty} x_1(t)e^{-st}dt + a_2\int_{-\infty}^{\infty} x_2(t)e^{-st}dt$$

$$= a_1X_1(s) + a_2X_2(s)$$

3) Time differentiation:

If $L\{x(t)\} = X(s)$ then $L\left\{\frac{d}{dt}x(t)\right\} = sX(s) - x(0)$; where $x(0)$ is value of $x(t)$ at $t=0$

proof

$$X(s) = L\{x(t)\} = \int_0^{\infty} x(t)e^{-st}dt$$

$$\therefore L\left\{\frac{d}{dt}x(t)\right\} = \int_0^{\infty}\frac{dx(t)}{dt}e^{-st} = \int_0^{\infty} e^{-st}\frac{dx(t)}{dt}\cdot dt$$

$$\left\{ \because \int uv = u\int v - \int\left[du\int v\right] \right.$$

$$u = e^{-st} \;\Big|\; v = \frac{dx(t)}{dt}$$

$$= \left[e^{-st}\cdot x(t)\right]_0^{\infty} - \int_0^{\infty} -se^{-st}x(t)\,dt$$

$$= e^{-\infty}x(\infty) - e^{0}x(0) + s\int_0^{\infty} x(t)e^{-st}dt$$

$$= s\int_0^{\infty} x(t)e^{-st}dt - x(0) = sX(s) - x(0)$$

(4) Time Integration:

If $L[x(t)] = X(s)$ then $L\{\int x(t)\,dt\} = \frac{X(s)}{s} + \frac{[\int x(t)\,dt]_{t=0}}{s}$

proof

$$X(s) = L\{x(t)\} = \int_0^{\infty} x(t) e^{-st} dt$$

$$L\{\int x(t)\,dt\} = \int_0^{\infty} \left[\int x(t)\,dt\right] e^{-st} dt$$

$$= \left[\left[\int x(t)\,dt\right] \frac{e^{-st}}{-s}\right]_0^{\infty} - \int_0^{\infty} x(t) \cdot \frac{e^{-st}}{-s}\,dt$$

$$= \left[\int x(t)\,dt\right]\Big|_{t=\infty} \frac{e^{-\infty}}{-s} - \left[\int x(t)\,dt\right]\Big|_{t=0} \frac{e^{0}}{-s} + \frac{1}{s}\int_0^{\infty} x(t) e^{-st} dt$$

$$= \frac{1}{s}\left[\int x(t)\,dt\right]\Big|_{t=0} + \frac{1}{s}\int_0^{\infty} x(t) e^{-st} dt$$

$$= \frac{X(s)}{s} + \frac{\left[\int x(t)\,dt\right]\Big|_{t=0}}{s}$$

5) Frequency shifting:

If $L[x(t)] = X(s)$ then $L\{e^{\pm at} x(t)] = X(s \mp a)$

proof

$$X(s) = L\{x(t)\} = \int_{-\infty}^{\infty} x(t) e^{-st} dt$$

$$\therefore L\{e^{\pm at} x(t)\} = \int_{-\infty}^{\infty} e^{\pm at} x(t) \cdot e^{-st} dt$$

$$= \int_{-\infty}^{\infty} x(t) \cdot e^{-(s \mp a)t} dt$$

$$= X(s \mp a).$$

6) Time shifting

If $L\{x(t)\} = X(s)$ then $L\{x(t \pm a)\} = e^{\pm as} X(s)$

Proof:

$$X(s) = L\{x(t)\} = \int_{-\infty}^{\infty} x(t)e^{-st}dt$$

$$L\{x(t\pm a)\} = \int_{-\infty}^{\infty} x(t\pm a)e^{-st}dt = \int_{-\infty}^{\infty} x(\tau)e^{-s(\tau\mp a)}d\tau$$

put $t\pm a = \tau$
$\therefore t = \tau \mp a$
$dt = d\tau$

$$= \int_{-\infty}^{\infty} x(\tau)e^{-s\tau} \times e^{\pm as}d\tau = e^{\pm as}\int_{-\infty}^{\infty} x(\tau)e^{-s\tau}d\tau$$

$$= e^{\pm as}\int_{-\infty}^{\infty} x(t)e^{-st}dt = e^{\pm as}(X(s))$$

(7) Frequency differentiation:

If $L\{x(t)\} = X(s)$ then $L\{t\,x(t)\} = -\frac{d}{ds}X(s)$

Proof

$$X(s) = L\{x(t)\} = \int_{-\infty}^{\infty} x(t)e^{-st}dt$$

$$\frac{d}{ds}X(s) = \frac{d}{ds}\int_{-\infty}^{\infty} x(t)e^{-st}dt$$

$$= \int_{-\infty}^{\infty} x(t)\left(\frac{d}{ds}e^{-st}\right)dt = \int_{-\infty}^{\infty} x(t)(-te^{-st})dt$$

$$= \int_{-\infty}^{\infty} (-t\,x(t))\,e^{-st}dt = L\{-t\,x(t)\} = -L\{t\,x(t)\}$$

$$\therefore L\{t \times x(t)\} = -\frac{d}{ds}X(s).$$

(8) Frequency Integration:

If $L\{x(t)\} = X(s)$ then $L\{\frac{1}{t}x(t)\} = \int_s^{\infty} X(s)\,ds.$

Proof

$$X(s) = L\{x(t)\} = \int_{-\infty}^{\infty} x(t)e^{-st}dt$$

On integrating above eqn w.r.to s b/w limits s to ∞

$$\int_s^{\infty} X(s)\,ds = \int_s^{\infty}\left[\int_{-\infty}^{\infty} x(t)e^{-st}dt\right]ds = \int_{-\infty}^{\infty} x(t)\left[\int_s^{\infty} e^{-st}ds\right]dt$$

$$= \int_{-\infty}^{\infty} x(t)\left[\frac{e^{-st}}{-t}\right]_s^{\infty}dt = \int_{-\infty}^{\infty} x(t)\left[\frac{e^{-\infty}}{-t} - \frac{e^{-st}}{-t}\right]dt$$

$$= \int_{-\infty}^{\infty} x(t)\left[0 + \frac{e^{-st}}{t}\right]dt = \int_{-\infty}^{\infty}\left[\frac{1}{t}x(t)\right]e^{-st}dt = L\left\{\frac{1}{t}x(t)\right\}.$$

9) Time Scaling:

If $L[x(t)] = X(s)$ then $L\{x(at)\} = \frac{1}{|a|} X\left(\frac{s}{a}\right)$

Proof $X(s) = L\{x(t)\} = \int_{-\infty}^{\infty} x(t) e^{-st} dt$

$\therefore L\{x(at)\} = \int_{-\infty}^{\infty} x(at) e^{-st} dt = \int_{-\infty}^{\infty} x(\tau) e^{-s\left(\frac{\tau}{a}\right)} \frac{d\tau}{a}$ $\left\{ \begin{array}{l} \text{put } at = \tau \\ \therefore dt = \frac{d\tau}{a} \end{array} \right.$

$= \frac{1}{a} \int_{-\infty}^{\infty} x(\tau) e^{-\left(\frac{s}{a}\right)\tau} d\tau$

$= \frac{1}{a} X\left(\frac{s}{a}\right)$

The above transform is applicable for positive values of 'a'.

If 'a' happens to be negative

$L\{x(at)\} = \frac{-1}{a} X\left(\frac{s}{a}\right)$

In general, $L\{x(at)\} = \frac{1}{|a|} X\left(\frac{s}{a}\right)$.

(10) Periodicity:

If $x(t) = x(t+nT)$ and $x_1(t)$ be one period of $x(t)$ and $L\{x_1(t)\} = \int_0^T x_1(t) e^{-st} dt$

then $L\{x(t+nT)\} = \frac{1}{1-e^{-sT}} \int_0^T x_1(t) e^{-st} dt$

Proof $L\{x(t+nT)\} = \int_0^{\infty} x(t+nT) e^{-st} dt$

$= \int_0^T x_1(t) e^{-st} dt + \int_T^{2T} x_1(t-T) e^{-s(t+T)} dt + \int_{2T}^{3T} x_1(t-2T) e^{-s(t+2T)} dt + \cdots$

$+ \cdots + \int_{PT}^{(P+1)T} x_1(t-PT) e^{-s(t+PT)} dt + \cdots$

$= \sum_{P=0}^{\infty} \int_{PT}^{(P+1)T} x_1(t-PT) e^{-s(t+PT)} dt$

$= \sum_{P=0}^{\infty} \int_0^T x_1(t) e^{-st} \cdot e^{-PsT} dt = \int_0^T x_1(t) e^{-st} \left(\sum_{P=0}^{\infty} e^{-PsT}\right) dt$

$= \int_0^T x_1(t) e^{-st} \left(\sum_{P=0}^{\infty} e^{-sT}\right)^P dt$

$= \int_0^T x_1(t) e^{-st} \left(\frac{1}{1-e^{-sT}}\right) dt = \frac{1}{1-e^{-sT}} \int_0^T x_1(t) e^{-st} dt$.

- **INVERSE LAPLACE TRANSFORM: (PARTIAL FRACTION EXPANSION)**

The universe laplace transform by partial fraction method of all three cases.

Case i) when s-domain signal $X(s)$ has distinct poles

Let $X(s) = \dfrac{K}{s(s+P_1)(s+P_2)}$

By partial fraction

$$X(s) = \frac{K_1}{s} + \frac{K_2}{s+P_1} + \frac{K_3}{s+P_2}$$

The residues K_1, K_2, K_3 are given by

$$K_1 = X(s) \times s \Big|_{s=0}, \quad K_2 = X(s) \times (s+P_1) \Big|_{s=-P_1}$$

$$K_3 = X(s) \times (s+P_2) \Big|_{s=-P_2}.$$

$$L^{-1}\{X(s)\} = L^{-1}\left\{\frac{K_1}{s} + \frac{K_2}{s+P_1} + \frac{K_3}{s+P_2}\right\}$$

$$x(t) = K_1 L^{-1}\left\{\frac{1}{s}\right\} + K_2 L^{-1}\left\{\frac{1}{s+P_1}\right\} + K_3 L^{-1}\left\{\frac{1}{s+P_2}\right\}$$

$$= K_1 u(t) + K_2 e^{-P_1(t)} u(t) + K_3 e^{-P_2 t} u(t).$$

Case ii) when s-domain signal $X(s)$ has multiple poles

$$X(s) = \frac{K}{s(s+P_1)(s+P_2)^2}$$

$$X(s) = \frac{K_1}{s} + \frac{K_2}{s+P_1} + \frac{K_3}{(s+P_2)^2} + \frac{K_4}{s+P_2}$$

The residues K_1, K_2, K_3, K_4 are given by

$$K_1 = X(s) \times s \Big|_{s=0} \qquad K_2 = X(s) \times (s+P_1) \Big|_{s=-P_1}$$

$$K_3 = X(s) \times (s+P_2)^2 \Big|_{s=-p^2} \qquad K_4 = \frac{d}{ds}\left[X(s) \times (s+P_2)^2\right]_{s=-p_2}$$

$$L^{-1}\{X(s)\} = L^{-1}\left\{\frac{K_1}{s} + \frac{K_2}{s+P_1} + \frac{K_3}{(s+P_2)^2} + \frac{K_4}{s+P_2}\right\}$$

$$= k_1 u(t) + k_2 e^{-P_1 t} u(t) + k_3 t\, e^{-P_2 t} u(t) + k_4 e^{-P_2 t} u(t)$$

