UNIT-4

CONVOLUTION AND CORRELATION OF SIGNALS:

→ Convolution is used to find common area between two signals or two fns.

The convolution $f(t)$ of two time functions $f_1(t)$ and $f_2(t)$ is designed or defined as

$$f(t) = f_1(t) \otimes f_2(t)$$

$$= \int_{-\infty}^{\infty} f_1(\tau)\, f_2(t-\tau)\, d\tau$$

(or)

$$= \int_{-\infty}^{\infty} f_2(\tau)\, f_1(t-\tau)\, d\tau$$

→ Convolution is a mathematical operation and multiplication is one of the form of convolution.

Ex:

$$\begin{array}{r} 25 \\ \times 25 \\ \hline 625 \end{array}$$

In convolution method

(i) $f_1(\tau)$ & $f_2(\tau)$

$\downarrow$ 25 $\downarrow$ 25

(ii) $f_2(-\tau)$

$\Rightarrow 52$

(iii) $f_2(t-\tau)$

$f_2(-\tau)$ shifted to right side by t seconds.

→ Here 't' is varied from $-\infty$ to ∞

Convolution Procedure:

step 1: $f_1(t) \otimes f_2(t) = \int_{-\infty}^{\infty} f_1(\tau)\, f_2(t-\tau)\, d\tau$

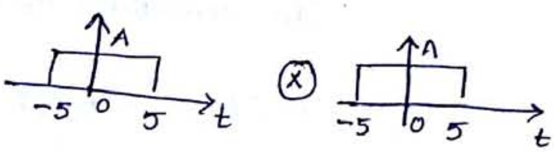
Here the independent variable convolution integral is τ, so replace t by τ to get $f_1(\tau)$ & $f_2(\tau)$

step 2: $f_1(\tau)$ is first function and $f_2(\tau)$ is the 2nd function. $f_2(-\tau)$ is the mirror image of the $f_2(\tau)$

step 3: $f_2(t-\tau)$ represents the function $f_2(-\tau)$ shifted to right side by t sec. 't' is varied from $-\infty$ to ∞ and find common area between two functions.

→ The value of convolution obtained at different values of 't' and may be plotted on a graph.

① Find the F.T of



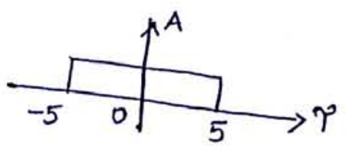
sol step 1:

$f_1(t) = A\, G_{10}(t) \longrightarrow f_1(\tau) = A\, G_{10}(\tau)$

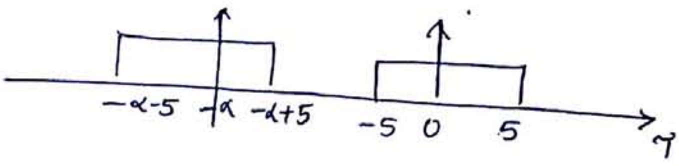
$f_2(t) = A\, G_{10}(t) \longrightarrow f_2(\tau) = A\, G_{10}(\tau)$

step 2:

$f_2(-\tau) = A\, G_{10}(-\tau) \longrightarrow$ even function

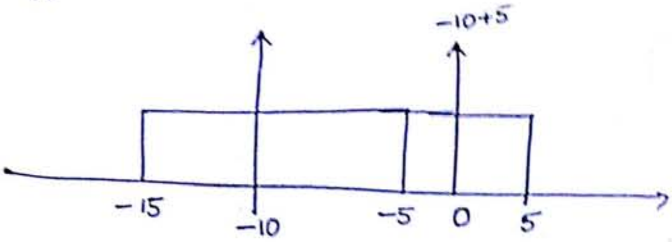


① $f_1(\tau)$ at $t = -\alpha$



$f_1(t) \otimes f_2(t) = 0$

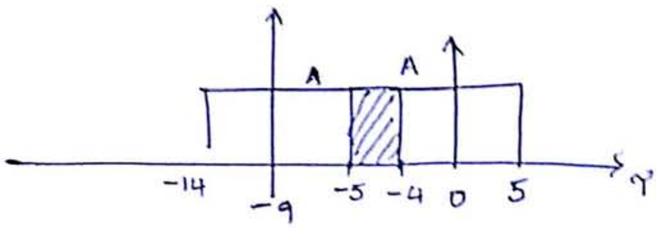
② at $t=-10$



$-10+5=-5$

$-10-5=-15$

③ at $t=-9$



$-9-5=-14$

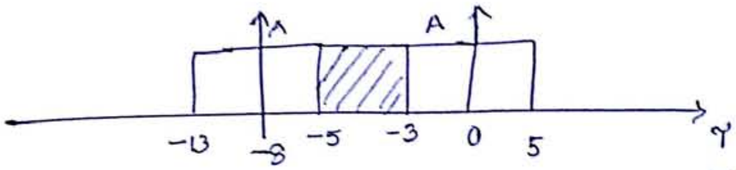
$-9+5=-4$

$$\Rightarrow \int_{-5}^{-4} A\cdot A\, d\gamma = A^2[\gamma]_{-5}^{-4}$$

$$= A^2[-4+5]$$

$$= A^2$$

④ at $t=-8$



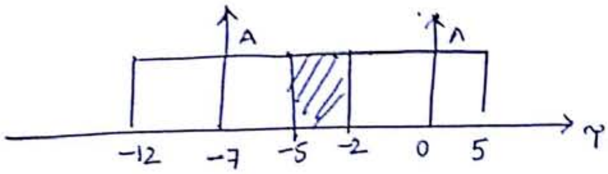
$-8+5=-3$

$-8-5=-13$

$$\Rightarrow \int_{-5}^{-3} A\cdot A\, d\gamma = A^2[\gamma]_{-5}^{-3}$$

$$= A^2[-3+5] \Rightarrow 2A^2.$$

⑤ at $t=-7$



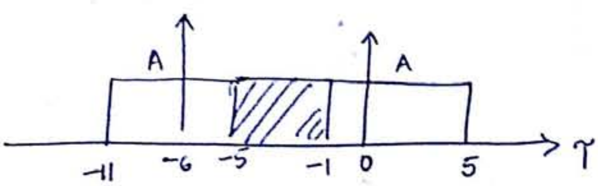
$-7+5=-2$

$-7-5=-12$

$$\Rightarrow \int_{-5}^{-2} A\cdot A\, d\gamma = A^2[\gamma]_{-5}^{-2}$$

$$= A^2[-2+5] = 3A^2.$$

⑥ at $t=-6$



$-6+5=-1$

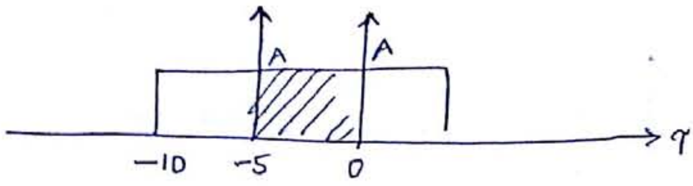
$-6-5=-11$

$$\Rightarrow \int_{-5}^{-1} A\cdot A\, d\gamma \Rightarrow A^2[-1+5] = 4A^2.$$

⑦ at $t=-5$

$-5-5=-10$

$-5+5=0$

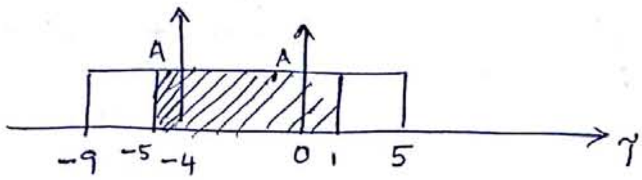


$$\int_{-5}^{0} A \cdot A \, d\tau \Rightarrow A^2 \cdot [\tau]_{-5}^{0} = A^2 \cdot 5$$

