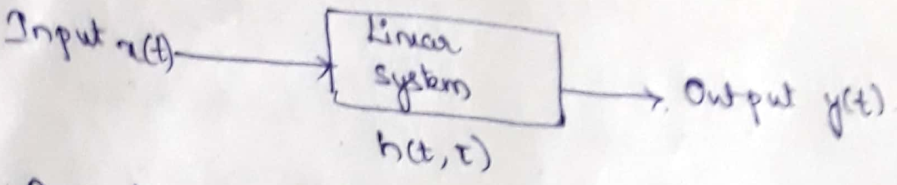
Linear Systems - fundamentals:



→ A system can be represented as

$$y(t) = L[x(t)] \quad \text{—— (1)}$$

where L is an operator representing the action of the system on $x(t)$.

→ A system is said to be linear if its response to a sum of i/ps $x_n(t)$, $n = 1, 2, \ldots N$, is equals to the sum of responses taken separately.

$$\therefore y(t) = L\left[\sum_{n=1}^{N} \alpha_n x_n(t)\right] = \sum_{n=1}^{N} \alpha_n L[x_n(t)]$$

$$= \sum_{n=1}^{N} \alpha_n y_n(t) \quad \text{—— (2)}$$

where α_n are arbitrary constants.

→ from the properties of the impulse fn, we have.

$$x(t) = \int_{-\infty}^{\infty} x(\xi)\, \delta(t-\xi)\, d\xi \quad \text{—— (3)}$$

→ By substituting eq. (3) in eq. (1), we obtain.

$$y(t) = L[x(t)] = L\left[\int_{-\infty}^{\infty} x(\xi)\, \delta(t-\xi)\, d\xi\right]$$

$$= \int_{-\infty}^{\infty} x(\xi) \cdot \underbrace{L[\delta(t-\xi)]}\, d\xi \quad \text{—— (4)}$$

→ Now, define a new fn. $h(t,\xi)$ as the impulse resp. of the linear system, ie

$$L[\delta(t-\xi)] = h(t,\xi).$$

eq. (4) becomes

$$y(t) = \int_{-\infty}^{\infty} x(\xi_1)\, h(t, \xi_1)\, d\xi_1 \quad \text{——— (5)}$$

which shows that the response of a general linear system is completely determined by its impulse response.

Mean & Mean Squared Value of System Response:

→ To find the mean of the sys. response, assume X(t) is WSS, we have

$$E[Y(t)] = E\left[\int_{-\infty}^{\infty} h(\xi_1)\, X(t-\xi_1)\, d\xi_1\right]$$

$$= \int_{-\infty}^{\infty} h(\xi_1) \cdot E[X(t-\xi_1)]\, d\xi_1$$

$$= \bar{X} \cdot \int_{-\infty}^{\infty} h(\xi_1)\, d\xi_1 = \bar{Y} = \text{const}$$

→ This expression indicates that the mean value of Y(t) equals the mean value of X(t) times the area under the impulse resp. if X(t) is WSS.

→ for mean-squared value of Y(t),

$$E[Y^2(t)] = E\left[\int_{-\infty}^{\infty} h(\xi_1) \cdot X(t-\xi_1)\, d\xi_1 \cdot \int_{-\infty}^{\infty} h(\xi_2)\, X(t-\xi_2)\, d\xi_2\right]$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E[X(t-\xi_1)\, X(t-\xi_2)] \cdot h(\xi_1)\, h(\xi_2)\, d\xi_1\, d\xi_2$$

→ If we assume, the i/p is WSS, then

$$E[X(t-\xi_1)\, X(t-\xi_2)] = R_{XX}(\xi_1 - \xi_2)$$

$$\therefore \overline{Y^2} = E[Y^2(t)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_{XX}(\xi_1-\xi_2) \cdot h(\xi_1)\, h(\xi_2)\, d\xi_1\, d\xi_2$$

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Auto correlation fn. of Response

Let $X(t)$ be WSS. The auto correlation fn. of $Y(t)$ is

$$R_{YY}(t, t+\tau) = E[Y(t)\,Y(t+\tau)].$$

$$= E\left[\int_{-\infty}^{\infty} h(\xi_1)\,x(t-\xi_1)\,d\xi_1 \cdot \int_{-\infty}^{\infty} h(\xi_2)\,x(t+\tau-\xi_2)\,d\xi_2\right]$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \underbrace{E[x(t-\xi_1)\,x(t+\tau-\xi_2)]} h(\xi_1)\,h(\xi_2)\,d\xi_1\,d\xi_2$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_{XX}(\tau+\xi_1-\xi_2)\,h(\xi_1)\,h(\xi_2)\,d\xi_1\,d\xi_2.$$

$$= R_{YY}(\tau)$$

Note:

→ ∴ $Y(t)$ is WSS if $X(t)$ is WSS, because $R_{YY}(\tau)$ does not depend on t & $E[Y(t)]$ is a const.