In general

$$X(s) = \frac{k}{s(s+P_1)(s+P_2)^q} \quad \text{then}$$

$$X(s) = \frac{k_1}{s} + \frac{k_2}{s+P_1} + \frac{k_3}{(s+P_2)^q} + \frac{k_{12}}{(s+P_2)^{q-1}} + \cdots + \frac{k_{(q-1)2}}{s+P_2}$$

$$k_{r2} = \frac{1}{r!} \frac{d^r}{dq^r} \left[X(s) \times (s+2)^q \right] ; \; r = 1, 2 \ldots q-1.$$

Case iii) When s-domain signal $X(s)$ has complex conjugate poles

Let $$X(s) = \frac{k}{(s+P_1)(s^2+bs+c)}$$

$$X(s) = \frac{k_1}{s+P_1} + \frac{k_2 s + k_3}{s^2+bs+c}$$

$$k_1 = X(s) \times (s+P_1)_{s=-P_1}$$

$$X(s) = \frac{k_1}{s+P_1} + \frac{k_2 s + k_3}{s^2 + 2\times\frac{b}{2}\cdot s + \left(\frac{b}{2}\right)^2 + c - \left(\frac{b}{2}\right)^2}$$

Arranging s^2+bs in $(x+y)^2$.

$$= \frac{k_1}{s+P_1} + \frac{k_2 s + k_3}{\left(s+\frac{b}{2}\right)^2 + \left(c - \frac{b^2}{4}\right)^2}$$

put $\frac{b}{2} = a$, $c - \frac{b^2}{4} = \Omega_0^2$

$$X(s) = \frac{k_1}{s+P_1} + k_2 \cdot \frac{s + a + \frac{k_3}{k_2} - a}{(s+a)^2 + \Omega_0^2}$$

$$= \frac{k_1}{s+P_1} + k_2 \cdot \frac{s+a+k_4}{(s+a)^2 + \Omega_0}$$

$\left(\because \text{put } \frac{k_3}{k_2} - a = k_4 \right)$

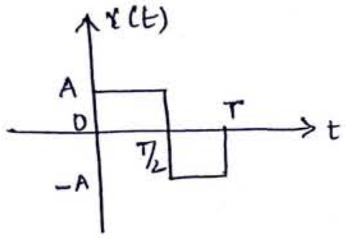
$$\therefore X(s) = \frac{k_1}{s+P_1} + k_2 \cdot \frac{s+a}{(s+a)^2+\Omega_0^2} + k_5 \frac{\Omega_0}{(s+a)^2+\Omega_0^2}$$

$\left(\frac{k_2 k_4}{\Omega_0} = k_5 \right).$

$$x(t) = k_1 e^{-P_1 t} u(t) + k_2 e^{-at} \cos \Omega_0 t\, u(t) + k_5 e^{-at} \sin \Omega_0 t\, u(t).$$

LAPLACE TRANSFORM OF CERTAIN SIGNALS USING WAVEFORM SYNTHESIS:

(a)



proof

$x(t) = A$ for $0<t<T/2$

$= -A$ for $T/2<t<T$

$$L\{x(t)\} = X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt = \int_0^T x(t) e^{-st} dt$$

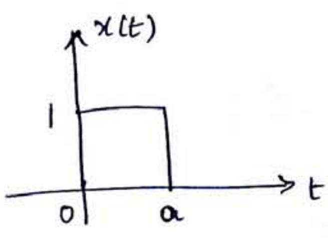
$$= \int_0^{T/2} A e^{-st} dt + \int_{T/2}^{T} (-A) e^{-st} dt = \left[\frac{Ae^{-st}}{-s} \right]_0^{T/2} + \left[\frac{-Ae^{-st}}{-s} \right]_{T/2}^{T}$$

$$= \left[\frac{Ae^{-sT/2}}{-s} - \frac{Ae^{0}}{-s} \right] + \left[\frac{Ae^{-sT}}{s} - \frac{Ae^{-sT/2}}{s} \right]$$

$$= -\frac{Ae^{-\frac{sT}{2}}}{s} + \frac{A}{s} + \frac{Ae^{-sT}}{s} - \frac{Ae^{-sT/2}}{s}$$

$$= \frac{A}{s}\left[1 - e^{-sT/2}\right]^2$$

(b)



proof

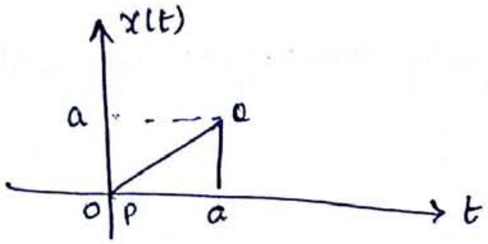
$x(t) = 1$ for $0 \le t \le a$

$= 0$ for $t>a$

$$L\{x(t)\} = X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt = \int_0^a 1 \times e^{-st} dt = \int_0^a e^{-st} dt$$

$$= \left[\frac{e^{-st}}{-s} \right]_0^a = \frac{e^{-as}}{-s} - \frac{e^{0}}{-s} = \frac{-e^{-as}}{s} + \frac{1}{s} = \frac{1}{s}(1 - e^{-as})$$

(c)



Consider the eqn of straight line $\frac{y-y_1}{y_1-y_2} = \frac{x-x_1}{x_1-x_2}$.

Here $y = x(t)$, $x = t$

Consider points P and Q as shown in figure

$$P = [0, 0], \quad Q[a, a]$$
$$t_1, x_1(t) \qquad t_2, x_2(t)$$

$$\therefore \frac{x(t)-0}{0-a} = \frac{t-0}{0-a} \Rightarrow x(t) = t$$

$\therefore$ $x(t) = t$ for $t = 0$ to a

$= 0$ for $t > a$

$$L\{x(t)\} = X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st} dt = \int_0^a e^{-st} \cdot t\, dt$$

$$= \left[t \times \frac{e^{-st}}{-s} - \int 1 \cdot \frac{e^{-st}}{-s} dt \right]_0^a$$

$$= \left[-\frac{te^{-st}}{s} - \frac{e^{-st}}{s^2} \right]_0^a$$

$$= \left[-\frac{ae^{-sa}}{s} - \frac{e^{-sa}}{s^2} + 0 + \frac{e^0}{s^2} \right]$$

$$= \frac{1}{s^2} - \frac{e^{-as}}{s^2} - \frac{ae^{-as}}{s}$$

$$= \frac{1}{s^2}\left[1 - e^{-as}(1+as) \right]$$

Problems on Laplace transforms:

(i) Determine the Laplace transform of continuous time signals and their ROC

(a) $x(t) = A\,u(t)$

$x(t) = A$ for $t \geq 0$ bcoz $u(t) = 1$ for $t \geq 0$
$= 0$ for $t < 0$

Laplace transform

$$L\{x(t)\} = X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st} dt = \int_{0}^{\infty} A \cdot e^{-st} dt = A\left[\frac{e^{-st}}{-s}\right]_0^{\infty}$$

$$= A\left[\frac{e^{-(\sigma+j\omega)t}}{-s}\right]_0^{\infty}$$

$$= A\left[\frac{e^{-(\sigma+j\omega)\times\infty}}{-s} + \frac{e^{0}}{s}\right] = A\left[\frac{e^{-\sigma\times\infty} \cdot e^{-j\omega\times\infty}}{-s} + \frac{e^{0}}{s}\right]$$

when $\sigma > 0$; $e^{-\sigma\times\infty} = e^{-\sigma\infty} = e^{-\infty} = 0$

when $\sigma < 0$; $e^{-\sigma\times\infty} = e^{\infty} = \infty$

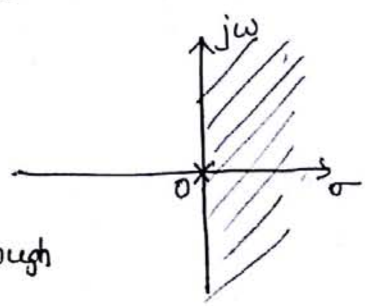
$\therefore$ we can say that $X(s)$ converges when $\sigma > 0$

When $\sigma > 0$, the $X(s)$ is given by

$$X(s) = A\left[\frac{0 \times e^{-j\omega\times\infty}}{-s} + \frac{1}{s}\right] = \frac{A}{s}$$

$\therefore L\{A\,u(t)\} = \frac{A}{s}$; with ROC as all point is s-plane to the right of line passing through $\sigma = 0$

(or ROC is right half of s-plane).



(2) $x(t) = t\,u(t)$

Sol

$$x(t) = t \quad \begin{cases} u(t) = 1 & \text{for } t \geq 0 \\ \quad\;\; = 0 & \text{otherwise} \end{cases}$$

$$\int uv = u\int v - \int\left[du \int v\right]$$

$$L\{x(t)\} = X(s) = \int_0^\infty x(t)\,e^{-st}\,dt$$

$$= \int_0^\infty t \cdot e^{-st}\,dt = t\left[\frac{e^{-st}}{-s}\right]_0^\infty - \int\left[\frac{e^{-st}}{-s}\right]_0^\infty$$

$$= t\cdot\left[\frac{e^{-st}}{-s}\right]_0^\infty - \left[\frac{e^{-st}}{s^2}\right]_0^\infty$$

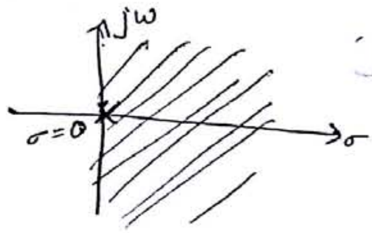
$$= t\left[\frac{e^{-(\sigma+j\omega)t}}{-s}\right]_0^\infty - \left[\frac{e^{-(\sigma+j\omega)t}}{s^2}\right]_0^\infty$$

$$= \left[\infty \times \frac{e^{-(\sigma+j\omega)\infty}}{-s} - 0 - \frac{e^{-(\sigma+j\omega)\infty}}{s^2} + \frac{e^0}{s^2}\right]$$

$$= \left[\infty \times \frac{e^{-\sigma\times\infty} \times e^{-j\omega\times\infty}}{-s} - \frac{e^{-\sigma\times\infty}\cdot e^{-j\omega\times\infty}}{s^2} + \frac{1}{s^2}\right]$$

when $\sigma > 0$, positive i.e $e^{-\sigma\times\infty} = e^{-\sigma\infty} = 0$

when $\sigma < 0$, negative i.e $e^{-\sigma\times\infty} = e^{\sigma\infty} = \infty$



It converges when $\sigma > 0$

$$= \left[\frac{\infty \times 0 \times e^{-j\omega\times\infty}}{-s} - \frac{0 \times e^{-j\omega\times\infty}}{s^2} + \frac{1}{s^2}\right] = \frac{1}{s^2}$$

$\therefore$ ROC is right half of s-plane.