⑧ at $t=-4$

$-4+5=1$

$-4-5=-9$

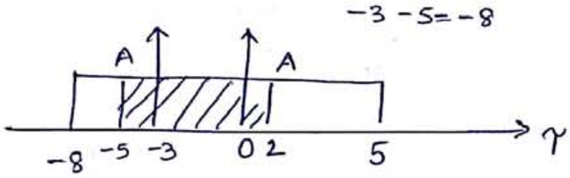


$$\int_{-5}^{1} A \cdot A \, d\tau \Rightarrow A^2 [\tau]_{-5}^{1} = 6A^2.$$

⑨ at $t=-3$

$-3+5=+2$

$-3-5=-8$



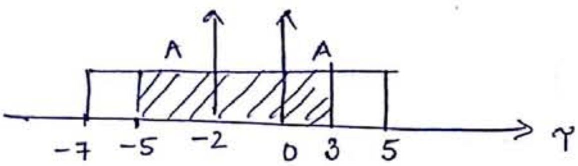
$$\int_{-5}^{2} A \cdot A \, d\tau \Rightarrow A^2 [\tau]_{-5}^{2}$$

$$\Rightarrow A^2 [2+5] = 7A^2.$$

⑩ at $t=-2$

$-2+5=+3$

$-2-5=-7$

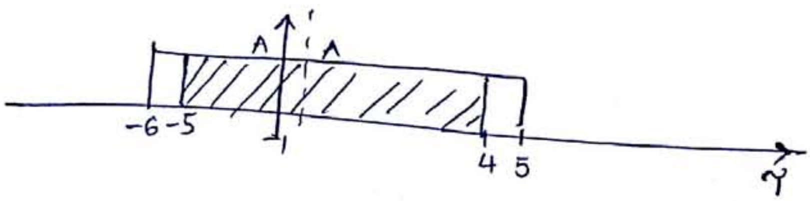


$$\int_{-5}^{3} A \cdot A \, d\tau \Rightarrow A^2 [\tau]_{-5}^{3} = A^2 [3+5] = 8A^2$$

⑪ at $t=-1$

$-1+5=4$

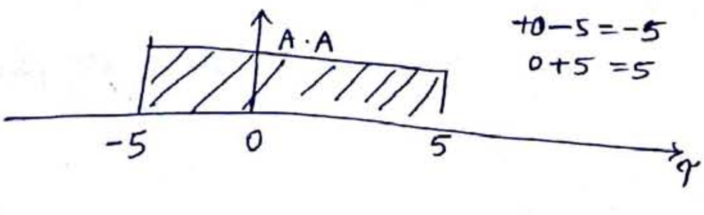
$-1-5=-6$



$$\Rightarrow \int_{-5}^{4} A \cdot A \, d\tau \Rightarrow A^2 [\tau]_{-5}^{-4} = A^2 [9]$$

$$= 9A^2$$

⑫ at $t=0$

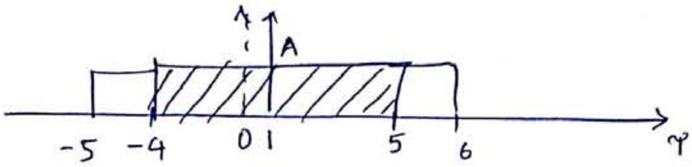


$+0-5=-5$

$0+5=5$

$$\Rightarrow \int_{-5}^{5} A \cdot A \, d\tau = A^2 [\tau]_{-5}^{5} = 10A^2$$

(13) at $t=1$



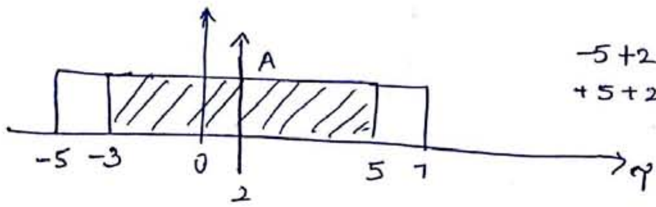
$-5+1=-4$

$+5+1=6$

$$\int_{-4}^{5} A\cdot A\, d\tau \Rightarrow A^2[\tau]_{-4}^{5}$$

$$\Rightarrow A^2[5+4]=9A^2.$$

(14) at $t=2$



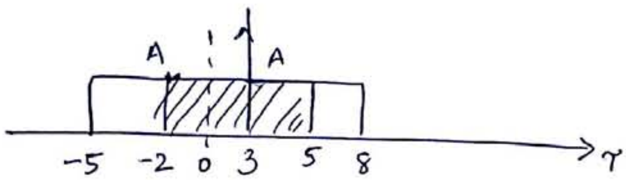
$-5+2=-3$

$+5+2=7$

$$\int_{-3}^{5} A\cdot A\, d\tau = A^2[\tau]_{-3}^{5}$$

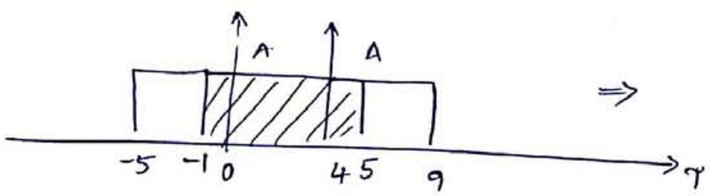
$$= 8A^2$$

(15) at $t=3$



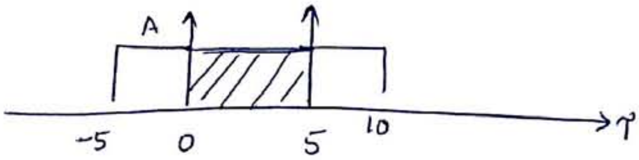
$$\int_{-2}^{5} A\cdot A\, d\tau \Rightarrow A^2[5+2]=7A^2.$$

(16) $t=4$



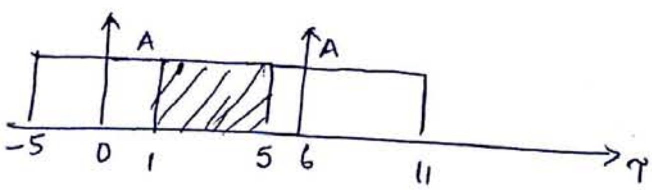
$$\Rightarrow \int_{-1}^{5} A\cdot A\, d\tau = A^2[5+1]=6A^2$$

(17) $t=5$



$$\int_{0}^{5} A\cdot A\, d\tau = A^2[5]$$

(18) $t=6$



$$\int_{1}^{5} A\cdot A\, d\tau \Rightarrow A^2[\tau]_{1}^{5} = A^2[4]$$

$||^{ly}$ for $t=7 \Rightarrow 3A^2$

$t=8 \Rightarrow 2A^2$

$t=9 \Rightarrow 1A^2$

$t=10 \Rightarrow 0$

$t=-10 \,;\, 10 = 0$

$t=-9 \,;\, 9 = 1A^2$

$t=-8 \,;\, 8 = 2A^2$

$t=-7 \,;\, 7 = 3A^2$

$t=-6 \,;\, 6 = 4A^2$

$t=-5 \,;\, 5 = 5A^2$

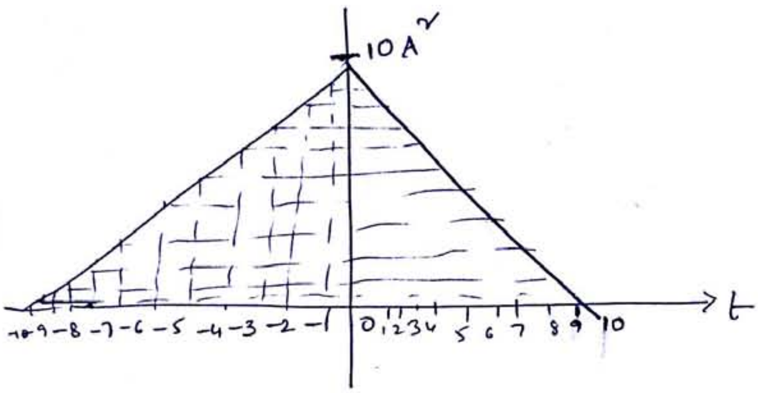
$t=-4 \,;\, 4 = 6A^2$

$t=-3 \,;\, 3 = 7A^2$

$t=-2 \,;\, 2 = 8A^2$

$t=-1 \,;\, 1 = 9A^2$

$t=0 \Rightarrow 10A^2$



Convolution : Mathematical way of combining two signals to form a third signal. i.e input signal, output signal and impulse response.