→ $R_{YY}(\tau)$ is the two fold convolution of the i/p auto correlation fn. with the n/w.s impulse response. i.e

$$\boxed{R_{YY}(\tau) = R_{XX}(\tau) * h(-\tau) * h(\tau)}$$

Cross correlation fn.s of i/p & o/p:

→ The cross correlation fn. of $X(t)$ & $Y(t)$ is

$$R_{XY}(t, t+\tau) = E[X(t)\,Y(t+\tau)].$$

$$= E\left[X(t)\cdot\int_{-\infty}^{\infty} h(\xi)\,x(t+\tau-\xi)\,d\xi\right]$$

$$= \int_{-\infty}^{\infty} E[X(t)\,X(t+\tau-\xi)]\,h(\xi)\,d\xi \quad \text{—①}$$

→ If $X(t)$ is WSS,

$$\therefore R_{XY}(\tau) = \int_{-\infty}^{\infty} R_{XX}(\tau-\xi)\,h(\xi)\,d\xi \quad \text{—②}$$

which is the convolution of $R_{xx}(t)$ with $h(\tau)$.

$$\therefore R_{xy}(\tau) = R_{xx}(\tau) * h(\tau).$$

Similarly,

$$R_{yx}(\tau) = \int_{-\infty}^{\infty} R_{xx}(\tau - \xi_1)\, h(-\xi_1)\, d\xi_1 \quad \text{--- } (3)$$

$$= R_{xx}(\tau) * h(-\tau).$$

→ from eqs (2) & (3);

It is clear that the cross correlation fns depend on τ and not on absolute time t.

$\therefore$ $X(t)$ & $Y(t)$ are jointly WSS, if $X(t)$ is WSS, because $Y(t)$ already is WSS.

→ $$R_{yy}(t) = \int_{-\infty}^{\infty} R_{xy}(\tau + \xi_1)\, h(\xi_1)\, d\xi_1$$

$$\therefore \boxed{R_{yy}(\tau) = R_{xy}(\tau) * h(-\tau)}$$

→ Similarly,

$$R_{yy}(t) = \int_{-\infty}^{\infty} R_{yx}(t - \xi_2)\, h(\xi_2)\, d\xi_2$$

$$\boxed{R_{yy}(t) = R_{yx}(\tau) * h(t)}$$

Gaussian Random Processes

Consider a continuous RP, and define N RVs

$X_1 = X(t_1), X_2 = X(t_2), \ldots X_i = X(t_i), \ldots X_N = X(t_N)$ corresponding to N time instants $t_1, \ldots t_i, \ldots t_N$.

→ If, for any $N = 1, 2, \ldots$ and any times $t_1, t_2, \ldots t_N$, these RVs are jointly gaussian, ie they have a joint density, then the process is called Gaussian.

→ The joint density is given by:

$$f_X(x_1, \ldots x_N; t_1, \ldots t_N) = \frac{\exp\{-(1/2)[x-\bar{X}]^t [C_X]^{-1} [x-\bar{X}]\}}{\sqrt{(2\pi)^N \cdot |[C_X]|}}$$

where
→ $[x-\bar{X}]$ & $[C_X]$ are matrices.

→ The mean values $\bar{X}_i$ of $X_i = X(t_i)$ are

$$\bar{X}_i = E[X_i] = E[X(t_i)]$$

→ The elements of the covariance matrix $[C_X]$ are

$$C_{ik} = C_{X_i X_k} = E[(X_i - \bar{X}_i)(X_k - \bar{X}_k)]$$

$$= E[\{X(t_i) - E[X(t_i)]\}\{X(t_k) - E[X(t_k)]\}]$$

$= C_{XX}(t_i, t_k)$ → the auto covariance of $X(t_i)$ & $X(t_k)$.

→ WKT

$$C_{XX}(t_i, t_k) = R_{XX}(t_i, t_k) - E[X(t_i)]E[X(t_k)]$$

→ If the gaussian process is not stationary the mean & auto covariance fn.s depend on absolute time.

for a WSS process, the mean will be const.

$$\overline{X_i} = E[X(t_i)] = \overline{X} \text{ (const.)}$$

while the auto covariance & auto correlation fn.s will depend only on time diff.s & not absolute time.

$$C_{XX}(t_i, t_k) = C_{XX}(t_k - t_i)$$

$$R_{XX}(t_i, t_k) = R_{XX}(t_k - t_i)$$

Poisson Random Process:

→ It is an example of Discrete Random Process.

→ It describes the no. of times that some event has occured as a fn. of time, where the events occur at random times.

→ Examples:

1. Arrival of a customer at a bank.
2. Super market check-out register
3. failure of some component in a system.
4. Emission of an electron from the surface of photo detector etc.

→ A single event occurs at a random time & the process amounts to count the no. of such occurences with time. For this reason, the process is also known as poisson counting process.

→ To define the Poisson RP, we require two conditions.

1. The event occur only once in any small interval of time
2. The occurence times be statistically independent so that the no. that occurs in any given time interval is independent of the no in any other non-overlapping time interval.

A consequence of the 2 conditions is that the no. of event occurences in any finite interval of time is described by the poisson distribution where the avg. rate of occurences is denoted by λ.