3) Given that $x(t) = e^{-3t}u(t)$:

Sol

$x(t) = e^{-3t}$ for $t \geq 0$

$$L\{x(t)\} = X(s) = \int_{-\infty}^{\infty} x(t)\cdot e^{-st}\,dt$$

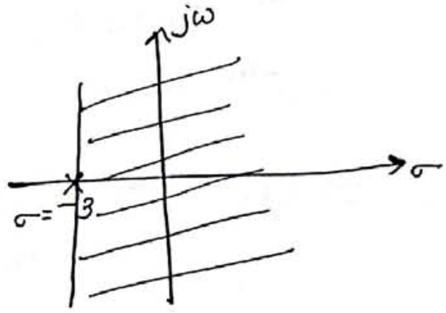
$$= \int_0^\infty x(t) \cdot e^{-st} dt = \int_0^\infty e^{-3t} \cdot e^{-st} dt = \int_0^\infty e^{-(s+3)t} dt.$$

$$= \left[\frac{e^{-(s+3)t}}{-(s+3)} \right]_0^\infty$$

$$= \left[\frac{e^{-(s+3)\infty}}{-(s+3)} + \frac{e^{-(s+3)\cdot 0}}{s+3} \right] \Rightarrow \frac{e^{-(\sigma+j\omega+3)\times\infty}}{-(s+3)} + \frac{1}{s+3}$$

$$= \left[\frac{e^{-(\sigma+3)\infty} \cdot e^{-j\omega\times\infty}}{-(s+3)} + \frac{1}{s+3} \right]$$

$\sigma + 3 = \sigma - (-3)$



if $\sigma > -3$ = positive $\therefore e^{-k\times\infty} = e^{-k\infty} = 0$

if $\sigma < -3$ = negative $\therefore e^{-k\times\infty} = e^{k\infty} = \infty$

$\therefore$ It converges at $\sigma > -3$.

$$x(s) = \left[\frac{0 \times e^{-j\omega\times\infty}}{-(s+3)} + \frac{1}{s+3} \right] = \frac{1}{s+3}$$

$L\{e^{-3t}u(t)\} = \frac{1}{s+3}$; with ROC as all pts in s-plane to right of line passing through $\sigma = -3$

(4) $x(t) = e^{-3t}u(-t)$

Sol $x(t) = e^{-3t}$ for $t \leq 0$

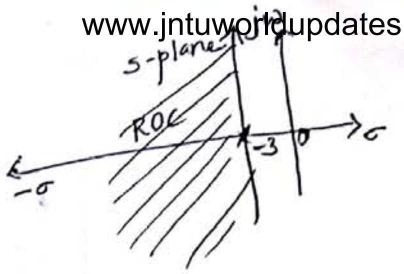
$$L\{x(t)\} = x(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt = \int_{-\infty}^{0} e^{-st} \cdot e^{-3t} dt$$

$$= \int_{-\infty}^{0} e^{-(s+3)t} dt$$

$$= \left. \frac{e^{-(s+3)t}}{-(s+3)} \right]_{-\infty}^{0} = \frac{e^{0}}{-(s+3)} + \frac{e^{+(s+3)\times\infty}}{+(s+3)} \Bigg]$$

$$= \frac{-1}{s+3} + \frac{e^{+(\sigma+j\omega+3)\times\infty}}{s+3}$$

$$= \frac{-1}{s+3} + \frac{e^{+(\sigma+3)\times\infty} \cdot e^{-j\omega\times\infty}}{s+3}$$



$k = \sigma+3 = \sigma-(-3)$

when $\sigma > -3$; positive $+(\sigma+3)\times\infty = \infty$

when $\sigma < -3$; negative $e^{-k\infty} = e^{k\infty} = 0$

$\therefore$ ROC it converges when $\sigma < -3$ $\Rightarrow \frac{-1}{s+3} + \frac{0 \times e^{-j\omega\times\infty}}{s+3}$

$L\{x(t)\} = L\{e^{-3t}u(-t)\} = \frac{-1}{s+3}$ { ROC as all pts in s plane to the left passing through $\sigma = -3$

---

(5) $x(t) = e^{-4|t|}$

$= e^{-4t}$ for $t \geq 0$

$= e^{+4t}$ for $t \leq 0$

$$L\{x(t)\} = X(s) = \int_{-\infty}^{\infty} x(t) \cdot e^{-st}dt = \int_{-\infty}^{0} e^{4t} \cdot e^{-st}dt + \int_{0}^{\infty} e^{-4t} \cdot e^{-st}dt$$

$$= \int_{-\infty}^{0} e^{-(s-4)t}dt + \int_{0}^{\infty} e^{-(s+4)t}dt$$

$$= \left[\frac{e^{-(s-4)t}}{-(s-4)}\right]_{-\infty}^{0} + \left[\frac{e^{-(s+4)t}}{-(s+4)}\right]_{0}^{\infty}$$

$$= \left[\frac{1}{-(s-4)} + \frac{e^{+(s-4)\times\infty}}{s-4} - \frac{e^{-(s+4)\times\infty}}{s+4} + \frac{1}{s+4}\right]$$

$$= \left[\frac{-1}{s-4} + \frac{e^{(\sigma+j\omega-4)\times\infty}}{s-4} - \frac{e^{-(\sigma+j\omega+4)\times\infty}}{s+4} + \frac{1}{s+4}\right]$$

$$= \left[\frac{-1}{s-4} + \frac{e^{(\sigma-4)\times\infty} \cdot e^{+j\omega\times\infty}}{s-4} - \frac{e^{-(\sigma+4)\times\infty} \cdot e^{-j\omega\times\infty}}{s+4} + \frac{1}{s+4}\right]$$

$\sigma-4 \Rightarrow \sigma-(+4)$

when $\sigma > 4$ positive $e^{k\times\infty} = \infty$

when $\sigma < 4$ negative $e^{-k\infty} = 0$

It converges when $\sigma < +4$

$\sigma+4 \Rightarrow \sigma-(-4)$

when $\sigma > -4$ positive $e^{-k\infty} = 0$

when $\sigma < -4$ negative $e^{+k\infty} = \infty$

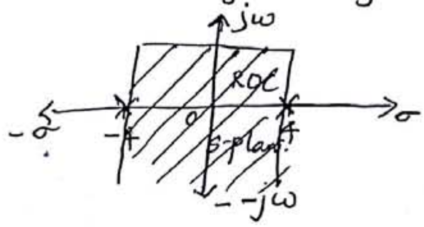
It converges when $\sigma > -4$

When σ lies between -4 and $+4$, the $X(s)$ is given by

$$X(s) = \left[\frac{-1}{s-4} + \frac{e^{(\sigma-4)\times\infty} e^{j\omega\times\infty}}{s-4} - \frac{e^{-(\sigma+4)\times\infty} \cdot e^{-j\omega\times\infty}}{s+4} + \frac{1}{s+4} \right]$$

$$= \frac{-1}{s-4} + \frac{1}{s+4} \Rightarrow \frac{-s-4+s-4}{s^2-16} = \frac{-8}{s^2-16}.$$

with ROC as all points in splane in between the lines passing through $\sigma = -4$ and $\sigma = 4$.



(6) Determine the laplace transform of following signals.

$x(t) = \sin\omega_0 t \, u(t)$

Sol $x(t) = \sin\omega_0 t$ for $t \geq 0$

$$L\{x(t)\} = X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt \qquad \because \sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

$$= \int_{0}^{\infty} \sin\omega_0 t \cdot e^{-st} dt$$

$$= \int_0^\infty \frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{2j} \cdot e^{-st} dt$$

$$= \frac{1}{2j}\left[\int_0^\infty e^{-(s-j\omega_0)t} dt - \int_0^\infty e^{-(s+j\omega_0)t} dt \right]$$

$$= \frac{1}{2j}\left\{ \left[\frac{e^{-(s-j\omega_0)t}}{-(s-j\omega_0)} \right]_0^\infty - \left[\frac{e^{-(s+j\omega_0)t}}{-(s+j\omega_0)} \right]_0^\infty \right\}$$

$$= \frac{1}{2j}\left[\frac{e^{-\infty}}{-(s-j\omega_0)} + \frac{e^0}{s-j\omega_0} + \frac{e^{-\infty}}{s+j\omega_0} - \frac{e^0}{s+j\omega_0} \right]$$

$$= \frac{1}{2j}\left[\frac{1}{s-j\omega_0} - \frac{1}{s+j\omega_0} \right] = \frac{1}{2j}\left[\frac{\cancel{s}+j\omega_0 - \cancel{s} + j\omega_0}{s^2 - j^2\omega_0^2} \right] \Rightarrow \frac{1}{2j} \cdot \frac{2j\omega_0}{s^2+\omega_0^2}.$$

$$= \omega_0 / s^2+\omega_0^2.$$

(7) $x(t) = \cos\omega_0 t \, u(t)$

Sol $x(t) = \cos\omega_0 t \quad ; t \geq 0$

$$L\{x(t)\} = X(s) = \int_0^\infty x(t) \cdot e^{-st} dt = \int_0^\infty \frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2} \cdot e^{-st} dt$$

$$= \frac{1}{2} \int_0^\infty \left[e^{-(s-j\omega_0)t} + e^{-(s+j\omega_0)t} \right] dt$$

$$= \frac{1}{2} \left[\frac{e^{-(s-j\omega_0)t}}{-(s-j\omega_0)} + \frac{e^{-(s+j\omega_0)t}}{-(s+j\omega_0)} \right]_0^\infty$$

$$= \frac{1}{2} \left[\frac{e^{-\infty}}{-(s-j\omega_0)} + \frac{e^0}{s-j\omega_0} - \frac{e^{-\infty}}{s+j\omega_0} + \frac{e^0}{s+j\omega_0} \right]$$

$$= \frac{1}{2} \left[\frac{1}{s-j\omega_0} + \frac{1}{s+j\omega_0} \right] = \frac{1}{2} \left[\frac{s+j\cancel{\omega_0}+s-j\cancel{\omega_0}}{s^2-j^2\omega_0^2} \right] = \frac{1}{\cancel{2}} \cdot \frac{\cancel{2}s}{s^2+\omega_0^2}$$

$$L\{\cos\omega_0 t \, u(t)\} = \frac{s}{s^2+\omega_0^2}$$

8) $x(t) = \cosh\omega_0 t \quad u(t)$

Sol $x(t) = \cosh\omega_0 t \quad \text{for } t \geq 0 \qquad \because \cosh\theta = \frac{e^\theta + e^{-\theta}}{2}$

$$L\{x(t)\} = X(s) = \int_{-\infty}^\infty x(t) \cdot e^{-st} dt$$

$$= \int_0^\infty \left(\frac{e^{\omega_0 t} + e^{-\omega_0 t}}{2} \right) e^{-st} dt$$

$$= \frac{1}{2} \int_0^\infty e^{\omega_0 t} \cdot e^{-st} dt + \frac{1}{2} \int_0^\infty e^{-\omega_0 t} \cdot e^{-st} dt$$

$$= \frac{1}{2} \left[\int_0^\infty e^{-(s-\omega_0)t} dt + \int_0^\infty e^{-(s+\omega_0)t} dt \right]$$

$$= \frac{1}{2} \left\{ \left[\frac{e^{-(s-\omega_0)t}}{-(s-\omega_0)} \right]_0^\infty + \left[\frac{e^{-(s+\omega_0)t}}{-(s+\omega_0)} \right]_0^\infty \right.$$

$$= \frac{1}{2} \left[\frac{e^{-\infty}}{-(s-\omega_0)} + \frac{1}{s-\omega_0} + \frac{e^{-\infty}}{-(s+\omega_0)} + \frac{1}{s+\omega_0} \right]$$

$$= \frac{1}{2}\left[\frac{1}{s-\omega_0} + \frac{1}{s+\omega_0}\right]$$

$$= \frac{1}{2}\left[\frac{s+\cancel{\omega_0}+s-\cancel{\omega_0}}{s^2-\omega_0^2}\right] = \frac{s}{s^2-\omega_0^2}.$$

