Unit-6 Random Processes

– Temporal Characteristics

Random Process:

→ The concept of a RP is based on enlarging the RV concept to include time.

→ Since a RV x is a fn. of the possible outcomes 's' of an experiment, it now becomes a fn. of both s & time.

→ In other words, a time fn. $x(t, s)$ to every outcome 's'.

→ The family of all such fn.s, denoted $X(t, s)$ is called a RP.

→ x was denoted as a specific value of RV X.

we shall often use short form notation $x(t)$ to represent a specific waveform of RP denoted by $X(t)$.

* → A RP $X(t, s)$ represents a family or ensemble of time fn.s when t & s are variables.

→ Each member time fn. is called a sample fn, ensemble member or sometimes a realization of the process.

* → Thus a RP represents a single time fn. when t is a variable & s is fixed at a specific value.

→ A RP also represents a RV when t is fixed and s is a variable.

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→ A RP can also represent a mere no. when t and s are both fixed.

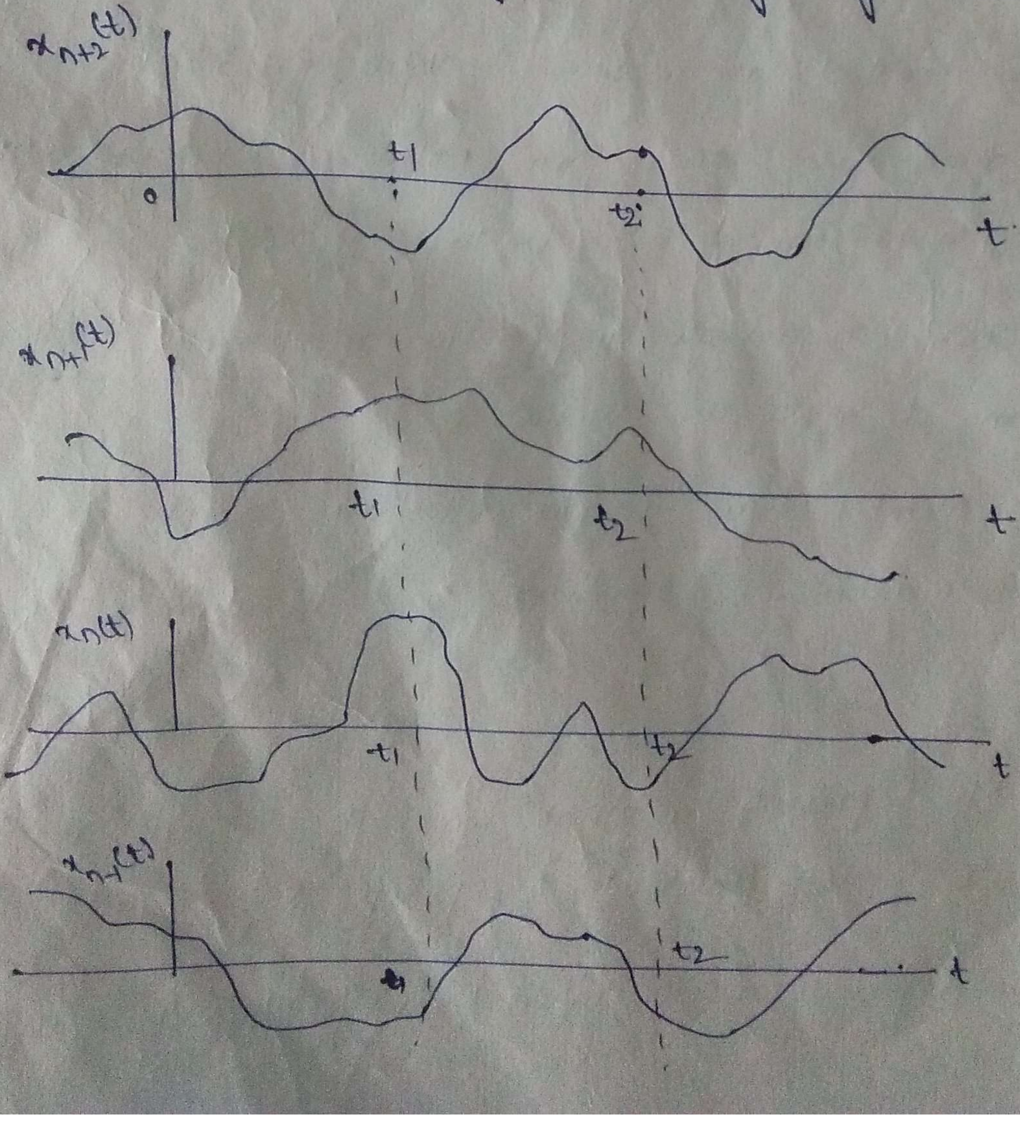
Classification of Processes:

→ RPs are classified according to the characteristics of t and the RV $X = X(t)$ at time t.

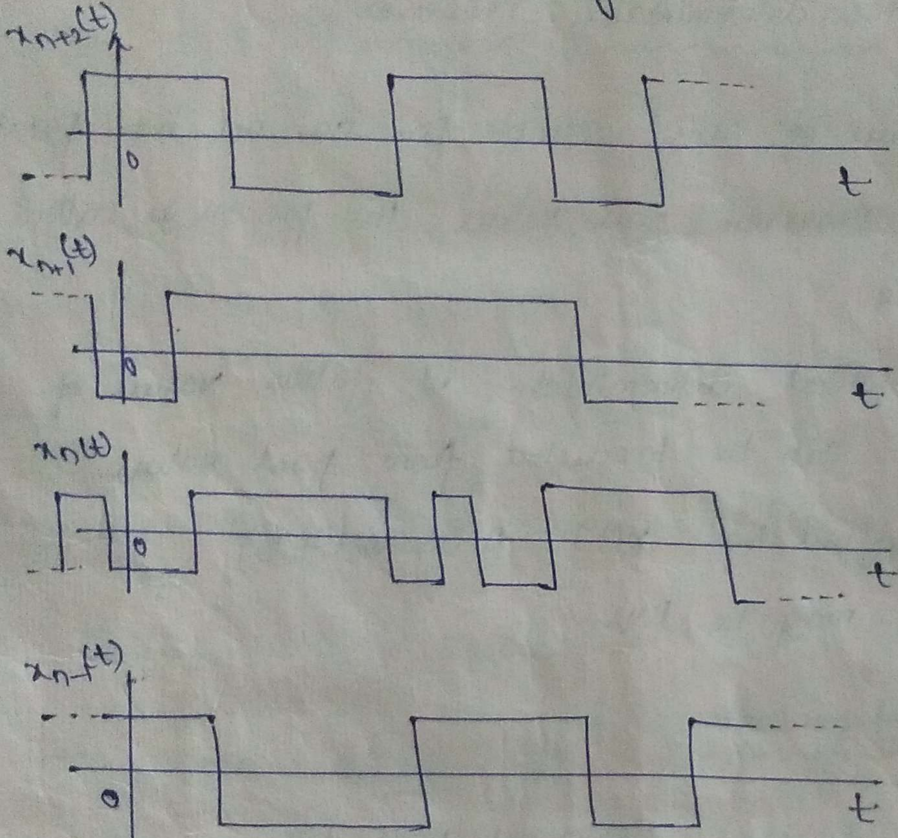
→ We consider only 4 cases based on t and x having values in the ranges $-\infty < t < \infty$ and $-\infty < x < \infty$.

1. If x is continuous & t can have any of a continuum of values, then $X(t)$ is called a <u>Continuous Random Process</u>.

<u>Ex</u>: Thermal noise generated by any realizable n/w.



(2) A 2nd class of RP, called a Discrete RP, corresponds to the RV X having only discrete values while t is continuous. Many sample fn.s have only 2 discrete values.
a) positive level b) negative level.



(3) A RP for which X is continuous but time has only discrete value is called a continuous Random Sequence.

→ It is formed by periodically sampling the ensemble members.

→ It is also frequently called a discrete-time (DT) random process.

→ ~~It is~~ These types of processes are important in the analysis of various DSP systems.

(4) A 4th class of RP, called a discrete Random Sequence, corresponds to both time and the RV being discrete.