→ It is used to express the input and output relationship of a LTI system

→ If two functions $x(t)$ and $y(t)$ in time domain are defined then convolution $z(t)$ is

$$z(t) = x(t) * y(t) = \int_{-\infty}^{\infty} x(\tau)\, y(t-\tau)\, d\tau .$$

(or)

$x(t) \otimes y(t)$.

↳ Read as $x(t)$ convolved with $y(t)$.

Convolution in time domain:

→ The convolution in time domain is equivalent to multiplication of their spectra in frequency domain i.e $x(t) \xleftrightarrow{} X(\omega)$ and $y(t) \longleftrightarrow Y(\omega)$ then

$$x(t) * y(t) \longleftrightarrow X(\omega) \cdot Y(\omega).$$

Proof $x(t)$ fourier transform is given by

$$F\{x(t)\} = X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t} dt$$

$$F\{x(t) * y(t)\} = \int_{-\infty}^{\infty} [x(t) * y(t)]\, e^{-j\omega t} dt$$

$$\text{But } x(t) * y(t) = \int_{-\infty}^{\infty} x(\tau)\, y(t-\tau)\, d\tau$$

$$\therefore F\{x(t) * y(t)\} = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} x(\tau)\, y(t-\tau)\, d\tau \right] e^{-j\omega t} dt$$

Interchanging the order of integration, we get

$$F\{x(t) * y(t)\} = \int_{-\infty}^{\infty} x(\tau) \left[\int_{-\infty}^{\infty} y(t-\tau)\, e^{-j\omega t} dt \right] d\tau$$

Using time shifting property, we get

$$\int_{-\infty}^{\infty} y(t-\tau)\, e^{-j\omega t} dt = Y(\omega)\, e^{-j\omega \tau}$$

$$F[x(t)*y(t)] = \int_{-\infty}^{\infty} x(\tau)Y(\omega)e^{-j\omega\tau}d\tau$$

$$= Y(\omega)\int_{-\infty}^{\infty} x(\tau)e^{-j\omega\tau}d\tau = Y(\omega)X(\omega).$$

$$x(t)*y(t) \longleftrightarrow X(\omega)Y(\omega)$$

$\downarrow$

This is time convolution theorem.

Convolution in frequency domain:

→ Multiplication of two functions in time domain is equivalent to convolution of their spectra in frequency domain.

$x(t) \longleftrightarrow X(\omega)$ and $y(t) \longleftrightarrow Y(\omega)$ then $2\pi x(t)y(t) \longleftrightarrow X(\omega)*Y(\omega)$ (or)

$x(t)y(t) \longleftrightarrow X(f)*Y(f)$.

proof

$$X(\omega) = F\{x(t)\} = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

$$x(t) = F^{-1}\{X(\omega)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega$$

$$F\{x(t)y(t)\} = \int_{-\infty}^{\infty} x(t)y(t)e^{-j\omega t}dt$$

$$= \int_{-\infty}^{\infty}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} X(\lambda)e^{j\lambda t}d\lambda\right]y(t)e^{-j\omega t}dt$$

Interchanging the order of integration we get

$$F\{x(t)y(t)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\lambda)\left[\int_{-\infty}^{\infty} y(t)\cdot e^{-j\omega t}e^{j\lambda t}dt\right]d\lambda$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\lambda)\left[\int_{-\infty}^{\infty} y(t)\cdot e^{-j(\omega-\lambda)t}dt\right]d\lambda$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\lambda)Y(\omega-\lambda)d\lambda. = \frac{1}{2\pi}[X(\omega)*Y(\omega)]$$

$$x(t)y(t) \longleftrightarrow \frac{1}{2\pi}X(\omega)*Y(\omega)$$

$$2\pi x(t)y(t) \longleftrightarrow X(\omega)*Y(\omega)$$

Graphical convolution:

→ The convolution by inspection provides the information needed without complicated calculations. This convolution by inspection procedure is called graphical convolution.

Procedure:

(1) $x(\tau)$ is the first function, where an independent variable 't' is replaced by dummy variable 'τ'.

(2) $y(-\tau)$ is the mirror image of $y(\tau)$ i.e $y(\tau)$ is flipped.

(3) $y(t-\tau)$ represents the function $y(-\tau)$ shifted to right by t seconds.

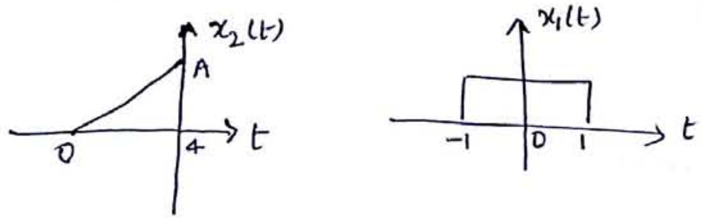
(4) For a particular value of $t=a$, integration of product $x(\tau)\, y(a-\tau)$ represents the area under the (common area) product curve.

$$\int_{-\infty}^{\infty} x(\tau)\, y(a-\tau)\, d\tau = \left[x(t) * y(t)\right]_{t=a}$$

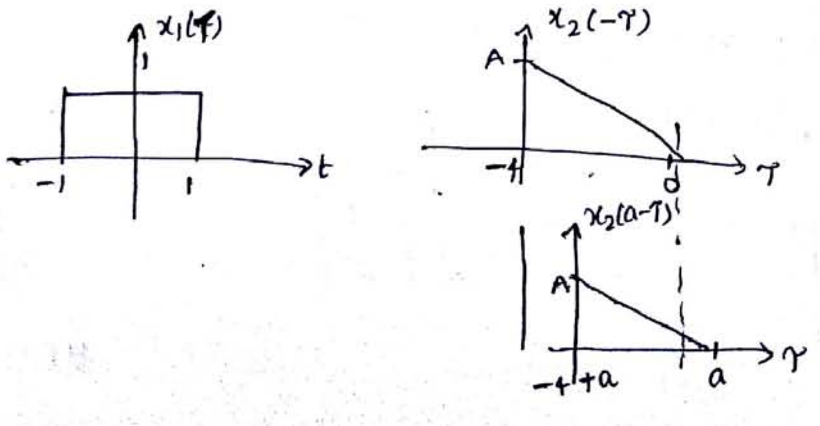
(5) The procedure is repeated for different values of t.
For negative value of t, the function $y(-\tau)$ is shifted left by t seconds.

6) The value of convolution obtained at different values of 't' (i.e +ve, −ve).