9) $x(t) = e^{-at}\sin\omega_0 t\, u(t)$

Sol $x(t) = e^{-at}\sin\omega_0 t$ for $t \geq 0$

$$L\{x(t)\} = X(s) = \int_0^\infty x(t)\cdot e^{-st}\,dt = \int_0^\infty e^{-at}\sin\omega_0 t\, e^{-st}\,dt = \int_0^\infty e^{-at}\left(\frac{e^{j\omega_0 t}-e^{-j\omega_0 t}}{2j}\right)e^{-st}\,dt$$

$$= \frac{1}{2j}\left[\int_0^\infty e^{-(s+a-j\omega_0)t} - e^{-(s+a+j\omega_0)t}\right]dt$$

$$= \frac{1}{2j}\left[\frac{e^{-(s+a-j\omega_0)t}}{-(s+a-j\omega_0)}\right]_0^\infty - \left[\frac{e^{-(s+a+j\omega_0)t}}{-(s+a+j\omega_0)}\right]_0^\infty$$

$$= \frac{1}{2j}\left[\frac{e^{-\infty}}{-(s+a-j\omega)} + \frac{e^0}{s+a-j\omega_0} + \frac{e^{-\infty}}{s+a+j\omega} - \frac{e^0}{s+a+j\omega_0}\right]$$

$$= \frac{1}{2j}\left[\frac{1}{s+a-j\omega_0} - \frac{1}{s+a+j\omega_0}\right] = \frac{1}{2j}\left[\frac{\cancel{s}+\cancel{a}+j\omega_0-\cancel{s}-\cancel{a}+j\omega_0}{(s+a)^2+\omega_0^2}\right]$$

$$= \frac{\cancel{2}\cancel{j}\omega_0}{\cancel{2}\cancel{j}\cdot(s+a)^2+\omega_0^2}$$

$$= \frac{\omega_0}{(s+a)^2+\omega_0^2}.$$

(10) $x(t) = e^{-at}\cos\omega_0 t\; u(t)$

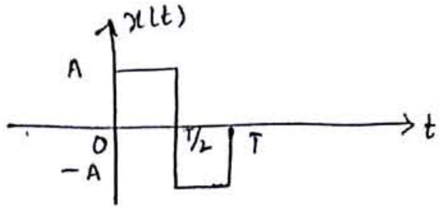
Sol $x(t) = e^{-at}\cos\omega_0 t$ for $t \geq 0$.

$\parallel^{ly}$

$$= \frac{s+a}{(s+a)^2+\omega_0^2}$$

(11) Determine the Laplace transform of following signals.

Sol)



$x(t) = A$ for $0<t\leq T/2$

$= -A$ for $T/2<t<T$

$$L\{x(t)\} = X(s) = \int_0^{T/2} A\cdot e^{-st}dt + \int_{T/2}^{T} -A\cdot e^{-st}dt$$

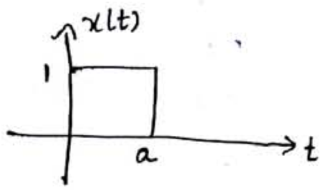
$$= A\left[\frac{e^{-st}}{-s}\right]_0^{T/2} - A\left[\frac{e^{-st}}{-s}\right]_{T/2}^{T}$$

$$= A\left[\frac{e^{-sT/2}}{-s} + \frac{1}{s}\right] - A\left[\frac{e^{-sT}}{-s} + \frac{e^{-sT/2}}{s}\right]$$

$$= -\frac{Ae^{-sT/2}}{s} + \frac{A}{s} + \frac{Ae^{-sT}}{s} - \frac{Ae^{-sT/2}}{s}$$

$$= \frac{A}{s}\left[1 + e^{-sT} - 2e^{-sT/2}\right] = \frac{A}{s}\left[1 - e^{-sT/2}\right]^2.$$

(12)



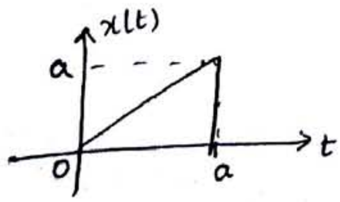
$x(t) = 1$ for $0<t<a$

$= 0$ otherwise

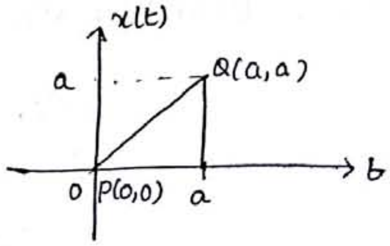
$$L\{x(t)\} = X(s) = \int_0^a e^{-st}dt = \left[\frac{e^{-st}}{-s}\right]_0^a$$

$$= \frac{e^{-as}}{-s} + \frac{1}{s} \Rightarrow \frac{1}{s}\left[1 - e^{-as}\right]$$

(13)



$$\frac{y-y_1}{y_1-y_2} = \frac{x-x_1}{x_1-x_2}.$$



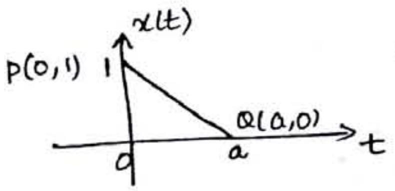
here $(x_1,y_1) = (0,0)$, $(x_2,y_2) = (a,a)$

$x = t$, $y = x(t)$.

$$\Rightarrow \frac{x(t)-0}{0-a} = \frac{t-0}{0-a} \Rightarrow \frac{x(t)}{-a} = \frac{t}{-a} \Rightarrow x(t) = t \quad \text{for } 0<t<a$$
$$= 0 \quad \text{otherwise}$$

$$L\{x(t)\} = X(s) = \int_{-\infty}^{\infty} x(t).e^{-st}dt$$
$$= \int_0^a t e^{-st}\,dt = \left\{ t\left[\frac{e^{-st}}{-s}\right]_0^a - \int \left.\frac{e^{-st}}{-s}\right]_0^a \right\}$$
$$= \left. t.\frac{e^{-st}}{-s}\right]_0^a - \left.\frac{e^{-st}}{s^2}\right]_0^a$$
$$= a.\frac{e^{-sa}}{-s} - \frac{e^{-as}}{s^2} + \frac{1}{s^2}$$
$$= \frac{1}{s^2}\left[1 - e^{-as} - s.e^{-as}.a\right]$$
$$= \frac{1}{s^2}\left[1 - e^{-as} - as.e^{-as}\right]$$
$$= \frac{1}{s^2}\left[1 - e^{-as}(1+as)\right]$$

(14)



$(x_1,y_1) = (0,1)$, $(x_2,y_2) = (a,0)$, $x = t$, $y = x(t)$.

$$\frac{y-y_1}{y_1-y_2} = \frac{x-x_1}{x_1-x_2} \Rightarrow \frac{x(t)-1}{1-0} = \frac{t-0}{0-a} \Rightarrow x(t)-1 = \frac{-t}{a}$$

$x(t) = 1 - t/a$ for 0 to a

$= 0$ for $t>a$

$$L\{x(t)\} = X(s) = \int_{-\infty}^{\infty} x(t)\cdot e^{-st}\,dt$$

$$= \int_0^a \left(1-\frac{t}{a}\right) e^{-st}\,dt = \int_0^a e^{-st}\,dt - \int_0^a \frac{t}{a} e^{-st}\,dt$$

$$= \left.\frac{e^{-st}}{-s}\right]_0^a - \frac{1}{a}\left[\int_0^a t\cdot e^{-st}\,dt\right]$$

$$= \frac{e^{-as}}{-s} + \frac{1}{s} - \frac{1}{a}\left[t\cdot\frac{e^{-st}}{-s} - \frac{e^{-st}}{s^2}\right]_0^a$$

$$= \frac{e^{-as}}{-s} + \frac{1}{s} - \frac{1}{\cancel{a}}\cdot\cancel{a}\cdot\frac{e^{-as}}{-s} + \frac{1e^{-sa}}{as^2} - \frac{1}{as^2}\Big]$$

$$= \cancel{\frac{e^{-as}}{-s}} + \frac{1}{s} + \cancel{\frac{e^{-as}}{s}} + \frac{e^{-as}}{as^2} - \frac{1}{as^2}$$

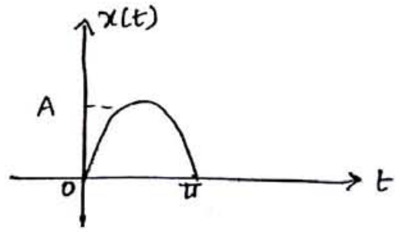
$$= \frac{1}{s} + \frac{e^{-as}}{as^2} - \frac{1}{as^2} = \frac{1}{as^2}\left[e^{-as} + as - 1\right]$$

(15) Determine the laplace transform of sine pulse

Sol

$x(t) = A\sin t$ for $0<t<\pi$

$= 0$ for $t>\pi$



$$L\{x(t)\} = X(s) = \int_0^{\pi} A\cdot\sin t\, e^{-st}\,dt$$

$$= A\left[\frac{e^{jt}-e^{-jt}}{2j} e^{-st}\right]dt = \frac{A}{2j}\left[\int_0^{\pi} e^{-(s-j)t}\,dt - \int_0^{\pi} e^{-(s+j)t}\,dt\right]_0$$

$$= \frac{A}{2j}\left[\frac{e^{-(s-j)t}}{-(s-j)}\right]_0^{\pi} - \frac{A}{2j}\left[\frac{e^{-(s+j)t}}{-(s+j)}\right]_0^{\pi}$$

$$= \frac{A}{2j}\left[\frac{e^{-(s+j)t}}{s+j} - \frac{e^{-(s-j)t}}{s-j}\right]_0^{\pi} = \frac{A}{2j}\left[\frac{(s-j)e^{-st}e^{-jt} - (s+j)e^{-st}e^{+jt}}{s^2-j^2}\right]_0^{\pi}$$