→ In Quantization process, we will get these processes.

Deterministic & Non deterministic Processes:

* If future values of any sample fn. can not be predicted exactly from observed past values, the process is called non-deterministic.

* A process is called deterministic, if future values of any sample fn. can be predicted from past values.

Ex: The RP defined by $x(t) = A\cos(\omega_0 t + \theta)$

Here A, θ, ω_0 may be RVs.

Stationarity & Independence:

→ The RV will possess statistical properties, such as a mean value, moments, variance etc., that are related to its density fn.

→ If two RVs are obtained from the RP for two time instants, they will have statistical properties related to their joint density fn.

*→ A RP is said to be stationary if all its statistical properties do not change with time.

→ To define stationarity, we must 1st define distribution & density fn.s as they apply to a RP $X(t)$.

→ for a particular time t_1, the distribution fn. associated with the RV $X_1 = X(t_1)$ will be denoted $F_X(x_1; t_1)$.

It is defined as

$$F_X(x_1; t_1) = P\{X(t_1) \le x_1\}.$$

→ for two RVs $X_1 = X(t_1)$ & $X_2 = X(t_2)$, the second-order joint distribution fn. is

$$F_X(x_1, x_2; t_1, t_2) = P\{X(t_1) \le x_1, X(t_2) \le x_2\}.$$

→ The N-th order joint distribution fn. is

$$F_X(x_1, x_2, \dots x_N; t_1, t_2, \dots t_N) = P\{X(t_1) \le x_1, \dots X(t_N) \le x_N\}.$$

→ Joint density fn. are found from appropriate derivates

$$f_X(x_1; t_1) = \frac{d}{dx_1} F_X(x_1; t_1)$$

$$f_X(x_1, x_2; t_1, t_2) = \frac{\partial^2}{\partial x_1 \partial x_2} F_X(x_1, x_2; t_1, t_2)$$

$$f_X(x_1 \dots x_N; t_1, \dots t_N) = \frac{\partial^N}{\partial x_1 \dots \partial x_N} F_X(x_1, \dots x_N; t_1 \dots t_N).$$

Statistical Independence:

Two processes $X(t)$ and $Y(t)$ are statistically independent if the RV group $X(t_1), X(t_2) \ldots X(t_N)$ is independent of the group $Y(t_1'), Y(t_2') \ldots Y(t_M')$ for any choice of times $t_1, t_2, \ldots t_N, t_1', t_2', \ldots t_M'$.

$$f_{XY}(x_1, \ldots x_N, y_1, \ldots y_M ; t_1, \ldots t_N, t_1', \ldots t_M')$$

$$= f_X(x_1, \ldots x_N ; t_1 \ldots t_N) \cdot f_Y(y_1, \ldots y_M ; t_1', \ldots t_M').$$

First - Order Stationary Processes:

→ A RP is called stationary to order one if its first order density fn. does not change with a shift in time origin.

→ In other words

$$f_X(x_1;t_1) = f_X(x_1; t_1+\Delta) \quad \text{—— (1)}$$

must be true for any t_1.

→ Consequences of eq. (1) are that

→ $f_X(x_1;t_1)$ is independent of t_1 &

→ the process mean value $E[X(t)]$ is a const.

$$\therefore E[X(t)] = \bar{X} = \text{constant} \quad \text{—— (2)}.$$

Proof: Consider the RVs $X_1 = X(t_1)$ & $X_2 = X(t_2)$.

for X_1:

$$E[X_1] = E[X(t_1)] = \int_{-\infty}^{\infty} x_1 f_X(x_1;t_1)\, dx_1$$

for X_2:

$$E[X_2] = E[x(t_2)] = \int_{-\infty}^{\infty} x_1 f_X(x_1; t_2)\, dx_1$$

Let $t_2 = t_1 + \Delta$ → we get

$$E[x(t_1+\Delta)] = E[x(t_1)].$$

which must be a const. because t_1 & Δ are arbitrary.