→ The prob. of exactly 'k' occurens over a time interval $(0,t)$ is

$$P[X(t)=k] = \frac{(\lambda t)^k e^{-\lambda t}}{k!}, \quad k=0,1,2\ldots$$

→ ∴ The prob. density of the no of occurences is

$$f_X(x) = \sum_{k=0}^{\infty} \frac{(\lambda t)^k e^{-\lambda t}}{k!} \delta(x-k)$$

→ The mean of the Poisson RP is

$$E[X(t)] = \int_{-\infty}^{\infty} x \cdot f_X(x)\, dx = \int_{-\infty}^{\infty} x \sum_{k=0}^{\infty} \frac{(\lambda t)^k e^{-\lambda t}}{k!} \delta(x-k)\, dx$$

$$= \sum_{k=0}^{\infty} \frac{k(\lambda t)^k e^{-\lambda t}}{k!} = \underline{\lambda t}$$

→ the mean squared value is

$$E[X^2(t)] = \int_{-\infty}^{\infty} x^2 f_X(x)\, dx = \sum_{k=0}^{\infty} \frac{k^2(\lambda t)^k e^{-\lambda t}}{k!}$$

$$= \lambda t[1+\lambda t]$$

→ The variance is also λt

→ The joint probability density fn is given by

$$f_X(x_1,x_2) = \sum_{k_1=0}^{\infty} \sum_{k_2=k_1}^{\infty} P(k_1,k_2)\, \delta(x_1-k_1)\, \delta(x_2-k_2)$$

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Random Processes – Spectral Characteristics.

The spectral description of a deterministic wave-form is obtained by Fourier Transforming the Wave-form.

→ ∴ The spectral properties of a deterministic signal $x(t)$ are contained in its F.T $X(\omega)$ given by.

$$X(\omega) = \int_{-\infty}^{\infty} x(t) \cdot e^{-j\omega t} \, dt.$$

→ The fn. $X(\omega)$ is the spectrum of $x(t)$, has the unit of volts per hertz. when $x(t)$ is a voltage.

→ It describes the way in which relative signal voltage is distributed with freq.

→ The F.T can be considered to be a voltage density spectrum applicable to $x(t)$. Both the amplitudes & phases of the frequencies present in $x(t)$ are described by $X(\omega)$.

→ If $X(\omega)$ is known then $x(t)$ can be recovered by means of the Inverse Fourier Transform.

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) \cdot e^{j\omega t} \, d\omega.$$

Power Density Spectrum:

→ For a RP $X(t)$, let $x_T(t)$ be defined as a sample fn. $x(t)$ that exists between $-T$ and T, ie

$$x_T(t) = \begin{cases} x(t) & -T < t < T \\ 0 & \text{elsewhere.} \end{cases}$$

→ The Fourier Transform $X_T(\omega)$ is given by.

$$X_T(\omega) = \int_{-T}^{T} x_T(t) \cdot e^{-j\omega t} \, dt = \int_{-T}^{T} x(t) \cdot e^{-j\omega t} \, dt$$

Energy contained in $x(t)$ in the interval $(-T,T)$ is

$$E(T) = \int_{-T}^{T} x_T^2(t)\,dt = \int_{-T}^{T} x^2(t)\,dt.$$

* By Parseval's theorem,

$$E(T) = \int_{-T}^{T} x^2(t)\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X_T(\omega)|^2\,d\omega. \quad \text{— (1)}$$

→ By dividing eq. (1) by $2T$, we obtain the avg. power $P(T)$ in $x(t)$ over the interval $(-T,T)$:

$$\therefore P(T) = \frac{1}{2T}\int_{-T}^{T} x^2(t)\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{|X_T(\omega)|^2}{2T}\,d\omega \quad \text{— (2)}$$

We observe that $\frac{|X_T(\omega)|^2}{2T}$ is a power density spectrum because the power results through its integration.

→ $P(T)$ is actually a random variable w.r.to the RP. By taking the expected value in eq. (2), we can obtain an avg. power P_{XX} for the Random Process.

$$\therefore P_{XX} = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} E[x^2(t)]\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \lim_{T\to\infty} \frac{E[|X_T(\omega)|^2]}{2T}\,d\omega \quad \text{— (3)}$$

↓ extending to all T values.

→ From eq. (3), we observed that.

(1) The avg. power P_{XX} in a RP $X(t)$ is given by the time avg. of its second moment.

$$P_{XX} = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} E[x^2(t)]\,dt = A\{E[x^2(t)]\}$$

For a WSS Process; $E[x^2(t)] = \overline{X^2} = \underline{\text{const}}$.

& $P_{XX} = \overline{X^2}$

P_{xx} can be obtained by a freq. domain integration

The power density spectrum for the RP is

$$S_{XX}(\omega) = \lim_{T\to\infty} \frac{E[|X_T(\omega)|^2]}{2T}$$

& the power is given by.

$$P_{XX} = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(\omega)\, d\omega.$$

Prob: Consider the RP $X(t) = A_0 \cos(\omega_0 t + \theta)$ where A_0 & ω_0 are real const.s & θ is a RV uniformly distributed on the interval $(0, \pi/2)$. Find the avg. power P_{xx} in $x(t)$.