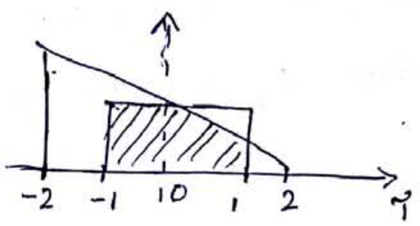
Q. Find the convolution of the functions $x_1(t)$ and $x_2(t)$



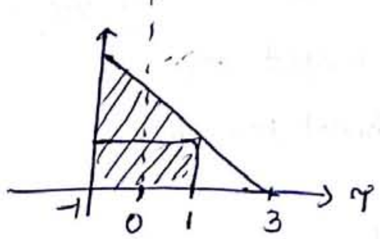
Sol



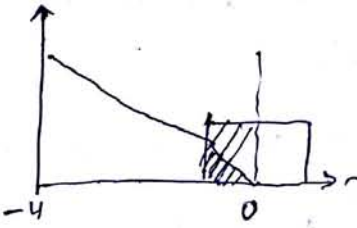
$[x_1(t) * x_2(t)]_{a=2}$



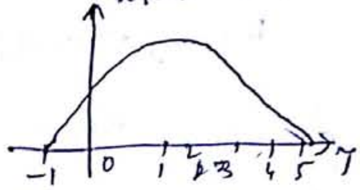
$[x_1(t) * x_2(t)]_{a=3}$



$[x_1(t) * x_2(t)]_{a=0}$



$x_1(t) * x_2(t)$



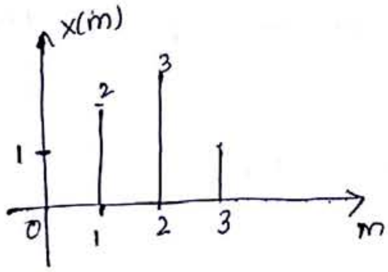
2) Determine the response of LTI system whose input $x(n)$ and impulse response $h(n)$ are given by $x(n) = \{\underset{\uparrow}{1}, 2, 3, 1\}$ and $h(n) = \{1, \underset{\uparrow}{2}, 1, -1\}$

Sol The input sequence starts at $n=0$ and impulse response starts at $n=-1$.

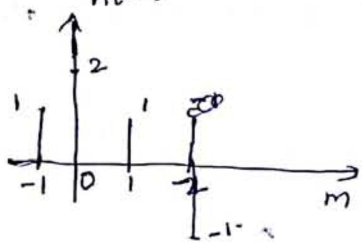
O/p sequence starts at $n = 0+(-1) = -1$

→ i/p & impulse response consists of 4 samples, so o/p consists of $4+4-1 = 7$ sa

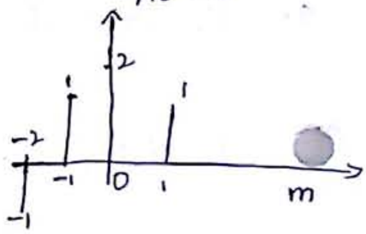
$x(m)$



$h(m)$



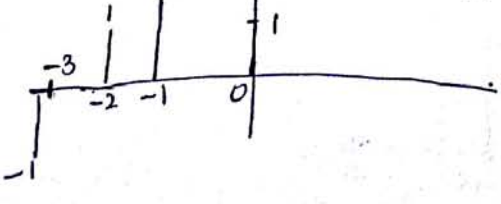
$h(-m)$



$$y(n) = \sum_{m=-\infty}^{\infty} x(m) h(n-m).$$

when $n=-1$; $y(-1) = \sum_{m=-\infty}^{\infty} x(m)\, h(-1-m)$

$h(-1-m)$



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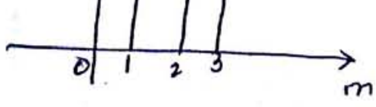
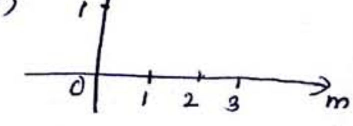
$x(m)$



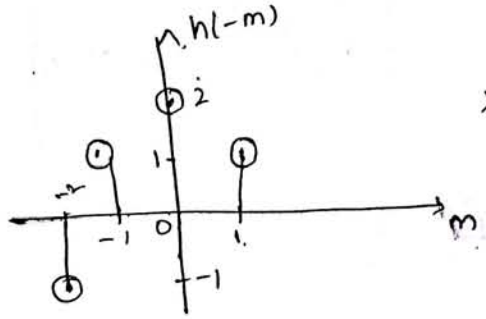
⇒



$y(-1) = 1$

when $n=0$, $y(0) = \sum_{m=-\infty}^{\infty} x(m)\, h(0-m)$

$y(0) = 2+2 = 4$



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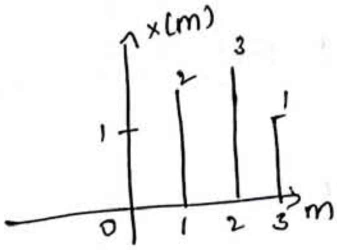

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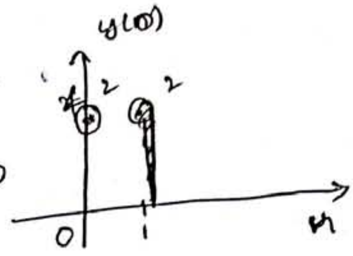
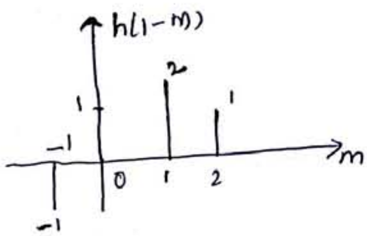


when $n=1$, $y(1) = \sum_{m=-\infty}^{\infty} x(m)\, h(1-m)$



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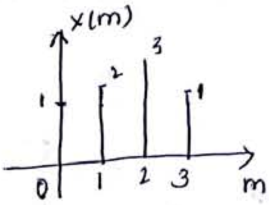

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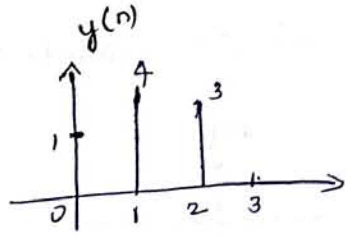
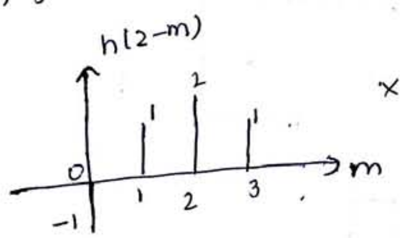


$y(1) = 1+4+3 = 8$

when $n=2$, $y(2)$



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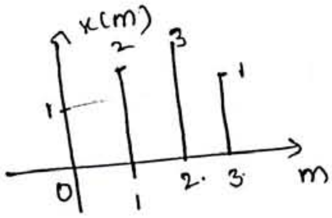
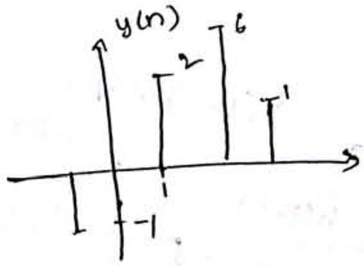
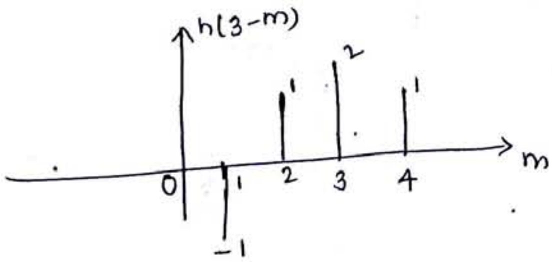


⇒



$y(2) = -1+2+6+1 = 8$

when $n=3$, $y(3)$



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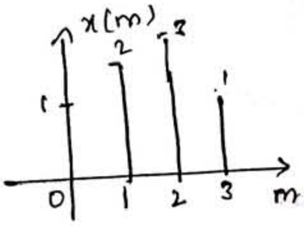
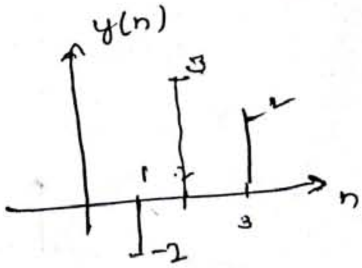
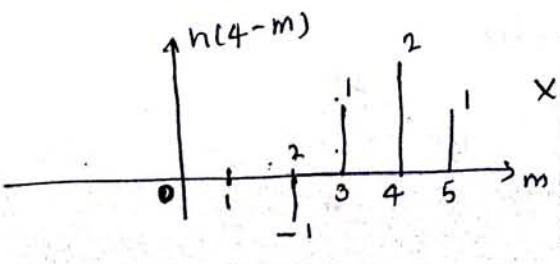