Second-Order and Wide-Sense stationary:

→ A process is called stationary to order two, if its second order density fn. satisfies

$$f_X(x_1, x_2; t_1, t_2) = f_X(x_1, x_2; t_1+\Delta, t_2+\Delta).$$

→ A second order stationary process is also first-order stationary because the second-order density fn determines the lower, first order density.

* → The correlation $E[X_1 X_2] = E[x(t_1)\, x(t_2)]$ of the RP is denoted by $R_{XX}(t_1, t_2)$ and called it as the auto correlation fn. of the RP $x(t)$.

$$R_{XX}(t_1, t_2) = E[x(t_1)\, x(t_2)] \quad \text{—— (1)}$$

→ The auto correlation fn. of a second-order stationary process is a fn. only of time differences and not absolute time. ie

$$t = t_2 - t_1.$$

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then eq. (1) becomes

$$R_{xx}(t_1, t_1+\tau) = E[x(t_1)\, x(t_1+\tau)]$$

$$= R_{xx}(\tau).$$

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→ The wide sense stationary (WSS) process, defined as that for which two conditions are true.

$$E[x(t)] = \bar{X} = \text{Const}$$

$$E[x(t)\, x(t+\tau)] = R_{xx}(\tau)$$

→ A process stationary to order 2 is clearly WSS.

Prob: S.T the RP $x(t) = A\cos(\omega_0 t + \theta)$ is WSS if it is assumed that A & ω_0 are constants and θ is a uniformly distributed RV on the interval $(0, 2\pi)$.

Sol: The mean value is

$$E[x(t)] = \int_0^{2\pi} A\cos(\omega_0 t+\theta)\cdot \frac{1}{2\pi}\, d\theta.$$

$$= \frac{A}{2\pi} \cdot \sin(\omega_0 t+\theta)\Big]_0^{2\pi}$$

$$= \frac{A}{2\pi}\left[\sin\omega_0 t - \sin\omega_0 t\right] = 0. = \text{Const}.$$

The auto-correlation fn.,

let $t_1 = t$ & $t_2 = t+\tau$

$$R_{xx}(t, t+\tau) = E[x(t)\, x(t+\tau)]$$

$$= E[A\cos(\omega_0 t+\theta)\cdot A\cos(\omega_0 t+\omega_0\tau+\theta)]$$

$$= \frac{A^2}{2} E\left[\cos(\omega_0 \tau) + \cos(2\omega_0 t + \omega_0 \tau + 2\theta)\right].$$

$$= \frac{A^2}{2} \cdot \cos(\omega_0 \tau) + \frac{A^2}{2} \underbrace{E\left[\cos(2\omega_0 t + \omega_0 \tau + 2\theta)\right]}_{0}.$$

$$= \frac{A^2}{2} \cos(\omega_0 \tau) = R_{XX}(\tau).$$

ie the auto correlation fn. depends only on τ and the mean value is a const.

$\therefore$ $X(t)$ is WSS.

→ For two RPs $X(t)$ and $Y(t)$, the correlation is denoted by $R_{XY}(t_1, t_2)$ and it is called as the cross correlation fn. & is given by

$$R_{XY}(t_1, t_2) = E\left[X(t_1)\, Y(t_2)\right].$$

is a fn. only of time difference $\tau = t_2 - t_1$ & not absolute time ie

$$R_{XY}(t, t+\tau) = E\left[X(t)\, Y(t+\tau)\right] = R_{XY}(\tau).$$

N-Order & strict-sense stationarity: (SSS).

→ A RP is stationary to order N if its N-th order density fn. is invariant to a time origin shift, ie if

$$f_X(x_1, \ldots x_N; t_1, \ldots t_N) = f_X(x_1, \ldots x_N; t_1+\Delta, \ldots t_N+\Delta).$$

→ A process stationary to all orders $N = 1, 2, \ldots$ is called strict sense stationary.

Time Averages & Ergodicity:

→ The time avg. of a quantity is defined as

$$A[\cdot] = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} [\cdot]\, dt$$

→ Here A is used to denote time avg. analogous to E for the statistical avg.