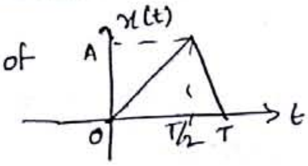
$$= \frac{A}{2j(s^2+1)}\left[(s-j)e^{-st}e^{-jt} - (s+j)e^{-st}e^{jt}\right]_0^{\pi}$$

$$= \frac{A}{2j(s^2+1)}\left[(s-j)e^{-s\pi}e^{-j\pi} - (s+j)e^{-s\pi}e^{j\pi} - (s-j)e^{-0} + (s+j)e^{0}\right]$$

$$= \frac{A}{2j(s^2+1)}\left[-(s-j)e^{-s\pi} + (s+j)e^{-s\pi} - (s-j) + (s+j)\right] = \frac{A}{2j(s^2+1)}\left(2je^{-\pi s} + 2j\right)$$

$$\Rightarrow \frac{A}{s^2+1}\left(e^{-\pi s}+1\right)$$

(16) Determine laplace transform of



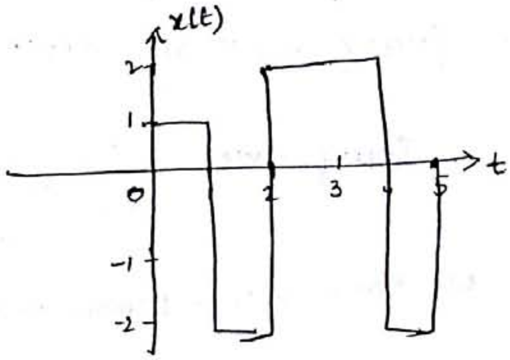
Sol

$$x(t) = \frac{2At}{T} \; ; \; 0<t<T/2$$

$$= 2A - \frac{2At}{T} \quad T/2<t<T.$$

$$L\{x(t)\} = X(S) = \frac{2A}{TS^2}\left(1 - e^{-ST/2}\right)^2.$$

(17) Determine laplace transform of



Sol

$$x(t) = \begin{cases} 1 & \text{for } 0<t<1 \\ -2 & \text{for } 1<t<2 \\ 2 & \text{for } 2<t<4 \\ -2 & \text{for } 4<t<5 \\ 0 & \text{for } t>5 \end{cases}$$

$$L\{x(t)\} = X(S) = \int_0^1 e^{-St}dt + \int_1^2 -2e^{-St}dt + \int_2^4 2e^{-St}dt + \int_4^5 -2e^{-St}dt$$

$$= \left.\frac{e^{-St}}{-S}\right]_0^1 - 2\left[\frac{e^{-St}}{-S}\right]_1^2 + 2\left[\frac{e^{-St}}{-S}\right]_2^4 - 2\left[\frac{e^{-St}}{-S}\right]_4^5$$

$$= \frac{e^{-S}}{-S} + \frac{1}{S} + \frac{2e^{-2S}}{S} - \frac{e^{-S}}{S} - \frac{2e^{-4S}}{S} + \frac{2e^{-2S}}{S} + \frac{2e^{-5S}}{S} - \frac{2e^{-4S}}{S}$$

$$= \frac{1}{S}\left[1 - 2e^{-S} + 4e^{-2S} - 4e^{-4S} + 2e^{-5S}\right]$$

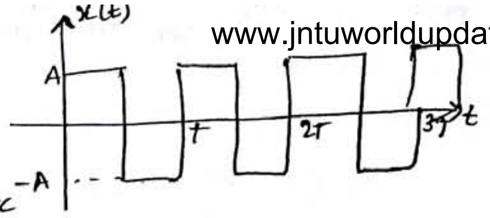
(18) Determine the Laplace transform of $\delta(t)$.

Sol

$$\delta(t) = 1 \text{ for } t=0$$

$$= 0 \text{ otherwise.}$$

$$L\{x(t)\} = X(S) = \left.\frac{e^{-St}}{-S}\right|_{t=0} =$$



→ Determine the Laplace transform of periodic square wave

Sol) The given waveform satisfy the condition

$x(t+nT) = x(t)$ and so it is periodic

$x_1(t) = A$ for $t = 0$ to $T/2$

$= -A$ for $t = T/2$ to T

From periodicity property of Laplace transform

If $X(S) = L\{x(t)\}$ and if $x(t) = x(t+nT)$ then $X(S) = \frac{1}{1-e^{-ST}} \int_0^T x_1(t) e^{-st} dt$

$$\therefore L\{x(t)\} = X(S) = \frac{1}{1-e^{-ST}} \int_0^T x_1(t) e^{-st} dt$$

we know $x_1(t)$ = Laplace transform $= \frac{A}{S}\left[1 - e^{-ST/2}\right]^2$

$$X(S) = \frac{1}{1-e^{-ST}} \left[\frac{A}{S}\left(1-e^{-ST/2}\right)\right]^2$$

$$= \frac{1}{\left(1-e^{-ST/2}\right)\left(1+e^{-ST/2}\right)} \left[\frac{A}{S}\left(1-e^{-ST/2}\right)^2\right]$$

$$= \frac{A}{S}\left[\frac{1-e^{-ST/2}}{1+e^{-ST/2}}\right]$$

Initial Value Theorem:

The initial value theorem states that, if $x(t)$ and its derivative are Laplace transformable then $\underset{t\to 0}{Lt}\, x(t) = \underset{s\to\infty}{Lt}\, sX(s)$.

i.e $x(0) = \underset{t\to 0}{Lt}\, x(t) = \underset{s\to\infty}{Lt}\, sX(s)$.

Proof

We know that

$$L\left\{\frac{dx(t)}{dt}\right\} = sX(s) - x(0)$$

On taking limits $s\to\infty$ on both sides of equation.

$$\underset{s\to\infty}{Lt}\, L\left\{\frac{dx(t)}{dt}\right\} = \underset{s\to\infty}{Lt}\left[sX(s) - x(0)\right]$$

$$\Rightarrow \underset{s\to\infty}{Lt}\int_0^\infty \frac{dx(t)}{dt} e^{-st}\,dt = \underset{s\to\infty}{Lt}\left[sX(s) - x(0)\right]$$

$$\Rightarrow \int_0^\infty \frac{dx(t)}{dt}\left(\underset{s\to\infty}{Lt}\, e^{-st}\right)dt = \underset{s\to\infty}{Lt}\, sX(s) - x(0)$$

$$\Rightarrow 0 = \underset{s\to\infty}{Lt}\, sX(s) - x(0)$$

$$\therefore x(0) = \underset{s\to\infty}{Lt}\, sX(s).$$

$$\boxed{\therefore \underset{t\to 0}{Lt}\, x(t) = \underset{s\to\infty}{Lt}\, sX(s)}$$

Final value theorem:

The final value theorem states that if $x(t)$ and its deriative are laplace transformable then $\underset{t\to\infty}{Lt}\, x(t) = \underset{s\to 0}{Lt}\, sX(s)$.

i.e Final value of signal $x(\infty) = \underset{t\to\infty}{Lt}\, x(t) = \underset{s\to 0}{Lt}\, sX(s)$.

proof

$$L\left\{\frac{dx(t)}{dt}\right\} = sX(s) - x(0)$$

By taking limits $s\to 0$ on both sides of above eqn we get

$$\underset{s\to 0}{Lt}\, L\left\{\frac{dx(t)}{dt}\right\} = \underset{s\to 0}{Lt}\left\{sX(s) - x(0)\right\}$$

$$\underset{s \to 0}{Lt} \int_0^\infty \frac{dx(t)}{dt} e^{-st} dt = \underset{s \to 0}{Lt} \left[sX(s) - x(0) \right]$$

$$\int_0^\infty \frac{dx(t)}{dt} \left(\underset{s \to 0}{Lt} e^{-st} \right) dt = \underset{s \to 0}{Lt}\, sX(s) - x(0)$$

Not fn of 's'

Z - TRANSFORMS :

FUNDAMENTAL DIFFERENCES BETWEEN CONTINUOUS & DISCRETE TIME SIGNAL
(Ref Unit - I page No: 9)

→ DISCRETE TIME SIGNAL REPRESENTATION USING COMPLEX EXPOENTIAL AND SINUSDIOAL COMPONENTS.
(Ref Unit - I - page No: 10, 11, 12)

→ PERIODICITY PROPERTIES OF DT COMPLEX FUNCTIONS:

→ A DT complex exponential is periodic if it repeats after certain 'N' number of samples.

consider $x(n) = e^{j\omega_0 n}$

$$\therefore x(n+N) = e^{j\omega_0 (n+N)}$$
$$= e^{j\omega_0 n} \cdot e^{j\omega_0 N}$$

For periodicity $x(n) = x(n+N)$

i.e $\cancel{e^{j\omega_0 n}} = \cancel{e^{j\omega_0 n}} \cdot e^{j\omega_0 N} \Rightarrow e^{j\omega_0 N} = 1.$

Expressing $e^{j\omega_0 N}$ in terms of sinusoidal functions

$e^{j\omega_0 N} = \cos\omega_0 N + j\sin\omega_0 N$ by Euler's identity $e^{j\theta} = \cos\theta + j\sin\theta$

→ For periodicity $e^{j\omega_0 N} = 1$ gives $\cos\omega_0 N + j\sin\omega_0 N = 1$. This eqn is satisfied for

$$\omega_0 N = 2\pi, 4\pi, 6\pi, 8\pi \ldots$$
$$= 2\pi K \text{ where } K = \text{integer}.$$

$$\text{(or)} \quad \frac{\omega_0}{2\pi} = \frac{K}{N}. \rightarrow (1)$$

$$\text{but } \omega_0 = 2\pi f \Rightarrow \frac{\omega_0}{2\pi} = f \rightarrow (2)$$

Substitute (2) in (1)

$$f = \frac{K}{N}.$$

$\therefore$ $\boxed{\text{For complex exponential to be periodic, } \frac{\omega_0}{2\pi} = \frac{K}{N} \text{ (i.e rational)}}$

Z-TRANFORM (CONCEPT):

The z-transform of $x(n)$ is denoted by $X(z)$ and defined as

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$ → where z is complex variable.

→ $x(n)$ & $X(z)$ is called z-transform pair represented as

$$x(n) \xleftrightarrow{z} X(z).$$

→ Purpose of learning z-transform is to analyse the DT signals and systems, digital filter design and for synthesis of digital filter / systems.

→ For any input sequence, the z-transform is complex. It has real & imaginary parts.

$$x(n) \longrightarrow \boxed{\text{z-transform}} \longrightarrow X(z)$$

Real & Imaginary parts.

DISTINCTION BETWEEN LAPLACE, FOURIER AND Z TRANSFORM:

z-transform given as

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} \quad \text{ROC}: r_2 < |z| < r_1$$

where 'z' is defined as $z = re^{j\Omega}$ in which 'r' is magnitude of z i.e $|z|$ and Ω = angle of z i.e $\angle z$

→ putting for $z = re^{j\Omega}$ in $X(z)$ we get

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) (re^{j\Omega})^{-n}$$

$$= \sum_{n=-\infty}^{\infty} [x(n) r^{-n}] e^{-j\Omega n} \quad \to (1)$$

→ Fourier transform is given by $X(\Omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\Omega n}$ →(2) comparing $X(\Omega)$ with $X(z)$ we find that $X(z)$ indicates fourier transform of $x(n) r^{-n}$.

→ Let $X(z)$ of eq ① is evaluated on unit circle. Then $|z| = r = 1$ i.e $r^{-n} = 1$

$$\therefore X(z)\big|_{z=e^{j\Omega}} = \sum_{n=-\infty}^{\infty} x(n) e^{-j\Omega n}.$$

→ Fourier transform by definition

$$\boxed{X(z)\big|_{z=e^{j\Omega}} = X(\Omega)}$$

→ R/Betn FT & Z-T

→ Laplace Transform of i/p sgl $x(t)$ defined as

$$L\{x(t)\} = X(S) = \int_0^\infty x(t) e^{-st} dt \rightarrow ①$$

On substituting $s = \sigma + j\Omega$ in above eqn ①

$$L\{x(t)\} = X(S) = \int_{-\infty}^{\infty} x(t) e^{-(\sigma + j\Omega)t} dt \rightarrow ②$$

but definition of fourier transform of $x(t)$ is

$$F\{x(t)\} = X(j\Omega) = \int_{-\infty}^{\infty} x(t) \cdot e^{-j\Omega t} dt \rightarrow ③$$

By comparing eqations ② & (3) we can say that

By putting $\sigma = 0$ in laplace transform we get fourier transform of continuous time signal.

$$\therefore X(j\Omega) = X(S)\big|_{s=j\Omega}.$$

REGION OF CONVERGENCE IN Z-TRANSFORM:

→ Since z-transform is an infinite power series, it exists only for those values of z for which the series converges.

→ The ROC of $X(z)$ is the set of all values of z, for which $X(z)$ attains a finite value.

CONSTRAINTS ON ROC FOR VARIOUS CLASSES OF SIGNALS: (OR)

PROPERTIES OF ROC with proofs:

(1) The ROC is a ring or disk in the z-plane centered at the origin.