Sol:

$P_{xx} = A\{E[x^2(t)]\}$

$$E[x^2(t)] = E[A_0^2 \cos^2(\omega_0 t + \theta)]$$

$\cos 2\theta = 2\cos^2\theta - 1$

$$= E\left[\frac{A_0^2}{2} + \frac{A_0^2}{2}\cos(2\omega_0 t + 2\theta)\right]$$

$$= \frac{A_0^2}{2} + \frac{A_0^2}{2}\int_0^{\pi/2} \cos(2\omega_0 t + 2\theta)\cdot\frac{1}{\pi/2}\cdot d\theta$$

$\left.\frac{\sin(2\omega_0 t + 2\theta)}{2}\right]_0^{\pi/2}$

$$= \frac{A_0^2}{2} - \frac{A_0^2}{\pi}\cdot\sin(2\omega_0 t)$$

$$\therefore A\{E[x^2(t)]\} = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\left[\frac{A_0^2}{2} - \frac{A_0^2}{\pi}\sin(2\omega_0 t)\right]dt$$

$$= \frac{A_0^2}{2}$$

$$\therefore P_{xx} = A\{E[x^2(t)]\} = A_0^2/2$$

Properties of the Power Density Spectrum:

1. $S_{XX}(\omega) \geq 0$.

2. $S_{XX}(-\omega) = S_{XX}(\omega)$; $x(t)$ real.

3. $S_{XX}(\omega)$ is real.

4. $\frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(\omega)\, d\omega = A\{E[x^2(t)]\}$.

5. $S_{\dot{X}\dot{X}}(\omega) = \omega^2 S_{XX}(\omega)$

It says that the power density spectrum of the derivative $\dot{x}(t) = \frac{d}{dt}x(t)$ is ω^2 times the power spectrum of $x(t)$.

** 6. $\frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(\omega).e^{j\omega\tau}\, d\omega = A[R_{XX}(t, t+\tau)]$

$\therefore S_{XX}(\omega) = \int_{-\infty}^{\infty} A[R_{XX}(t,t+\tau)].e^{-j\omega\tau}\, d\tau$.

It states that the power density spectrum & the time avg. of the auto correlation fn. form a FT pair.

→ If $x(t)$ is atleast WSS,

$$A[R_{XX}(t,t+\tau)] = R_{XX}(\tau)$$

then prop. 6 indicates that the power spectrum &. the auto correlation fn. form a F.T pair.

Thus

$$S_{XX}(\omega) = \int_{-\infty}^{\infty} R_{XX}(\tau).e^{-j\omega\tau}\, d\tau.$$

&

$$R_{XX}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(\omega).e^{j\omega\tau}\, d\omega.$$

for a WSS Process.

Band width of the power Density spectrum:

- Assume that x(t) is a low pass process. ie its spectral components are clustered near $\omega=0$ & have decreasing magnitudes at higher ~~freqs~~. freq's. Such processes are also called Base band.

→ Then the area of $S_{XX}(\omega)$ is not necessarily unity, $S_{XX}(\omega)$ has characteristics similar to the prob. density fn.

→ The normalized power spectrum is a measure of its spread that call as rms bandwidth, which we denote W_{rms} (rad/sec).

→ Since $S_{XX}(\omega)$ is an even fn. for a real process, its 'mean value' is 'zero' & its standard deviation is the sqr. root of its second moment.

→ ∴ The rms (root mean squared) Bandwidth is given by

$$W_{rms}^2 = \frac{\int_{-\infty}^{\infty} \omega^2 \cdot S_{XX}(\omega)\, d\omega}{\int_{-\infty}^{\infty} S_{XX}(\omega)\, d\omega}.$$

→ If the process x(t) is real, $S_{XX}(\omega)$ will be real & have even symmetry about $\omega=0$, We define a mean freq. $\overline{\omega_0}$ by.

$$\overline{\omega_0} = \frac{\int_0^{\infty} \omega\, S_{XX}(\omega)\, d\omega}{\int_0^{\infty} S_{XX}(\omega)\, d\omega}$$

& rms band width by

$$W_{rms}^2 = \frac{4\int_0^{\infty} (\omega-\overline{\omega_0})^2 \cdot S_{XX}(\omega)\, d\omega}{\int_0^{\infty} S_{XX}(\omega)\, d\omega}.$$

Relationship b/w Power Spectrum & Auto Correlation fn:

The inverse F.T of the power density spectrum is the time avg. of the auto correlation fn. i.e.

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(\omega).\, e^{j\omega\tau}\, d\omega = A\left[R_{XX}(t,t+\tau)\right]$$

This expression will now be proved.

→ We know that $\quad |X_T(\omega)|^2 = X_T(\omega).X_T^*(\omega)$

$$S_{XX}(\omega) = \lim_{T\to\infty} \frac{E[|X_T(\omega)|^2]}{2T}$$

from the def. of $X_T(\omega) = \begin{cases} x(t) & -T < t < T \\ 0 & \text{elsewhere.} \end{cases}$

$$S_{XX}(\omega) = \lim_{T\to\infty} E\left[\frac{1}{2T}.\underbrace{\int_{-T}^{T} X(t_1).\, e^{j\omega t_1}\, dt_1}_{X_T(\omega)} . \underbrace{\int_{-T}^{T} X(t_2).e^{-j\omega t_2}\, dt_2}_{X_T^*(\omega)}\right]$$

$$= \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T}\int_{-T}^{T} E[X(t_1)X(t_2)] . e^{-j\omega(t_2-t_1)}\, dt_2\, dt_1.$$

WKT, $\quad E[X(t_1)X(t_2)] = R_{XX}(t_1,t_2) \qquad -T<(t_1 \,\&\, t_2)<T$

$$\therefore S_{XX}(\omega) = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T}\int_{-T}^{T} R_{XX}(t_1,t_2). e^{-j\omega(t_2-t_1)}\, dt_1\, dt_2 \quad —(1)$$

→ take I.F.T of eq. (1).