→ The mean value $\bar{x} = A[x(t)] = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} x(t)\, dt.$

time auto correlation fn. $\mathcal{R}_{xx}(\tau) = A[x(t)x(t+\tau)]$

$$= \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} x(t)x(t+\tau)\, dt.$$

*→ $\bar{x}$ and $\mathcal{R}_{xx}(\tau)$ are actually RVs.

By taking the expected values do these, we will get

$$E[\bar{x}] = \bar{X}$$

$$E[\mathcal{R}_{xx}(\tau)] = R_{XX}(\tau)$$

*→ If $\bar{x} = \bar{X}$ —①

$\mathcal{R}_{xx}(\tau) = R_{XX}(\tau)$ —②

ie, the time averages $\bar{x}$ & $\mathcal{R}_{xx}(\tau)$ equal the statistical averages $\bar{X}$ & $R_{XX}(\tau)$ respectively.

→ The Ergodic theorem allows the validity of eq.s ① & ②

→ The processes that satisfy the ergodic theorem are called ergodic processes.

→ Two RPs are called jointly ergodic if they are individually ergodic & also have a time cross-correlation fn. that equals the statistical cross-correlation fn.

$$R_{xy}(\tau) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} x(t)\, y(t+\tau)\, dt = R_{xy}(\tau).$$

Mean Ergodic Processes;

→ A process $X(t)$ with a const. mean value $\bar{X}$ is called mean-ergodic, or ergodic in the mean, if its statistical avg $\bar{X}$ equals the time avg. $\bar{x}$ of any sample fn. $x(t)$ with probability 1 for all sample fn. is ie if

$$E[X(t)] = \bar{X} = A[x(t)] = \bar{x}$$

with prob. 1 for all $x(t)$.

Correlation-Ergodic Processes;

→ A stationary continuous process $X(t)$ with auto correlation fn. $R_{XX}(\tau)$ is called auto correlation-ergodic or ergodic in the auto correlation if, and only if, $\forall \tau$

$$\lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} x(t)\, x(t+\tau)\, d\tau = R_{XX}(\tau)$$

→ ~~Replace $x(t)$ by~~ Consider $W(t) = X(t)\, X(t+\lambda)$ where λ is a time off-set.

$$E[W(t)] = E\left[X(t)\, X(t+\lambda)\right] = R_{XX}(\lambda)$$

$$R_{WW}(\tau) = E[W(t)\, W(t+\tau)]$$

$$= E[X(t)\, X(t+\lambda)\, X(t+\tau)\, X(t+\tau+\lambda)]$$

$$C_{WW}(\tau) = R_{WW}(\tau) - \{E[W(t)]\}^2 = R_{WW}(\tau) - R_{XX}^2(\lambda)$$

→ Two processes $x(t)$ and $Y(t)$ may be called cross-correlation-ergodic, or ergodic in the correlation, if the time cross-correlation fn. is equal to the statistical cross-correlation fn.

$$\mathcal{R}_{xy}(\tau) = R_{xy}(\tau)$$

Correlation Functions:

Auto-correlation fn. & its properties:

→ The auto correlation fn. of a RP $x(t)$ is the correlation $E[x_1 x_2]$ of two RVs $x_1 = x(t_1)$ and $x_2 = x(t_2)$ defined by the process at times t_1 and t_2. Mathematically,

$$R_{xx}(t_1, t_2) = E[x(t_1)\, x(t_2)]$$

→ for time $t_1 = t$ and $t_2 = t+\tau$, with τ a real no,

$$\therefore R_{xx}(t, t+\tau) = E[x(t)\, x(t+\tau)]$$

→ for a wide-sense stationary processes

$$R_{xx}(\tau) = E[x(t)\, x(t+\tau)]$$

*→ Properties:

1. $|R_{xx}(\tau)| \leq R_{xx}(0)$

   ie $R_{xx}(\tau)$ is bounded by its value at the origin.

2. $R_{xx}(-\tau) = R_{xx}(\tau)$

   indicates that an autocorrelation fn. has even symmetry.

3. $R_{xx}(0) = E[x^2(t)]$

   The bound, is equal to the mean-squared value called the power in the process.