⇒



$y(3) = -2+3+2 = 3$

when $n=4$, $y(4)$



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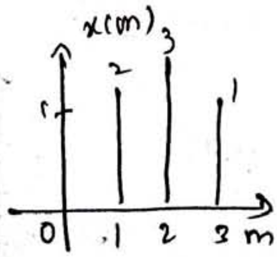
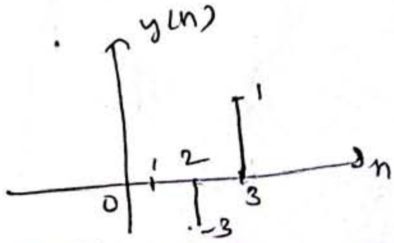
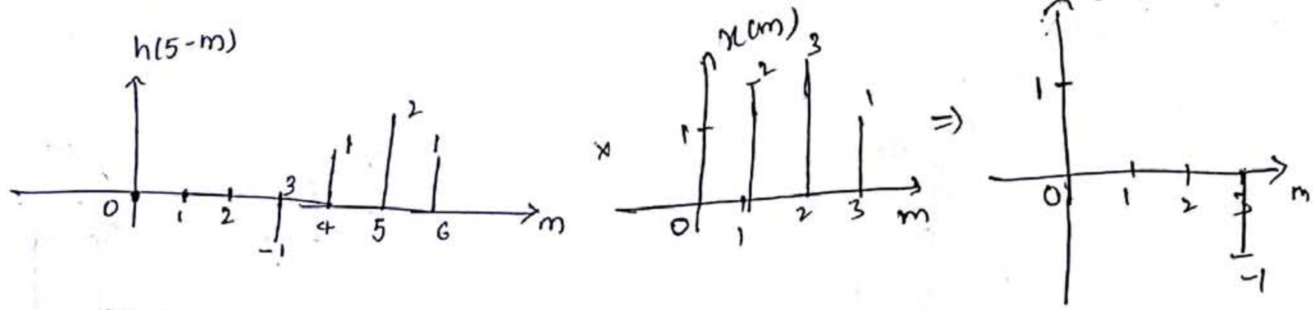


⇒



$y(4) = -3+1 = -2$

when $n = 5$

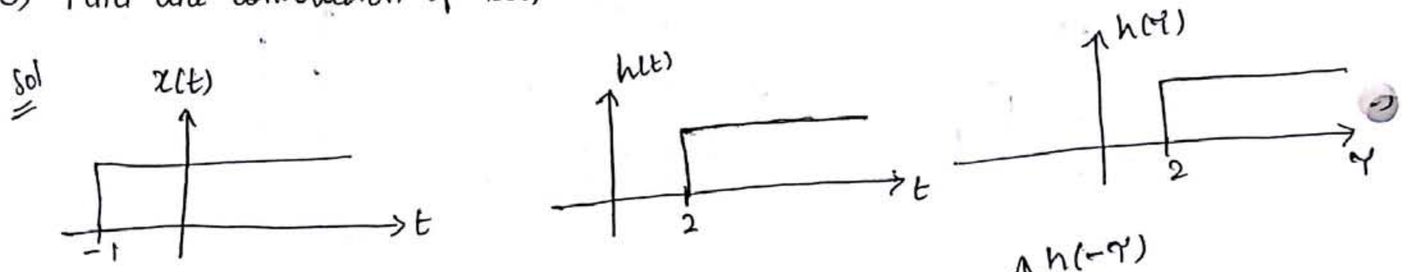


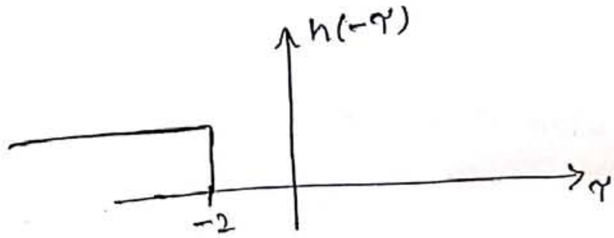
$y(5) = -1$

Output sequence $y(n) = \{1, \underset{\uparrow}{4}, 8, 8, 3, -2, -1\}$

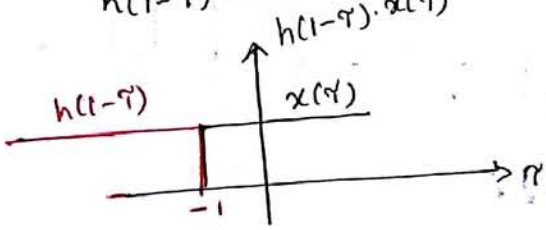
(3) Find the convolution of $x(t) = u(t+1)$ and $h(t) = u(t-2)$

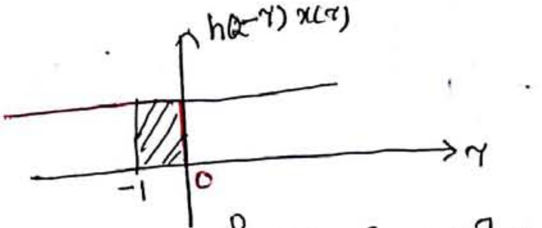
Sol





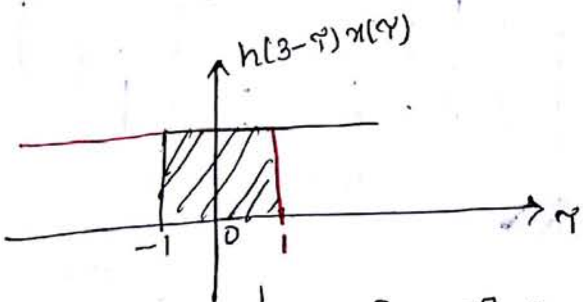
$h(1-\tau)\,x(\tau)$

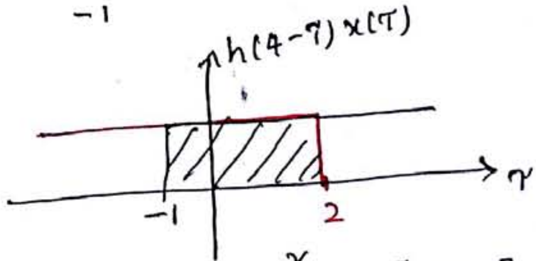




$y(1) =$ no overlap $= 0$

$$y(2) = \int_{-1}^{0} 1 \cdot d\tau = [0-(-1)] = 1$$

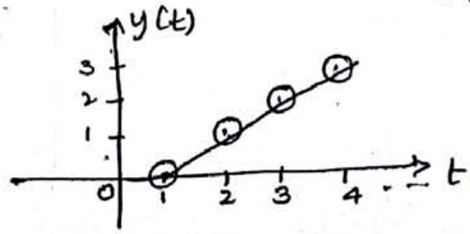




$$y(3) = \int_{-1}^{1} 1 \cdot d\tau = [1-(-1)] = 2$$

$$y(4) = \int_{-1}^{\tau} 1 \cdot d\tau = [2-(-1)] = 3$$

$\therefore\ y(t) =$



Condition

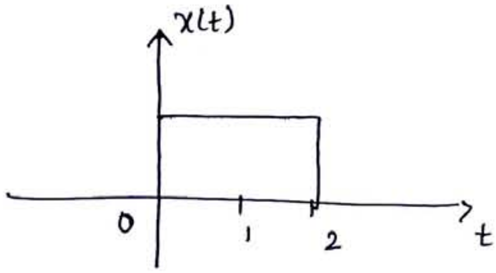
$y(t) = 0$ for $t \leq 0$

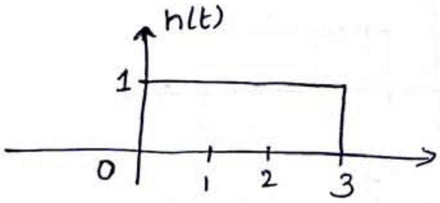
$y(t) = t-1$ for $t > 0$

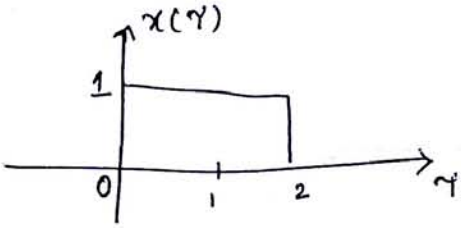
(7) Find the convolution of $x(t)$ and $h(t)$