The I.F.T will exist since the power spectrum is real, non-negative & absolutely integrable.

∴ The inverse F.T is

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(\omega)\, e^{j\omega\tau}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T}\int_{-T}^{T} R_{XX}(t,t_1) . e^{-j\omega(t_1-t)}\, dt.\, dt_1 . e^{j\omega\tau}\, d\omega.$$

$$= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T}\int_{-T}^{T} R_{XX}(t,t_1) \cdot \underbrace{\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{j\omega(\tau - t_1 + t)}\, d\omega}_{\downarrow}\, dt_1\, dt$$

$$2\pi\delta(\tau - t_1 + t)$$

$$= 2\pi\delta(t_1 - t - \tau)$$

$$\therefore \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(\omega)\, d\omega = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T}\int_{-T}^{T} R_{XX}(t,t_1)\, \underbrace{\delta(t_1 - t - \tau)}_{\downarrow}\, dt_1\, dt$$

exists only at

$t_1 = t + \tau$

$$\therefore = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} R_{XX}(t, t+\tau)\, dt.$$

$$\therefore \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(\omega)\, d\omega = A[R_{XX}(t, t+\tau)].$$

which indicates the Inverse F.T of the PDS is the time Avg. of the process auto correlation.

$$\Rightarrow S_{XX}(\omega) = \int_{-\infty}^{\infty} A[R_{XX}(t,t+\tau)] \cdot e^{-j\omega\tau}\, d\tau.$$

shows that $S_{XX}(\omega)$ & $A[R_{XX}(t,t+\tau)]$ form a F.T pair.

$$\therefore A[R_{XX}(t,t+\tau)] \longleftrightarrow S_{XX}(\omega)$$

* for the important case, where $X(t)$ is atleast WSS.

$A[R_{XX}(t,t+\tau)] = R_{XX}(\tau)$ & we get.

$$\boxed{\begin{aligned} \therefore S_{XX}(\omega) &= \int_{-\infty}^{\infty} R_{XX}(\tau)\, e^{-j\omega\tau}\, d\tau \quad \longrightarrow (A) \\ \text{(or)} \quad R_{XX}(\tau) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(\omega)\, e^{j\omega\tau}\, d\omega \quad \longrightarrow (B) \end{aligned}}$$

$$R_{XX}(\tau) \longleftrightarrow S_{XX}(\omega)$$

The expressions (A) & (B) are usually called the

* Wiener ~~Kinc~~ Khintchine relations.

Find the power spectrum for the RP $x(t) = A\cos(\omega_0 t + \theta)$ that has the auto correlation fn.

$$R_{xx}(\tau) = \frac{A_0^2}{2}\cos\omega_0\tau. \quad \text{where } A_0 \text{ \& } \omega_0 \text{ are}$$

constants.

Sol:

$$S_{xx}(\omega) = FT[R_{xx}(\tau)].$$

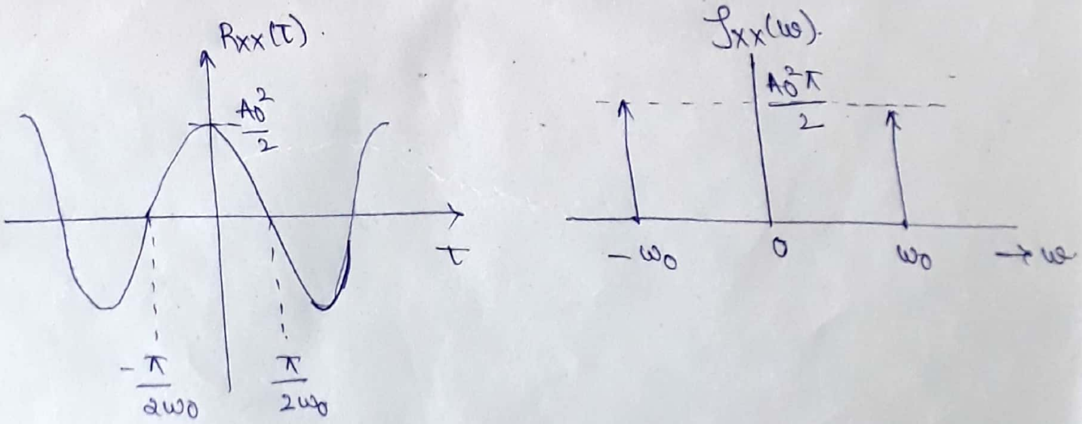
We can write as $R_{xx}(\tau) = \frac{A_0^2}{2}\left[\frac{e^{j\omega_0\tau} + e^{-j\omega_0\tau}}{2}\right]$

$$S_{xx}(\omega) = \int_{-\infty}^{\infty} \frac{A_0^2}{4}(e^{j\omega_0\tau} + e^{-j\omega_0\tau}).\ e^{-j\omega\tau}\,d\tau.$$

$$= \frac{A_0^2}{4}\left[\int_{-\infty}^{\infty} e^{-j(\omega+\omega_0)\tau}\,d\tau + \int_{-\infty}^{\infty} e^{-j(\omega-\omega_0)\tau}\,d\tau\right]$$

$$= \frac{A_0^2}{4}\left[2\pi\delta(\omega+\omega_0) + 2\pi\delta(\omega-\omega_0)\right]$$

$$S_{xx}(\omega) = \frac{A_0^2\pi}{2}\left[\delta(\omega+\omega_0) + \delta(\omega-\omega_0)\right]$$