(4) If $E[x(t)] = \bar{x} \neq 0$ and $x(t)$ is ergodic with no periodic components then

$$\lim_{|\tau| \to \infty} R_{xx}(\tau) = \bar{x}^2$$

(5) If $x(t)$ has a periodic component, then $R_{xx}(\tau)$ will have a periodic component with the same period.

(6) If $x(t)$ is ergodic, zero-mean and has no periodic component, then

$$\lim_{|\tau| \to \infty} R_{xx}(\tau) = 0.$$

(7) $R_{xx}(\tau)$ cannot have an arbitrary shape.

Prob: For a stationary ergodic process with no periodic components, the auto correlation fn. is given by

$$R_{xx}(\tau) = 25 + \frac{4}{1+6\tau^2}$$

Find the mean & variance of the process.

sol: from prop (4)

$$\lim_{|\tau| \to \infty} R_{xx}(\tau) = \bar{x}^2$$

$$\Rightarrow 25 + \frac{4}{1+\infty} = 25 = \bar{x}^2$$

$$\Rightarrow \bar{x} = \pm 5$$

from Prop. 3,

$$R_{XX}(0) = E[X^2(t)].$$

$$\Rightarrow 25 + \frac{4}{1+(0)} = 29 = E[X^2(t)].$$

$\therefore$ The variance is $\sigma_X^2 = E[X^2(t)] - E[X(t)]^2$.

$$= 29 - 25 = \underline{\underline{4}}.$$

Prob: $X(t)$ be a wide-sense stationary RP with auto correlation fn. $R_{XX}(\tau) = e^{-a|\tau|}$ where $a>0$ is a const.

Assume $X(t)$ "amplitude modulates" a "carrier" $\cos(\omega_0 t+\theta)$, where ω_0 is a const. & θ is a random variable uniform on $(-\pi, \pi)$ that is statistically independent of $X(t)$. Determine the auto-correlation fn. of $Y(t)$.

Sol: $R_{YY}(t, t+\tau) = E[Y(t) \cdot Y(t+\tau)]$.

$\therefore$ $Y(t) = X(t)\cos(\omega_0 t+\theta)$.

$\therefore$ $R_{YY}(t, t+\tau) = E[X(t)X(t+\tau)\cos(\omega_0 t+\theta) \cdot \cos(\omega_0 t+\omega_0\tau+\theta)]$

$$= \frac{R_{XX}(\tau)}{2} E[\cos(\omega_0\tau) + \cos(2\omega_0 t+\omega_0\tau+2\theta)].$$

$$\Rightarrow E[\cos(2\omega_0 t+\omega_0\tau+2\theta)] = \int_{-\pi}^{\pi} \frac{1}{2\pi} \cdot [\cos\{(2\omega_0 t+\omega_0\tau+2\theta)\}]\, d\theta.$$

$$= 0.$$

$$\therefore R_{YY}(t, t+\tau) = \frac{1}{2} R_{XX}(\tau) \cdot \cos\omega_0\tau = R_{YY}(\tau).$$

Cross-correlation fn. & its properties:

→ The cross-correlation fn. of two RPs $X(t)$ & $Y(t)$, by setting $t_1 = t$ & $\tau = t_2 - t_1$, is given by

$$R_{XY}(t, t+\tau) = E[X(t)\,Y(t+\tau)].$$

→ If $X(t)$ & $Y(t)$ are atleast jointly WSS, $R_{XY}(t, t+\tau)$ is independent of absolute time & we can write

$$R_{XY}(\tau) = E[X(t)\,Y(t+\tau)].$$

*→ If $R_{XY}(t, t+\tau) = 0$

then $X(t)$ and $Y(t)$ are called orthogonal processes.

→ If the two processes are statistically independent,

then $R_{XY}(t, t+\tau) = E[X(t)] \cdot E[Y(t+\tau)]$.

→ If, in addition to being independent, $X(t)$ & $Y(t)$ are atleast WSS, then $R_{XY}(\tau) = \bar{X}\,\bar{Y} = \text{const}$.