(2) The ROC cannot contain any poles.

(3) If x(n) is a finite duration, causal sequence then the ROC is the entire z-plane except at z=0.

(4) If x(n) is a finite duration, anti-causal sequence then the ROC is the entire z-plane except at $z=\infty$

(5) If x(n) is a finite duration, two sided sequence then the ROC is entire z-plane except at $z=0$ & $z=\infty$.

(6) If x(n) is an infinite duration, two sided sequence the ROC will consist of a ring in z-plane, bounded on interior and exterior by a pole not containing any poles.

(7) The ROC of an LTI stable system contains unit circle.

(8) The ROC must be connected region.

Case i) Finite duration, right sided (causal) signal:

Let x(n) be finite duration signal with N-samples, defined in range $0 \leq n \leq (N-1)$

$$\therefore x(n) = \{x(0), x(1), x(2) \ldots x(N-1)\}$$

$$X(z) = \sum_{n=0}^{N-1} x(n) z^{-n}$$

$$= x(0) + x(1) z^{-1} + \ldots + x(N-1) z^{-(N-1)}$$

$$= x(0) + \frac{x(1)}{z} + \frac{x(2)}{z^2} + \ldots + \frac{x(N-1)}{z^{N-1}}$$

When $z=0$, all the terms except the first term become infinite. Hence the X(z) exists for all values of z except $z=0$.

$\therefore$ ROC for finite duration right sided is entire z-plane except $z=0$

Case ii) finite duration, left sided (anticausal) signal:

Let $x(n)$ be finite duration signal with N-samples, defined in the range $-(N-1) \leq n < 0$

$$\therefore x(n) = \{x(-N-1)), \ldots x(-2), x(-1), x(0)\}$$

z-transform of $x(n)$ is

$$X(z) = \sum_{n=-(N-1)}^{0} x(n) z^{-n}$$

$$x(z) = x(-(N-1)) z^{(N-1)} + \ldots + (x(-2)) z^{2} + x(-1) z + x(0)$$

when $z = \infty$, all terms except the last term become infinite. Hence the $x(z)$ exists for all values of z, except $z = \infty$.

$\therefore$ ROC of $x(z)$ is entire z-plane except $z = \infty$.

Case iii) Finite duration, two sided (non-causal) signal:

Let $x(n)$ be a finite duration signal with N-samples, defined in the range $-M \leq n \leq M$ where $M = \frac{N-1}{2}$.

$$\therefore x(n) = \{x(-M), \ldots, x(-2), x(-1), x(0), x(1), x(2) \ldots x(M)\}$$

z-transform of $x(n)$ is

$$X(z) = \sum_{n=-M}^{M} x(n) z^{-n}$$

$$= x(-M) z^{M} + \ldots + x(-2) z^{2} + x(-1) z + x(0) + x(1) z^{-1} + x(2) z^{-2} + \cdots + x(M) z^{-M}$$

$$= x(-M) z^{M} + \ldots + x(-2) z^{2} + x(-1) z + x(0) + \frac{x(1)}{z} + \frac{x(2)}{z^{2}} + \cdots + \frac{x(M)}{z^{M}}.$$

when $z = 0$, the terms with negative power of z attain infinity and when $z = \infty$, the terms with positive power of z attain infinity. Hence $x(z)$ converges for all values of z, except at $z = 0$ & $z = \infty$.

ROC is entire z-plane except $z = 0$ & $z = \infty$.

Case iv) Infinite duration, right sided (causal) signal :

Let $x(n) = r_1^n \; ; n \geq 0$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} = \sum_{n=0}^{\infty} r_1^n z^{-n} = \sum_{n=0}^{\infty} (r_1 z^{-1})^n$$

If $0 < |r_1 z^{-1}| < 1$, then $\sum_{n=0}^{\infty} (r_1 z^{-1})^n = \frac{1}{1 - r_1 z^{-1}}$

{ Using infinite geometric series sum formula

$$\sum_{n=0}^{\infty} C^n = \frac{1}{1-C}$$

if $0 < |C| < 1$

$$\therefore X(z) = \frac{1}{1 - \frac{r_1}{z}} = \frac{z}{z - r_1}$$

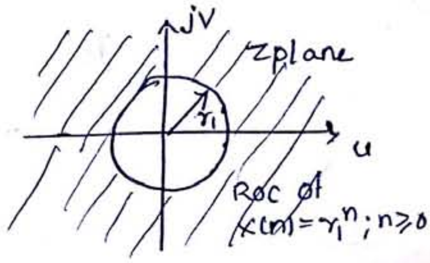
Condition to be satisfied for convergence of $X(z)$ is

$0 < |r_1 z^{-1}| < 1$

$\therefore |r_1 z^{-1}| < 1$

$\frac{|r_1|}{|z|} < 1 \Rightarrow |z| > |r_1|$

represent a circle of radius r_1 in z-plane.



$\rightarrow$ $X(z)$ converges for all points external to the circle of radius r_1 in z-plane.

$\therefore$ ROC of $X(z)$ is exterior of the circle of radius r_1 in z-plane.

Case v) : Infinite duration, left sided (anticausal) signal:

Let $x(n) = r_2^n \; ; n \leq 0$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} = \sum_{n=-\infty}^{0} r_2^n z^{-n} = \sum_{n=0}^{\infty} (r_2^{-1} z)^n$$

If $0 < |r_2^{-1} z| < 1$ then $\sum_{n=0}^{\infty} (r_2^{-1} z)^n = \frac{1}{1 - r_2^{-1} z}$

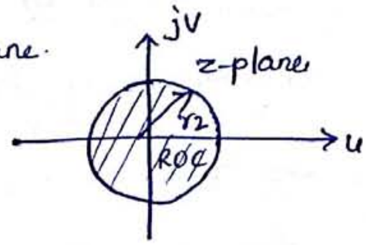
$$\therefore X(z) = \frac{1}{1 - r_2^{-1} z} = \frac{1}{1 - \frac{z}{r_2}} = \frac{r_2}{r_2 - z}$$

$$= \frac{-r_2}{z - r_2}$$

Condition to be satisfied for convergence of $X(z)$ is

$0 < |r_2^{-1} z^{-1}| < 1 \Rightarrow \therefore |r_2^{-1} z| < 1 \Rightarrow \frac{|z|}{|r_2|} < 1 \Rightarrow |z| < |r_2|$

The term $|r_2|$ represent a cle of radius of r_2 in z-plane. $X(z)$ converges for all pts internal to the cle of radius r_2 in z-plane. $\therefore$ ROC of $X(z)$ is interior of the circle of radius r_2.



Case vi) Infinite duration, two sided (anticausal) signal:

Let $x(n) = r_1^n u(n) + r_2^n u(-n)$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} = \sum_{n=-\infty}^{0} r_2^n z^{-n} + \sum_{n=0}^{\infty} r_1^n z^{-n}$$

$$= \sum_{n=0}^{\infty} r_2^{-n} z^n + \sum_{n=0}^{\infty} r_1^n z^{-n}$$

$$= \sum_{n=0}^{\infty} (r_2^{-1} z)^n + \sum_{n=0}^{\infty} (r_1 z^{-1})^n$$

$$= \frac{1}{1 - r_2^{-1} z} + \frac{1}{1 - r_1 z^{-1}}$$

Using infinite geometric series sum formula if $0 < |r_2^{-1} z| < 1$ & $0 < |r_1 z^{-1}| < 1$

The term $\sum_{n=0}^{\infty} (r_2^{-1} z)^n$ converges if

$0 < |r_2^{-1} z| < 1$

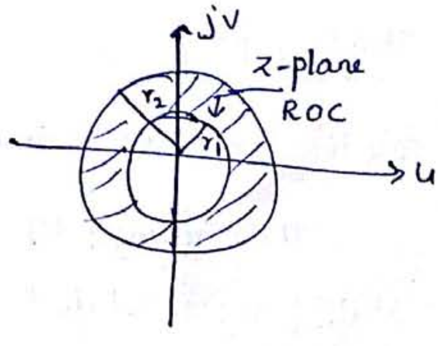
$\therefore |r_2^{-1} z| < 1 \Rightarrow \underline{|r_2^{-1} z|} \Rightarrow \frac{|z|}{|r_2|} < 1 \Rightarrow |z| < |r_2|.$

The term $\sum_{n=0}^{\infty} (r_1 z^{-1})^n$ converges if

$0 < |r_1^{-1} z| < 1$

$\therefore |r_1 z^{-1}| < 1$

$\frac{|r_1|}{|z|} < 1 \Rightarrow |z| > |r_1|.$



$\therefore$ ROC is the region between two circles of radius r_1 & r_2.

PROPERTIES OF Z-TRANSFORM:

Linearity Property:

→ States that z-transform of linear weighted combination of discrete time signals is equal to similar linear weighted combination of z transform of individual discrete time signals.

Let $Z\{x_1(n)\} = X_1(z)$ and $Z[x_2(n)] = X_2(z)$ then by linearity property

$$Z[a_1x_1(n) + a_2x_2(n)] = a_1X_1(z) + a_2X_2(z) \quad (a_1, a_2 \text{ are constants}).$$

proof:

z-transform definition

$$X_1(z) = Z\{x_1(n)\} = \sum_{n=-\infty}^{\infty} x_1(n)z^{-n}$$

$$X_2(z) = Z\{x_2(n)\} = \sum_{n=-\infty}^{\infty} x_2(n)z^{-n}$$

$$Z[a_1x_1(n) + a_2x_2(n)\} = \sum_{n=-\infty}^{\infty}[a_1x_1(n) + a_2x_2(n)]z^{-n}$$

$$= \sum_{n=-\infty}^{\infty}[a_1x_1(n)z^{-n} + a_2x_2(n)z^{-n}]$$

$$= \sum_{n=-\infty}^{\infty} a_1x_1(n)z^{-n} + \sum_{n=-\infty}^{\infty} a_2x_2(n)z^{-n}$$

$$= a_1\sum_{n=-\infty}^{\infty} x_1(n)z^{-n} + a_2\sum_{n=-\infty}^{\infty} x_2(n)z^{-n}$$

$$= a_1X_1(z) + a_2X_2(z).$$

Shifting property:

case i) Two sided z-transform

The shifting property of z-transform states that z-transform of a shifted signal shifted by 'm' units of time is obtained by multiplying z^m to z-transform of unshifted signal.