(a) auto correlation fn.   (b) power spectrum density

For a WSS RP $x(t)$, the auto correlation fn is given by

$$R_{xx}(\tau) = \begin{cases} A_0\left[1-\left(|\tau|/T\right)\right] & -T \le t \le T \\ 0 & \text{elsewhere.} \end{cases}$$

where $T>0$ & A_0 are constants. find the power spectrum.

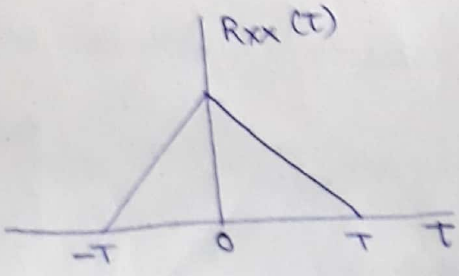
Sol:

$$S_{xx}(\omega) = \int_{-\infty}^{\infty} R_{xx}(\tau)\cdot e^{-j\omega\tau}\, d\tau$$

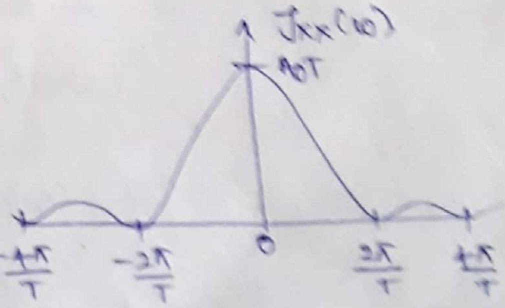
$$= A_0 \int_{-T}^{0} [1+\tau/T]\, e^{-j\omega\tau}\, d\tau + A_0 \int_{0}^{T} [1-\tau/T]\, e^{-j\omega\tau}\, d\tau$$

$$\Rightarrow A_0 \cdot T\ Sa^2\left(\frac{\omega T}{2}\right).$$

F.T [Triangular fn] = Sinc fn.



a) auto correlation fn.



b) power spectrum fn.

Cross Power Density Spectrum & Its Properties:

consider a real RP $W(t)$ given by the sum of two other real processes $X(t)$ & $Y(t)$

$$\therefore \; W(t) = X(t) + Y(t). \quad \text{— (1)}$$

→ The auto correlation of $W(t)$ is

$$R_{WW}(t, t+\tau) = E[W(t)\,W(t+\tau)]$$

$$= E[\{X(t)+Y(t)\}\{X(t+\tau)+Y(t+\tau)\}]$$

$$= R_{XX}(t,t+\tau) + R_{XY}(t,t+\tau) + R_{YX}(t,t+\tau) + R_{YY}(t,t+\tau) \quad \text{— (2)}$$

→ take time avg. of both sides of eq. (2) & F.T, the resulting expression we have

$$S_{WW}(\omega) = S_{XX}(\omega) + S_{YY}(\omega) + F\{A[R_{XY}(t,t+\tau)]\} + F\{A[R_{YX}(t,t+\tau)]\}. \quad \text{— (3)}$$

$S_{XX}(\omega) \rightarrow$ power spectrum of $X(t)$

$S_{YY}(\omega) \rightarrow$ power spectrum of $Y(t)$.

& $F\{A[R_{XY}(t,t+\tau)]\}$ & $F\{A[R_{YX}(t,t+\tau)]\}$ are Cross power density spectrums.

Cross power density spectrum:

→ for two real RPs $X(t)$ & $Y(t)$, define $x_T(t)$ & $y_T(t)$ as ensemble member, ie

$$x_T(t) = \begin{cases} x(t) & -T<t<T \\ 0 & \text{elsewhere} \end{cases} \quad \& \quad y_T(t) = \begin{cases} y(t) & -T<t<T \\ 0 & \text{elsewhere.} \end{cases}$$

The F.Ts are denoted by.

$$x_T(t) \longleftrightarrow X_T(\omega)$$

$$y_T(t) \longleftrightarrow Y_T(\omega).$$

→ The cross power $P_{xy}(T)$ in the two processes with in the interval $(-T,T)$ by.

$$P_{xy}(T) = \frac{1}{2T}\int_{-T}^{T} x_T(t)\, y_T(t)\, dt$$

$$= \frac{1}{2T}\int_{-T}^{T} x(t)\, y(t)\, dt.$$

→ Using parseval's theorem, we have.

$$P_{xy}(T) = \frac{1}{2T}\int_{-T}^{T} x(t)\, y(t)\, dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{X_T^*(\omega)\, Y_T(\omega)}{2T}\, d\omega \quad \text{—} (1)$$

→ This cross power is a random quantity, since its value will vary depending on which ensemble member is considered.