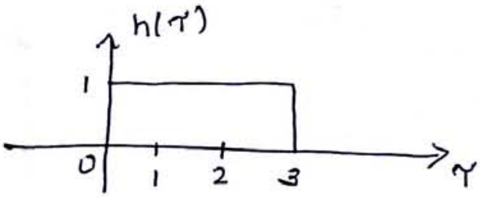
$$x(t) = 1 \quad 0 \le t < 2$$
$$= 0 \quad \text{otherwise}$$
$$h(t) = 1 \quad 0 \le t \le 3$$
$$= 0 \quad \text{otherwise}$$

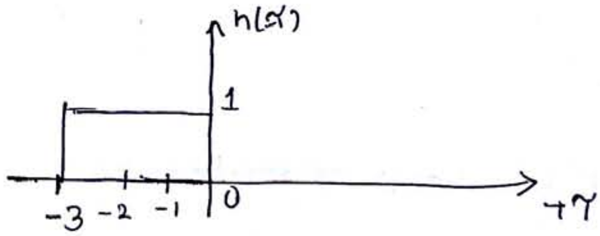
Sol



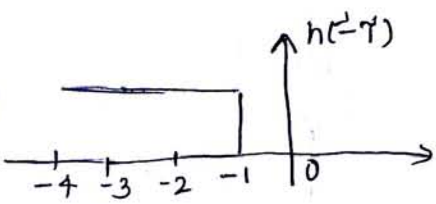


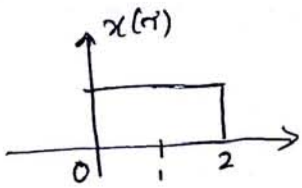




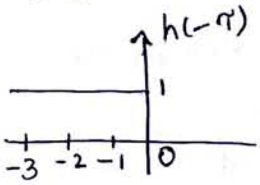


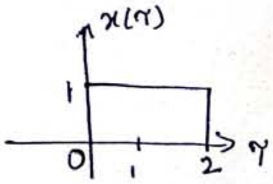
$t = -1$





$t = 0$

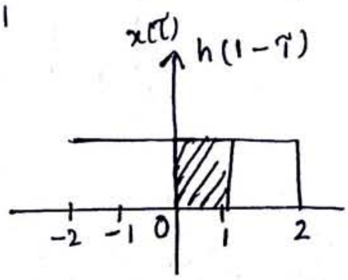


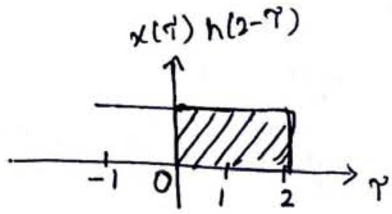


No overlap. $y(t) = 0$

$y(t) = 0$ for $t \le 0$

$t = 1$



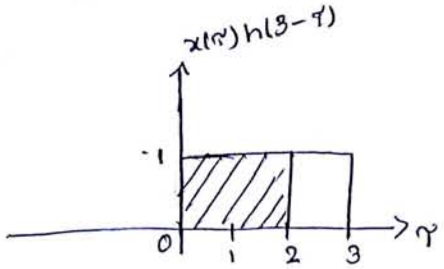


for $0 < t \le 1$

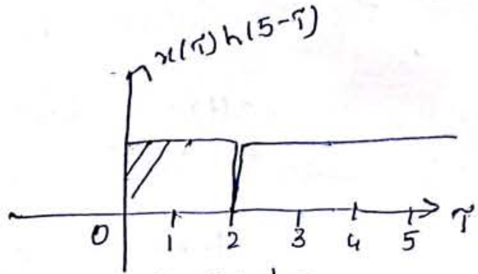
$$y(1) = \int_0^1 1 \cdot d\tau = [\tau]_0^1 = 1$$

for $0 < t \le 2$

$$y(2) = \int_0^2 1 \cdot d\tau = [\tau]_0^2 = 2.$$

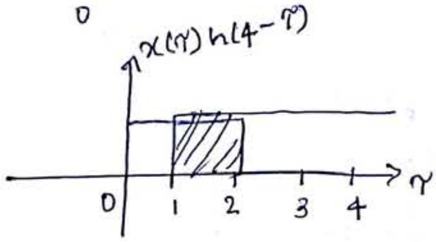


$$y(3) = \int_0^2 d\tau = 2$$

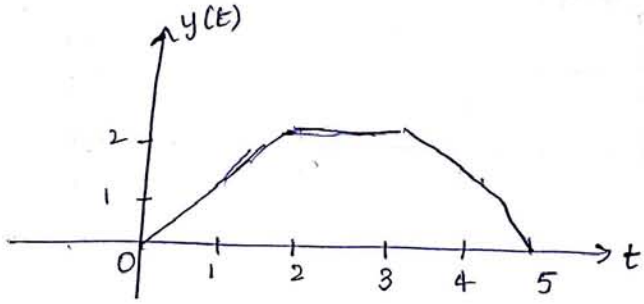


no overlap.

$y(5) = 0$.



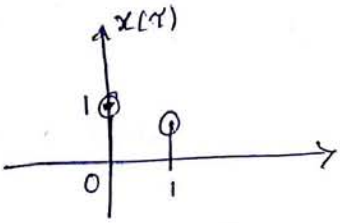
$$y(4) = \int_1^2 d\tau = [2-1] = 1$$

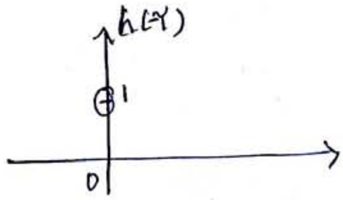


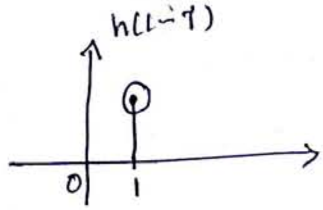
→ (5) Find the linear convolution $x(n) = \{1, 0.5\}$, $h(n) = \{1\}$

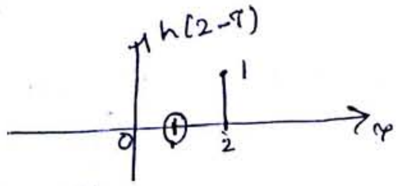
Sol $n_1 = 0$, $n_2 = 0$, So sequence starts at 0

$N_1 = 2$, $N_2 = 1$, $N_1 + N_2 - 1 = 2 + 1 - 1 = 2$





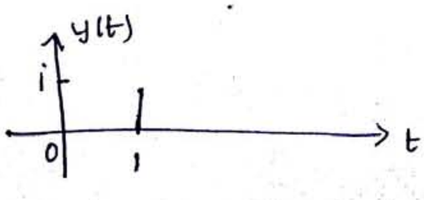




$y(0) = 1$

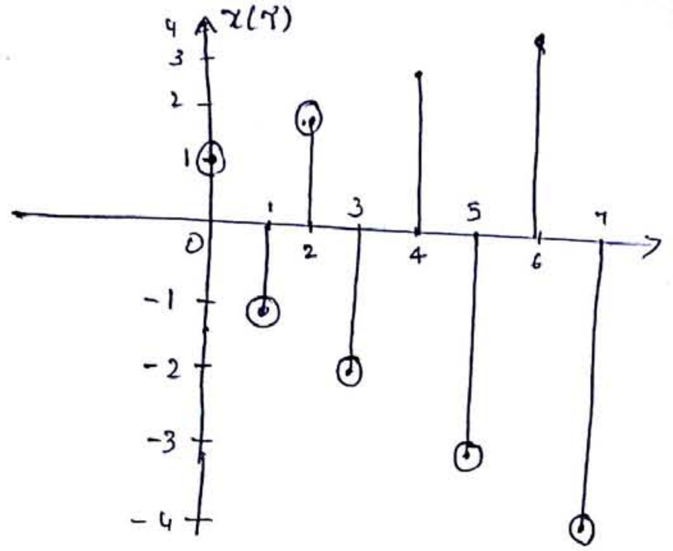
$y(1) = 0 + 0.5 \times 1 = 0.5$

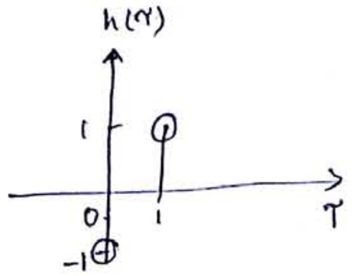
$y(2) = 0$.