Let $Z\{x(n)\} = X(z)$ then $Z\{x(n-m)\} = z^{-m}X(z)$ & $Z\{x(n+m)\} = z^{m}X(z)$

Proof:

$$X(z) = z\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$z\{x(n-m)\} = \sum_{n=-\infty}^{\infty} x(n-m) z^{-n}$$

Let $n-m=p \therefore n = p+m$

when $n \to -\infty, p \to -\infty$

$n \to \infty, \ p \to \infty$

$$= \sum_{p=-\infty}^{\infty} x(p) z^{-(m+p)}$$

$$= \sum_{p=-\infty}^{\infty} x(p) z^{-p} \cdot z^{-m}$$

$$= z^{-m} \sum_{p=-\infty}^{\infty} x(p) z^{-p} = z^{-m} X(z)$$

$$z\{x(n+m)\} = \sum_{n=-\infty}^{\infty} x(n+m) z^{-n}$$

$$= \sum_{p=-\infty}^{\infty} x(p) z^{-(p-m)}$$

$$= \sum_{p=-\infty}^{\infty} x(p) z^{-p} \cdot z^{m}$$

$$= z^{m} \sum_{p=-\infty}^{\infty} x(p) z^{-p} \qquad (\because \text{if } p \to n)$$

$$= z^{m} \cdot X(z).$$

Case ii) One sided z-transform:

Let $x(n)$ be discrete time signal defined in range $0<n<\infty$

$$z[x(n)] = X(z)$$

By shifting property

$$z[x(n-m)] = z^{-m} X(z) + \sum_{i=1}^{m} x(-i) z^{-(m-i)}$$

$$z[x(n+m)] = z^{m} X(z) - \sum_{i=0}^{m-1} x(i) z^{(m-i)}$$

Proof

$$x(z) = z\{x(n)\} = \sum_{n=0}^{\infty} x(n) z^{-n}$$

$$z\{x(n-m)\} = \sum_{n=0}^{\infty} x(n-m) z^{-m}$$

multiply by z^m & z^{-m}

$$= \sum_{n=0}^{\infty} x(n-m) z^{-n} z^{m}, z^{-m}$$

$$= z^{-m} \sum_{n=0}^{\infty} x(n-m) z^{-(n-m)}$$

$$= z^{-m} \sum_{p=-m}^{\infty} x(p) z^{-p} \quad \left\{ \text{Let } n-m=p \right.$$

$$= z^{-m} \sum_{p=0}^{\infty} x(p) z^{-p} + \sum_{p=-m}^{-1} x(p) z^{-p} \cdot z^{-m}$$

$$= z^{-m} \sum_{p=0}^{\infty} x(p) z^{-p} + z^{-m} \sum_{p=1}^{m} x(-p) z^{p}$$

Let $p=n$, in 1^{st} summation
$p=i$ in 2^{nd} "

$$= z^{-m} \sum_{n=0}^{\infty} x(n) z^{-n} + z^{-m} \sum_{i=1}^{m} x(-i) z^{i}$$

$$= z^{m} x(z) + \sum_{i=1}^{m} x(-i) z^{-(m-i)}$$

$$z\{x(n+m)\} = \sum_{n=0}^{\infty} x(n+m) z^{-n} = \sum_{n=0}^{\infty} x(n+m) z^{-n} z^{m} z^{-m} \text{ (multiply by } z^m \text{ \& } z^{-m})$$

$$= z^{m} \sum_{n=0}^{\infty} x(n+m) z^{-(n+m)}$$

$$= z^{m} \sum_{p=m}^{\infty} x(p) z^{-p}$$

Let $n+m=p$
when $n \to 0$, $p \to m$
$n \to \infty$, $p \to \infty$

$$= z^{m} \sum_{p=0}^{\infty} x(p) z^{-p} - z^{m} \sum_{p=0}^{m-1} x(p) z^{-p}$$

$$= z^{m} \sum_{n=0}^{\infty} x(n) z^{-n} - z^{m} \sum_{i=0}^{m-1} x(i) z^{-i}$$

$$= z^{m} x(z) - \sum_{i=0}^{m-1} x(i) z^{m-i} \quad \left\{ \because \text{Let } p=n \text{ in } 1^{st},\ p=i \text{ in } 2^{nd}. \right.$$

(3) Multiplication by 'n' (or Differentiation in z-domain)

Proof

If $Z[x(n)] = X(z)$ then $Z\{n\,x(n)\} = -z\frac{d}{dz}X(z)$

In general $Z[n^m x(n)] = \left(-z\frac{d}{dz}\right)^m X(z)$

$$X(z) = Z[x(n)] = \sum_{n=-\infty}^{\infty} x(n)z^{-n}$$

$$Z[n\,x(n)] = \sum_{n=-\infty}^{\infty} n\,x(n)z^{-n} = \sum_{n=-\infty}^{\infty} n\,x(n)z^{-n} z\, z^{-1}$$

$$= -z\sum_{n=-\infty}^{\infty} x(n)\left[-nz^{-n-1}\right]$$

$$= -z\sum_{n=-\infty}^{\infty} x(n)\left[\frac{d}{dz}z^{-n}\right] \qquad \left\{\because \frac{d}{dz}z^{-n} = -nz^{-n-1}\right.$$

$$= -z\frac{d}{dz}\sum_{n=-\infty}^{\infty} x(n)z^{-n}$$

$$Z[n\,x(n)] = -z\frac{d}{dz}X(z). \qquad \left\{\text{Interchanging summation \& differentiation}\right.$$

(4) Multiplication by an exponential sequence, a^n (scaling in z-domain)

If $Z\{x(n)\} = X(z)$ then $Z\{a^n x(n)\} = X(a^{-1}z)$

Proof

$$Z\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n)z^{-n}$$

$$Z[a^n x(n)] = \sum_{n=-\infty}^{\infty} a^n x(n)z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} x(n)(a^{-1}z)^{-n}$$

$$= \sum_{n=-\infty}^{\infty} x(n)(a^{-1}z)^{-n}$$

$$= X(a^{-1}z).$$

(5) Time Reversal :

If $z\{x(n)\} = X(z)$ then $z\{x(-n)\} = X(z^{-1})$

proof

$$z[x(n)] = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$z[x(-n)] = \sum_{n=-\infty}^{\infty} x(-n) z^{-n}$$

Let $p = -n$
when $n \to -\infty$, $p \to \infty$
$n \to \infty$, $p \to -\infty$

$$= \sum_{p=-\infty}^{\infty} x(p) z^{p}$$

$$= \sum_{p=-\infty}^{\infty} x(p) (z^{-1})^{-p}$$

$$= X(z^{-1})$$

(6) Conjugation :

If $z\{x(n)\} = X(z)$ then $z\{x^*(n)\} = X^*(z^*)$

proof

$$X(z) = z\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$z\{x^*(n)\} = \sum_{n=-\infty}^{\infty} x^*(n) z^{-n}$$

$$= \left[\sum_{n=-\infty}^{\infty} x(n) (z^*)^{-n} \right]^*$$

$$= [X(z^*)]^*$$

$$= X^*(z^*)$$

INVERSE Z-TRANSFORM:

Let $X(z)$ be z-transform of discrete time signal $x(n)$. The inverse z-transform is the process of recovering the discrete time sgl $x(n)$ from its z-transform $X(z)$.

→ The signal $x(n)$ can be uniquely determined from $X(z)$ & its ROC.

It can be determined by three methods

(i) Direct evaluation by contour integration (or residue method)
(ii) partial fraction expansion method
(iii) power series expansion method.

① Determine the inverse z-transform of fn $X(z) = \dfrac{3+2z^{-1}+z^{-2}}{1-3z^{-1}+2z^{-2}}$ by following three methods and prove that it is unique.

Sol (i) Residue Method:

$$X(z) = \frac{3+2z^{-1}+z^{-2}}{1-3z^{-1}+2z^{-2}} = \frac{\cancel{z^{-2}}(3z^2+2z+1)}{\cancel{z^{-2}}(z^2-3z+2)}$$

$$\therefore X(z) = \frac{3z^2+2z+1}{z^2-3z+2}$$

$$\begin{array}{r|l} & 3 \\ \hline z^2-3z+2 & 3\cancel{z^2}+2z+1 \\ & \underline{\pm 3z^2 \mp 9z \pm 6} \\ & 11z-5 \end{array}$$

$$= 3 + \frac{11z-5}{z^2-3z+2}$$

$$= 3 + \frac{11z-5}{(z-1)(z-2)}$$

Let $X_1(z) = 3$, $X_2(z) = \dfrac{11z-5}{(z-1)(z-2)}$; i.e $X(z) = X_1(z) + X_2(z)$

$$x(n) = z^{-1}\{X(z)\} = z^{-1}\{X_1(z)\} + z^{-1}\{X_2(z)\}$$

$$= z^{-1}\{3\} + z^{-1}\left\{\frac{11z-5}{(z-1)(z-2)}\right\}$$

$$= 3\delta(n) + \sum_{i=1}^{N}\left[(z-p_i)X_2(z)z^{n-1}\right]_{z=p_i}$$ Using residue theorem.

$$= 3\delta(n) + (z-1)\frac{11z-5}{(z-1)(z-2)} \cdot z^{n-1}\bigg|_{z=1} + (z-2)\cdot\frac{11z-5}{(z-1)(z-2)}\cdot z^{n-1}\bigg|_{z=2}$$

$$= 3\delta(n) + \frac{11-5}{1-2}(1)^{n-1} + \frac{11\times 2-5}{2-1}\,2^{n-1}$$

$$\therefore\ x(n) = 3\delta(n) - 6u(n-1) + 17(2)^{n-1}u(n-1)$$

$$= 3\delta(n) + \left[-6 + 17(2)^{n-1}\right]u(n-1).$$

when $n=0$, $x(0) = 3-0+0 = 3$

when $n=1$, $x(1) = 0-6+17\times 2^0 = 11$

when $n=2$, $x(2) = 0-6+17\times 2^1 = 28$

when $n=3$ $x(3) = 0-6+17\times 2^2 = 62$

$$\therefore\ x(n) = \{3, 11, 28, 62, 130 \ldots\}$$

Method 2:

$$x(z) = \frac{3+2z^{-1}+z^{-2}}{1-3z^{-1}+2z^{-2}} = \frac{3z^2+2z+1}{z^2-3z+2}$$

$$\therefore\ \frac{x(z)}{z} = \frac{3z^2+2z+1}{z(z-1)(z-2)}$$

Let $\frac{x(z)}{z} = \frac{3z^2+2z+1}{z(z-1)(z-2)} = \frac{A_1}{z} + \frac{A_2}{z-1} + \frac{A_3}{z-2}$

$$A_1 = z.\frac{x(z)}{z}\bigg|_{z=0} = \cancel{z}\cdot\frac{3z^2+2z+1}{\cancel{z}(z-1)(z-2)}\bigg|_{z=0} = 0.5$$

$$A_2 = (z-1)\frac{x(z)}{z}\bigg|_{z=1} = \cancel{(z-1)}\frac{3z^2+2z+1}{z\cancel{(z-1)}(z-2)}\bigg|_{z=1} = -6$$

$$A_3 = (z-2)\frac{x(z)}{z}\bigg|_{z=2} = \cancel{(z-2)}\cdot\frac{3z^2+2z+1}{z(z-1)\cancel{(z-2)}}\bigg|_{z=2} = 8.5$$

$$\frac{x(z)}{z} = \frac{0.5}{z} - \frac{6}{z-1} + \frac{8.5}{z-2}.$$

$$\therefore X(z) = 0.5 - 6 \cdot \frac{z}{z-1} + 8.5 \, \frac{z}{z-2}$$

On taking inverse z-transform

$$x(n) = 0.5\,\delta(n) - 6\,u(n) + 8.5\,(2)^n u(n)$$

$$= 0.5\,\delta(n) + \left[-6 + 8.5(2)^n\right] u(n)$$

When $n=0$, $x(0) = 0.5 - 6 + 8.5 \times 2^0 = 3$

$n=1$, $x(1) = 0 - 6 + 8.5 \times 2^1 = 11$

$n=2$ $x(2) = 0 - 6 + 8.5 \times 2^2 = 28$

$n=3$, $x(3) = 0 - 6 + 8.5 \times 2^3 = 62$

$$\therefore x(n) = \{\underset{\uparrow}{3}, 11, 28, 62 \cdots\}$$

Method 3: Power Series Expansion method.