→ Properties:

1. $R_{XY}(-\tau) = R_{YX}(\tau)$
2. $|R_{XY}(\tau)| \le \sqrt{R_{XX}(0)\,R_{YY}(0)}$
3. $|R_{XY}(\tau)| \le \frac{1}{2}[R_{XX}(0) + R_{YY}(0)]$

from prop 2 & 3, we have

$$\sqrt{R_{XX}(0)\,R_{YY}(0)} \le \frac{1}{2}[R_{XX}(0) + R_{YY}(0)]$$

*Prob: Let two RPs X(t) & Y(t) be defined by

$$X(t) = A\cos(\omega_0 t) + B\sin(\omega_0 t)$$

$$Y(t) = B\cos(\omega_0 t) - A\sin(\omega_0 t).$$

where A & B are RVs & ω_0 is a const.

X(t) is a WSS, if A & B are uncorrelated, zero-mean RVs with the same variance. Y(t) is also WSS. Find the cross-correlation fn. $R_{XY}(t, t+\tau)$ & S.T X(t) & Y(t) are jointly WSS.

Sol:

$$R_{XY}(t, t+\tau) = E[X(t)\,Y(t+\tau)]$$

$$= E\Big[AB\cos(\omega_0 t)\cos(\omega_0 t + \tau\omega_0)$$

$$+ B^2 \sin\omega_0 t \cdot \cos(\omega_0 t + \tau\omega_0)$$

$$- A^2 \cos\omega_0 t \cdot \sin(\omega_0 t + \omega_0 \tau)$$

$$- AB \sin\omega_0 t \cdot \sin(\omega_0 t + \omega_0 \tau)\Big].$$

$$= E[AB] \cdot \cos(2\omega_0 t + \omega_0 \tau)$$

$$+ E[B^2] \cdot \sin\omega_0 t \cdot \cos(\omega_0 t + \omega_0 \tau)$$

$$- E[A^2] \cdot \cos\omega_0 t \cdot \sin(\omega_0 t + \omega_0 t).$$

Given $E[A] = E[B] = 0$.

Uncorrelated $\rightarrow$ $E[AB] = E[A]E[B] = 0$.

$\sigma_A^2 = \sigma_B^2 = \sigma^2$ $\Rightarrow$ $E[A^2] - \underbrace{E[A]^2}_{0} = \sigma^2$

$\Rightarrow$ $E[A^2] = \sigma^2$

Illy $E[B^2] = \sigma^2$

$\therefore R_{XY}(t, t+\tau) = -\sigma^2 \sin(\omega_0 \tau)$

thus $x(t)$ & $y(t)$ are jointly WSS, because

$$R_{xy}(t, t+\tau) = R_{xy}(\tau).$$

* covariance functions:

→ The auto covariance fn. is defined by

$$C_{xx}(t, t+\tau) = E\left[\{x(t) - E[x(t)]\}\{x(t+\tau) - E[x(t+\tau)]\}\right]$$

$$\Rightarrow C_{xx}(t, t+\tau) = R_{xx}(t, t+\tau) - E[x(t)]\,E[x(t+\tau)] \quad \text{—} (1)$$

→ the cross-covariance fn. is defined by.

$$C_{xy}(t, t+\tau) = E\left[\{x(t) - E[x(t)]\}\{y(t+\tau) - E[y(t+\tau)]\}\right]$$

$$\Rightarrow C_{xy}(t, t+\tau) = R_{xy}(t, t+\tau) - E[x(t)]\,E[y(t+\tau)] . \quad \text{—} (2)$$

→ for jointly WSS processes,

eq (1) $\Rightarrow$ $C_{xx}(\tau) = R_{xx}(\tau) - \bar{X}^2$

eq (2) $\Rightarrow$ $C_{xy}(\tau) = R_{xy}(\tau) - \bar{X}\,\bar{Y}$

→ For a WSS process, variance does not depend on time & is given with $\tau = 0$.

$$\sigma_x^2 = E\left[\{x(t) - E[x(t)]\}^2\right] = R_{xx}(0) - \bar{X}^2.$$

$E[(x-\bar{x})^2]$

→ for uncorrelated RPs,

$$C_{xy}(t, t+\tau) = 0$$

$$\Rightarrow R_{xy}(t, t+\tau) = E[x(t)]\,E[y(t+\tau)]$$