→ The Avg. cross power, denoted $\overline{P}_{xy}(T)$, by taking the expected value in eq. (1). The result is.

$$\overline{P}_{xy}(T) = \frac{1}{2T}\int_{-T}^{T} R_{xy}(t,t)\, dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{E\left[X_T^*(\omega)\, Y_T(\omega)\right]}{2T}\, d\omega$$

→ Finally, we form the total avg. cross power P_{xy} by letting $T\to\infty$;

$$P_{xy} = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} R_{xy}(t,t)\, dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \lim_{T\to\infty} \frac{E\left[X_T^*(\omega)\cdot Y_T(\omega)\right]}{2T}\, d\omega$$

→ The integrand involving ω can be defined as the cross-power density spectrum, ie

$$S_{xy}(\omega) = \lim_{T\to\infty} \frac{E\left[X_T^*(\omega)\, Y_T(\omega)\right]}{2T}$$

The cross power formula is

$$P_{XY} = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{XY}(\omega)\, d\omega .$$

→ The other cross power density spectrum is given by

$$S_{YX}(\omega) = \lim_{T \to \infty} \frac{E\left[Y_T^*(\omega) . X_T(\omega) \right]}{2T}$$

→ Cross power is given by

$$P_{YX} = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{YX}(\omega)\, d\omega = P^*_{XY} .$$

→ Total power $P_{XY} + P_{YX}$ can be interpreted as the additional power two processes are capable of generating, over & above their individual powers due to the fact that they are not orthogonal.

Properties of Cross Power Density Spectrum:

1. $S_{XY}(\omega) = S_{YX}(-\omega) = S^*_{YX}(+\omega)$
2. $Re[S_{XY}(\omega)]$ & $Re[S_{YX}(\omega)]$ are even fn.s of ω.
3. $Im[S_{XY}(\omega)]$ & $Im[S_{YX}(\omega)]$ are odd fn.s of ω.
4. $S_{XY}(\omega) = 0$ & $S_{YX}(\omega) = 0$ if $X(t)$ & $Y(t)$ are orthogonal.
5. If $X(t)$ & $Y(t)$ are uncorrelated & have constant means $\overline{X}$ & $\overline{Y}$

$$S_{XY}(\omega) = S_{YX}(\omega) = 2\pi \overline{X}\,\overline{Y}\, \delta(\omega) .$$

$$A[R_{xy}(t,t+\tau)] \longleftrightarrow S_{xy}(\omega)$$

$$A[R_{yx}(t,t+\tau)] \longleftrightarrow S_{yx}(\omega)$$

For the case, of jointly WSS processes,

$$S_{xy}(\omega) = \int_{-\infty}^{\infty} R_{xy}(\tau).\, e^{-j\omega\tau}\, d\tau$$

$$S_{yx}(\omega) = \int_{-\infty}^{\infty} R_{yx}(\tau).\, e^{-j\omega\tau}\, d\tau$$

$$\therefore\ R_{xy}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{xy}(\omega).\, e^{j\omega\tau}\, d\omega$$

$$R_{yx}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{yx}(\omega).\, e^{j\omega\tau}\, d\omega$$

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prob: For the given cross power spectrum defined as

$$S_{xy}(\omega) = \begin{cases} a + j\dfrac{b\omega}{W} & -W < \omega < W \\ 0 & \text{elsewhere} \end{cases}$$

find the cross correlation fn.

Sol:

$$R_{xy}(\tau) = \frac{1}{2\pi}\int_{-W}^{W}\left(a + j\frac{b\omega}{W}\right).\, e^{j\omega\tau}\, d\omega$$

$$= \frac{a}{2\pi}\int_{-W}^{W} e^{j\omega\tau}\, d\omega + j\frac{b}{2\pi W}\int_{-W}^{W} \omega\, e^{j\omega\tau}\, d\omega$$

$$= \frac{a}{2\pi}\left[\frac{e^{j\omega\tau}}{j\tau}\Big|_{-W}^{W}\right] + j\frac{b}{2\pi W}\left\{e^{j\omega\tau}\left[\frac{\omega}{j\tau} - \frac{1}{(j\tau)^2}\right]\Big|_{-W}^{W}\right\}$$

$$= \frac{1}{\pi W\tau^2}\left[(aW\tau - b)\sin(W\tau) + bW\tau\cos(W\tau)\right]$$

Relationship b/w Cross power spectrum and cross correlation fn :

We have to show that

$$S_{xy}(\omega) = \int_{-\infty}^{\infty} \left\{ \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} R_{xy}(t, t+\tau)\, dt \right\} e^{-j\omega\tau}\, d\tau$$

→ We know that

$$X_T(\omega) = \int_{-T}^{T} x(t)\, e^{-j\omega t}\, dt.$$

$$Y_T(\omega) = \int_{-T}^{T} Y(t_1).\, e^{-j\omega t_1}\, dt_1.$$

Consider.