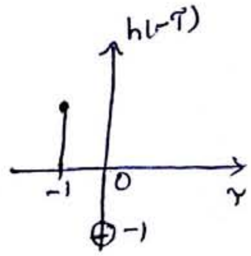


6) . Perform the linear convolution of $x(n) = \{1, -1, 2, -2, 3, -3, 4, -4\}$ $h(n) = \{-1, 1\}$

Sol





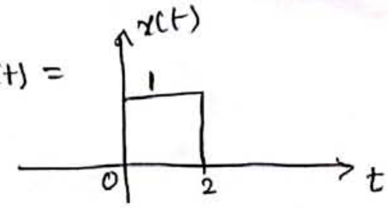


$n_1 = 0$, $n_2 = 0$ O/p sequence starts from 0

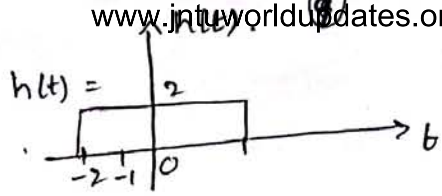
$N_1 = 8$, $N_2 = 2$, $N_1 + N_2 - 1 = 8 + 2 - 1 = 9$

$y(n) = \{-1, 2, -3, 4, -5, 6, -7, 8, 4\}$

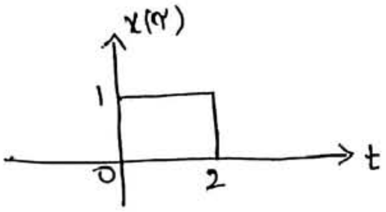
(7) Find the convolution of $x(t) =$

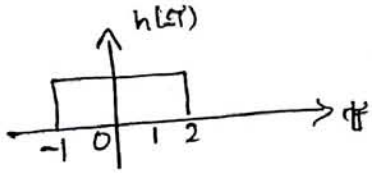


$h(t) =$

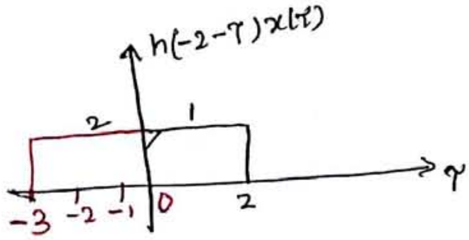


Sol



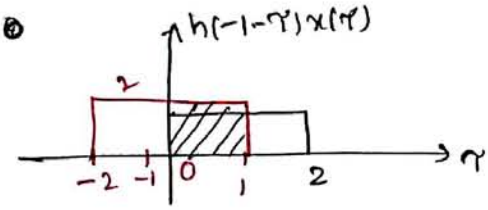


$y(-2) \Rightarrow 0$



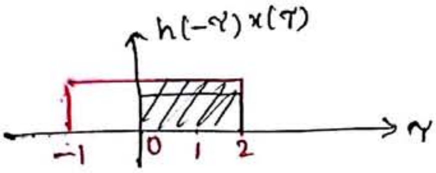
no overlap.

$y(-1) \Rightarrow$



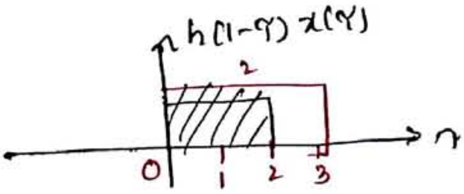
$$\Rightarrow \int_0^1 2.1\, dt = 2.$$

$y(0)$



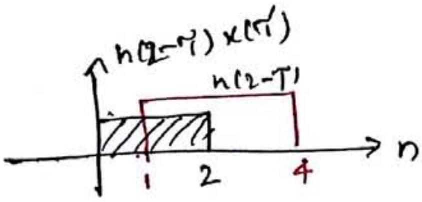
$$\Rightarrow \int_0^2 2.1\, dt = 2[2-0] = 4$$

$y(1) =$



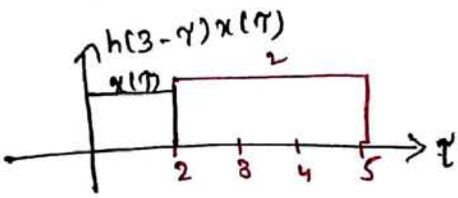
$$\Rightarrow \int_0^2 2.1\, dt = 2[2-0] = 4$$

$y(2) =$

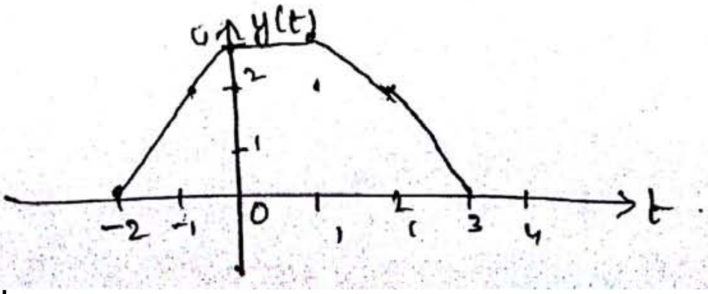


$$\Rightarrow \int_1^2 2.1\, dt = 2[2-1] = 2$$

$y(3) =$



$= \int$ no overlap $= 0$



Properties of Convolution:

Let us consider two signals $x_1(t)$ and $x_2(t)$. The convolution of two signals is given by eqnation.

$$x_1(t) * x_2(t) = \int_{-\infty}^{\infty} x_1(\tau)\, x_2(t-\tau)\, d\tau$$

1. Commutative Property:

Convolution obeys commutative property.

$$x_1(t) * x_2(t) = x_2(t) * x_1(t).$$

Proof

$$x_1(t) * x_2(t) = \int_{-\infty}^{\infty} x_1(\tau)\, x_2(t-\tau)\, d\tau \rightarrow ①$$

$\frac{\Sigma}{P}$

Let $\cancel{t}-\tau = p \Rightarrow \tau = t-p$ | when $\tau = -\infty$, $p = t+\infty = \infty$

$-d\tau = dp$ | when $\tau = \infty$, $p = t-\infty = -\infty$

Substituting in eq ① we get

$$x_1(t) * x_2(t) = -\int_{+\infty}^{-\infty} x_1(t-p)\, x_2(p)\, dp$$

$$= \int_{-\infty}^{\infty} x_2(p)\, x_1(t-p)\, dp$$

$$= x_2(t) * x_1(t)$$

2. Distributive Property:

$$x_1(t) * [x_2(t) + x_3(t)] = x_1(t) * x_2(t) + x_1(t) * x_3(t).$$

Proof

$$= x_1(t) * [x_2(t) + x_3(t)]$$

{ considering $x_4(t) = x_2(t) + x_3(t)$ }

$$= x_1(t) * x_4(t)$$

$$= \int_{-\infty}^{\infty} x_1(\tau)\, x_4(t-\tau)\, d\tau$$

$$= \int_{-\infty}^{\infty} x_1(\tau)\, [x_2(t-\tau) + x_3(t-\tau)]\, d\tau$$

$$= \int_{-\infty}^{\infty} x_1(\tau)\, x_2(t-\tau)\, d\tau + \int_{-\infty}^{\infty} x_1(\tau)\, x_3(t-\tau)\, d\tau$$

$$= [x_1(t) * x_2(t)] + [x_1(t) * x_3(t)] \quad \text{RHS.}$$

(3) Associative property:

$$x_1(t) * [x_2(t) * x_3(t)] = [x_1(t) * x_2(t)] * x_3(t) \quad / \quad x_1(n) * [x_2(n) * x_3(n)] = [x_1(n) * x_2(n)] * x_3(n)$$