Quotient: $3 + 11z^{-1} + 28z^{-2} + 62z^{-3} + 130z^{-4} + \cdots$

Divisor: $1 - 3z^{-1} + 2z^{-2}$

$$\cancel{3} + 2z^{-1} + z^{-2}$$
$$\cancel{3} - 9z^{-1} + 6z^{-2}$$
$$11\cancel{z^{-1}} - 5z^{-2}$$
$$\cancel{11}z^{-1} - 33z^{-2} + 22z^{-3}$$
$$28\cancel{z^{-2}} - 22z^{-3}$$
$$\cancel{28}z^{-2} - 84z^{-3} + 56z^{-4}$$
$$62\cancel{z^{-3}} - 56z^{-4}$$
$$\cancel{62}z^{-3} - 186z^{-4} + 124z^{-5}$$
$$130z^{-4} - 124z^{-5}$$
$$\vdots$$

$$\therefore X(z) = 3 + 11z^{-1} + 28z^{-2} + 62z^{-3} + 130z^{-4} + \cdots \quad \rightarrow (1)$$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} = \cdots + x(0) + x(1)z^{-1} + x(2)z^{-2} + \cdots \quad \rightarrow (2)$$

On comparing (1) & (2)

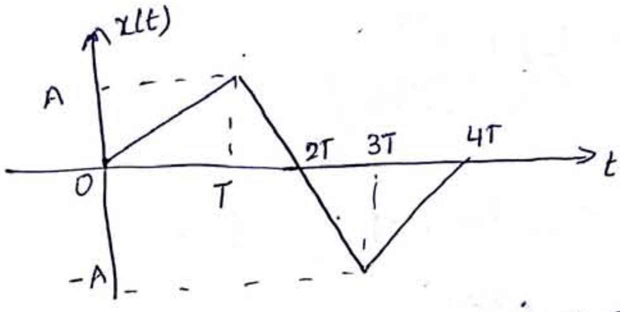
$x(0) = 3$, $x(1) = 11$, $x(2) = 28$, $x(3) = 62$

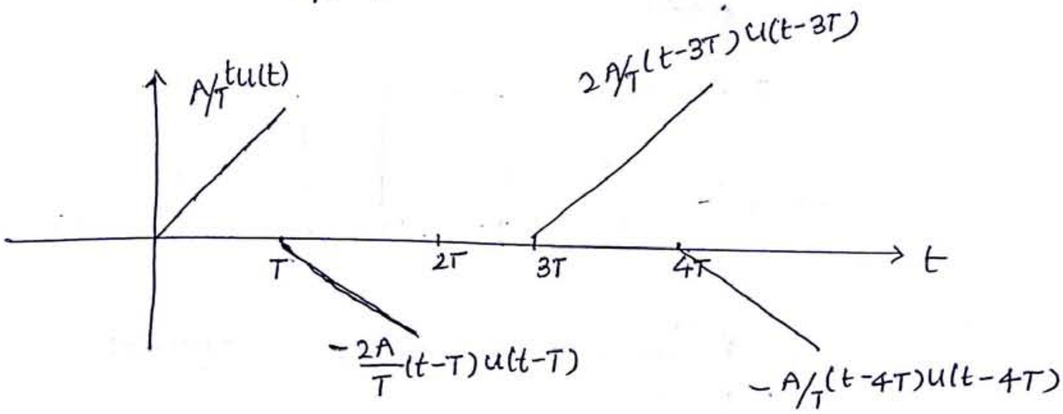
$$\therefore x(n) = \{3, 11, 28, 62 \cdots\}$$

Wave synthesis Using Laplace transform:

To express the function into singular functions and express it in synthesis

① Find the Laplace transform of the waveform.





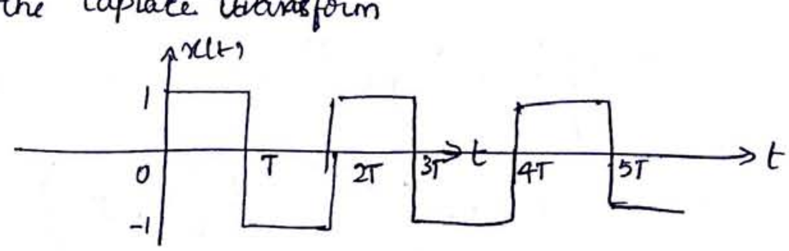
$$x(t) = \frac{A}{T} t\, u(t) - \frac{2A}{T}(t-T)\, u(t-T) + \frac{2A}{T}(t-3T)\, u(t-3T)$$
$$- \frac{A}{T}(t-4T)\, u(t-4T)$$

Taking Laplace transform on both sides

$$X(s) = \frac{A}{T}\frac{1}{s^2} - \frac{2A}{T}\frac{e^{-Ts}}{s^2} + \frac{2A}{T}\frac{e^{-3Ts}}{s^2} - \frac{A}{T}\frac{e^{-4Ts}}{s^2}$$

$$= \frac{A}{Ts^2}\left[1 - 2e^{-Ts} + 2e^{-3Ts} - e^{-4Ts}\right]$$

② Find the Laplace transform



$$x(t) = u(t) - 2u(t-T) + 2u(t-2T) - 2u(t-3T) + 2u(t-4T) - 2u(t-5T) + \ldots$$

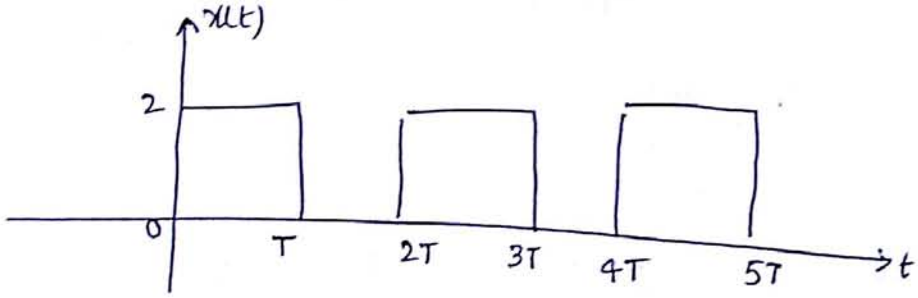
$$X(s) = \frac{1}{s} - 2\frac{e^{-Ts}}{s} + \frac{2e^{-2Ts}}{s} - \frac{2e^{-3Ts}}{s} + \frac{2e^{-4Ts}}{s} - \frac{2e^{-5Ts}}{s} + \ldots$$

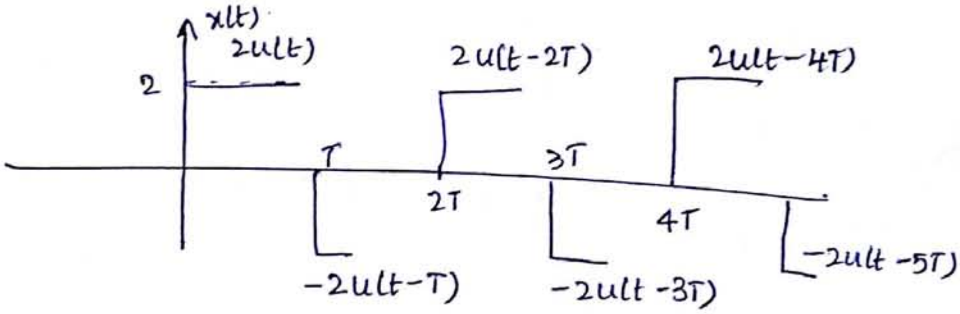
$$= \frac{1}{s}\left[1 - 2e^{-Ts}\left(1 - e^{-Ts} + e^{-2Ts} - e^{-3Ts} + e^{-4Ts} + \cdots\right]\right.$$

$$= \frac{1}{s}\left[1 - 2e^{-Ts}(1+e^{-Ts})^{-1}\right] = \frac{1}{s}\left(1 - \frac{2e^{-Ts}}{1+e^{-Ts}}\right) = \frac{1}{s}\left(\frac{1+e^{-Ts} - 2e^{-Ts}}{1+e^{-Ts}}\right)$$

$$= \frac{1}{s}\left(\frac{1-e^{-Ts}}{1+e^{-Ts}}\right)$$

(3) Find the Laplace transform of waveform





$$x(t) = 2u(t) - 2u(t-T) + 2u(t-2T) - 2u(t-3T) + 2u(t-4T) - 2u(t-5T) \cdots$$

Taking Laplace transform

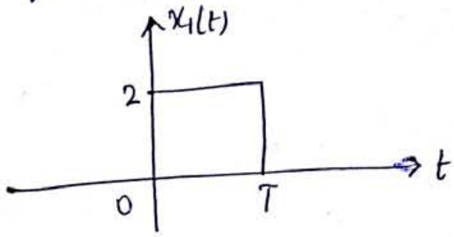
$$X(s) = \frac{2}{s} - \frac{2e^{-Ts}}{s} + \frac{2e^{-2Ts}}{s} - \frac{2e^{-3Ts}}{s} + \frac{2e^{-4Ts}}{s} - \frac{2e^{-5Ts}}{s} \cdots$$

$$= \frac{2}{s}\left[1 - e^{-Ts} + e^{-2Ts} - e^{-3Ts} + e^{-4Ts} - e^{-5Ts} \cdots\right]$$

$$= \frac{2}{s}\left[1+e^{-Ts}\right]^{-1} = \frac{2}{s}\left(\frac{1}{1+e^{-Ts}}\right)$$

Another method

The given waveform is periodic with period 2T.



periodicity property

$$X(s) = \frac{1}{1-e^{-2Ts}} X_1(s)$$

$$x_1(t) = 2\left[u(t) - u(t-T)\right]$$

$$X_1(s) = 2\left(\frac{1}{s} - \frac{e^{-Ts}}{s}\right) = \frac{2}{s}\left(1-e^{-Ts}\right)$$

$$X(s) = \frac{1}{1-e^{-2Ts}}\left(\frac{2}{s}\right)\left(1-e^{-Ts}\right)$$

$$= \frac{2}{s} \cdot \frac{1-e^{-Ts}}{(1+e^{-Ts})(1-e^{-Ts})} = \frac{2}{s}\left(\frac{1}{1+e^{-Ts}}\right)$$

periodicity property:

Used in determining the transform of periodic time functions.

$x(t) = x(t+nT)$. where T is period , $n=0,1,2\ldots$

$$X(s) = \int_0^{\infty} x(t) e^{-st}\, dt$$

$$X(s) = \int_0^T x(t) e^{-st} + \int_T^{2T} x(t) e^{-st} + \int_{2T}^{3T} x(t) e^{-st} dt + \cdots + \int_{nT}^{(n+1)T} x(t) e^{-st} dt + \cdots$$

$$= \int_0^T x(t) e^{-st} dt + e^{-sT}\int_0^T x(t+T) e^{-st} dt + \cdot + e^{-nsT}\int_0^T x(t+nT) e^{-st} dt$$

$$x(t) = x(t+T) = x(t+2T) \ldots$$

$$X(s) = \int_0^T x(t)\, e^{-st} dt + e^{-sT}\int_0^T x(t)\, e^{-st} dt + \cdots + e^{-nsT}\int_0^T x(t)\, e^{-st} dt + \cdots$$

$$= \left[1 + e^{-sT} + e^{-2sT} + \cdots + e^{-nsT}\right] \int_0^T x(t) e^{-st} dt$$

$$= \left[1 - e^{-sT}\right]^{-1} \int_0^T x(t)\, e^{-st} dt = \frac{1}{1 - e^{-sT}} X_1(s).$$

where $X_1(s) = \int_0^T x(t)\, e^{-st}\, dt$