→ ∴ $$X_T^*(\omega).\, Y_T(\omega) = \int_{-T}^{T} x(t).\, e^{j\omega t}\, dt.\, \int_{-T}^{T} Y(t_1).\, e^{-j\omega t_1}\, dt_1 \quad \text{--- (1)}$$

→ The cross power density spectrum, is given by.

$$S_{xy}(\omega) = \lim_{T \to \infty} \frac{E[X_T^*(\omega).\, Y_T(\omega)]}{2T}$$

from eq (1)

$$= \lim_{T \to \infty} \frac{1}{2T} E\left[\int_{-T}^{T} x(t).\, e^{j\omega t}\, dt.\, \int_{-T}^{T} Y(t_1).\, e^{-j\omega t_1}\, dt_1 \right].$$

$$= \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} \int_{-T}^{T} R_{xy}(t, t_1).\, e^{-j\omega(t_1 - t)}\, dt.\, dt_1.$$

→ Apply inverse F.T both sides,

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} S_{xy}(\omega).\, e^{j\omega\tau}\, d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} \int_{-T}^{T} R_{xy}(t, t_1).\, e^{-j\omega(t_1 - t)}\, dt.\, dt.\, e^{j\omega\tau}\, d\omega.$$

$$= \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} \int_{-T}^{T} R_{xy}(t, t_1).\, \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{j\omega(\tau - t_1 + t)}\, d\omega\, dt_1\, dt.$$

$$= \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} \int_{-T}^{T} R_{xy}(t, t_1).\, \delta(t_1 - \tau - t)\, dt_1.\, dt.$$

defined only at $t_1 = t + \tau$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} S_{xy}(\omega)\, e^{j\omega\tau}\, d\omega = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} R_{xy}(t,t+\tau)\,dt \quad — (2)$$

which is valid for $-T < t+\tau < T$.

$\therefore$ eq (2) indicates the the cross power spectrum and the time avg. cross correlation fn. form a F.T pair

→ For jointly WSS, the cross correlation fn. $R_{xy}(\tau)$ can be found from $S_{xy}(\omega)$ since its time average is just $R_{xy}(\tau)$.

Problem: Let the cross-correlation fn. of two processes $x(t)$ $y(t)$ be

$$R_{xy}(t,t+\tau) = \frac{AB}{2}\left\{ \sin(\omega_0\tau) + \cos[\omega_0(2t+\tau)] \right\}$$

where A, B and ω_0 are constants. Find the cross-power spectrum.

Sol:

$$S_{xy}(\omega) = F.T\{ A[R_{xy}(t,t+\tau)]\}$$

$$\therefore A[R_{xy}(t,t+\tau)] = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} R_{xy}(t,t+\tau)\,dt$$

$$= \frac{AB}{2}\sin(\omega_0\tau) + \frac{AB}{2}\lim_{T\to\infty}\frac{1}{2T}\underbrace{\int_{-T}^{T}\cos(\omega_0(2t+\tau))\,dt}_{0}$$

$$= \frac{AB}{2}\sin(\omega_0\tau)$$

$$\therefore S_{xy}(\omega) = F\left\{\frac{AB}{2}\sin\omega_0\tau\right\}$$

$$= -\frac{j\pi AB}{2}\left[\delta(\omega-\omega_0) - \delta(\omega+\omega_0)\right]$$

ectral Characteristics of System Response

power Density spectrum of Response:

→ The PSD $S_{YY}(\omega)$ of the response of a linear time-invariant system having a Transfer fn. $H(\omega)$ is given by.

$$S_{YY}(\omega) = S_{XX}(\omega) \cdot |H(\omega)|^2 \quad \text{— (1)}$$

where $S_{XX}(\omega)$ is the PSD of i/p $x(t)$.

$|H(\omega)|^2$ is the power transfer fn. of the system.

→ WKT,

$$S_{YY}(\omega) = \int_{-\infty}^{\infty} R_{YY}(\tau) \cdot e^{-j\omega\tau} d\tau \rightarrow (2)$$

$$\& \quad R_{YY}(\tau) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_{XX}(\tau + \epsilon_1 - \epsilon_2)\, h(\epsilon_1) h(\epsilon_2)\, d\epsilon_1 d\epsilon_2 \text{ — (3)}$$

Substitute eq. (3) in eq. (2)

We have

$$S_{YY}(\omega) = \int_{-\infty}^{\infty} h(\epsilon_1) \int_{-\infty}^{\infty} h(\epsilon_2) \cdot \int_{-\infty}^{\infty} R_{XX}(t+\epsilon_1-\epsilon_2) \cdot e^{-j\omega\tau} d\epsilon_2 d\epsilon_1$$

Let $\epsilon = t + \epsilon_1 - \epsilon_2 \Rightarrow d\epsilon = d\tau$

$$\therefore S_{YY}(\omega) = \int_{-\infty}^{\infty} h(\epsilon_1) e^{j\omega\epsilon_1} d\epsilon_1 \cdot \int_{-\infty}^{\infty} h(\epsilon_2) \cdot e^{-j\omega\epsilon_2} d\epsilon_2 \int_{-\infty}^{\infty} R_{XX}(\epsilon) \cdot e^{-j\omega\epsilon} d\epsilon$$

$$S_{YY}(\omega) = H^*(\omega) \cdot H(\omega)\, S_{XX}(\omega) = \underline{S_{XX}(\omega) \cdot |H(\omega)|^2}$$

The avg. power is $\quad P_{YY} = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{XX}(\omega) \cdot |H(\omega)|^2 d\omega$

Cross Power Density Spectrum of Input & Output:

The F.T.s of the cross correlation fn. may be written as

$$S_{XY}(\omega) = S_{XX}(\omega) \cdot H(\omega)$$

$$S_{YX}(\omega) = S_{XX}(\omega) \cdot H(-\omega).$$