Proof

Let $y_1(t) = x_1(t) * x_2(t)$

Let us replace by 'p'

$$y_1(p) = x_1(p) * x_2(p)$$

$$= \sum_{m=-\infty}^{\infty} x_1(m)\, x_2(p-m)$$

Let $y_2(t) = x_2(t) * x_3(t)$

$$y_2(t) = \sum_{q=-\infty}^{\infty} x_1(q)\, x_2(t-q)$$

$$y_2(t-m) = \sum_{q=-\infty}^{\infty} x_1(q)\, x_2(t-m-q)$$

where p, m, and q are dummy variables.

RHS: $[x_1(t) * x_2(t)] * x_3(t)$

$$= y_1(t) * x_3(t)$$

$$= \sum_{p=-\infty}^{\infty} y_1(p)$$

Proof

Let $y_1(n) = x_1(n) * x_2(n)$

Let us replace n by p

$$y_1(p) = x_1(p) * x_2(p)$$

$$= \sum_{m=-\infty}^{\infty} x_1(m)\, x_2(p-m)$$

Let $y_2(n) = x_2(n) * x_3(n)$

$$\therefore y_2(n) = \sum_{q=-\infty}^{\infty} x_1(q)\, x_2(n-q)$$

$$\therefore y_2(n-m) = \sum_{q=-\infty}^{\infty} x_1(q)\, x_2(n-q-m)$$

LHS: $[x_1(n) * x_2(n)] * x_3(n)$

$$= y_1(n) * x_3(n)$$

$$= \sum_{p=-\infty}^{\infty} y_1(p)\, x_3(n-p)$$

$$= \sum_{p=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} x_1(m)\, x_2(p-m)\, x_3(n-p)$$

$$= \sum_{m=-\infty}^{\infty} x_1(m) \sum_{p=-\infty}^{\infty} x_2(p-m)\, x_3(n-p).$$

Let $p-m = q$
$\therefore p = m+q$

when $p=-\infty$, $q = p-m = -\infty - m = -\infty$

when $p=+\infty$, $q = p-m = +\infty - m = +\infty$.

On replacing $(p-m)$ by 'q' and 'p' by $q+m$.

$$= \sum_{m=-\infty}^{\infty} x_1(m) \sum_{q=-\infty}^{\infty} x_2(q)\, x_3(n-q-m)$$

$$= \sum_{m=-\infty}^{\infty} x_1(m)\, y_2(n-m)$$

$$= x_1(n) * y_2(n)$$

$$= x_1(n) * [x_2(n) * x_3(n)]$$

$=$ RHS

(4) Shift property:

If $x_1(t) * x_2(t) = z(t)$ then $x_1(t) * x_2(t-T) = z(t-T)$

Proof

$$x_1(t) * x_2(t-T) = \int_{-\infty}^{\infty} x_1(\tau)\, x_2(t-T-\tau)\, d\tau$$

$$= z(t-T)$$

$||^{ly}$ $x_1(t-T) * x_2(t) = z(t-T)$ and $x_1(t-T_1) * x_2(t-T_2) = z(t-T_1-T_2)$.

(5) Convolution with impulse:

Convolution of a signal $x(t)$ with unit impulse is the signal $x(t)$ itself.

$$x(t) * \delta(t) = x(t).$$

Proof

$$x(t) * \delta(t) = \int_{-\infty}^{\infty} x(\tau)\, \delta(t-\tau)\, d\tau$$

$$= x(t)$$

$\left\{ \begin{array}{l} \delta(t-\tau) = 1 \text{ for } t=\tau \\ \quad = 0 \text{ otherwise} \end{array} \right.$

(6) Convolution with shifted impulse:

Convolution of a signal $x(t)$ with shifted impulse $\delta(t-t_0)$ is equal to $x(t-t_0)$

$$x(t) * \delta(t-t_0) = x(t-t_0)$$

Proof

$$x(t) * \delta(t-t_0) = \int_{-\infty}^{\infty} x(\tau)\, \delta(t-\tau-t_0)\, d\tau$$

$$= x(\tau)\big|_{\tau = t-t_0} = x(t-t_0).$$

(7) Convolution with unit step:

Convolution of a signal $x(t)$ with unit step signal $u(t)$ is given by

$$x(t) * u(t) = \int_{-\infty}^{t} x(\tau)\, d\tau$$

Proof

$$x(t) * u(t) = \int_{-\infty}^{\infty} x(\tau)\, u(t-\tau)\, d\tau$$

$$= \int_{-\infty}^{t} x(\tau)\, d\tau$$

$\because u(t-\tau) = 1$ for $\tau < t \Rightarrow 0$ for $\tau > t$

Correlations:

→ Correlation is basically used to compare two signals or it is a measure of the degree to which two signals are similar.

→ Two types

(1) Cross-correlation

(2) Auto-correlation.

Cross Correlation:

The cross correlation between a pair of signals /sequence $x_1(t)$ and $y(t)$ is given by

$$r_{xy}(\tau) = \int_{-\infty}^{\infty} x(t)\, y^*(t-\tau)\, dt \Big/ r_{xy}(l) = \sum_{n=-\infty}^{\infty} x(n)\, y(n-l) \quad \text{where } l = 0, \pm 1, \pm 2, \cdots \rightarrow (1)$$

$$= \int_{-\infty}^{\infty} x(t+\tau)\, y^*(t)\, dt$$

shift lag parameter.

→ The subscript xy indicates that $x(n)$ is the reference sequence that remains unshifted in time and $y(n)$ is shifted 'l' units in time w.r.to $x(n)$.

→ If we want to fix $y(n)$ and shift $x(n)$ then

$$r_{yx}(l) = \sum_{n=-\infty}^{\infty} y(n)\, x(n-l)$$

$$= \sum_{n=-\infty}^{\infty} y(n+l)\, x(n) \rightarrow (2)$$

→ If time shift $l=0$ then we get

$$r_{xy}(0) = r_{yx}(0) = \sum_{n=-\infty}^{\infty} x(n)\, y(n).$$

Comparing eq(1) with eq(2) we find that

$$r_{xy}(l) = r_{yx}(-l)$$

↳ folded version of $r_{xy}(l)$ abt $l=0$

We can rewrite the eq(1) as

$$r_{xy}(l) = \sum_{n=-\infty}^{\infty} x(n)\, y[-(l-n)]$$

$$= x(l) * y(-l)$$

Auto Correlation :

It is the correlation of a sequence within itself.

$$r_{xx}(l) = \sum_{n=-\infty}^{\infty} x(n)x(n-l)$$

(or)

$$r_{xx}(l) = \sum_{n=-\infty}^{\infty} x(n+l)\,x(n).$$

If time shift $l=0$, then

$$r_{xx}(0) = \sum_{n=-\infty}^{\infty} x^2(n).$$

Auto Correlation Signal :

$$R_{xx}(\tau) = \int_{-\infty}^{\infty} x(t)\,x^*(t-\tau)\,dt$$

$$= \int_{-\infty}^{\infty} x(t+\tau)\,x^*(t)\,dt$$

For real values

$$R_{xx}(\tau) = \int_{-\infty}^{\infty} x(t)\,x(t-\tau)\,dt$$

$$= \int_{-\infty}^{\infty} x(t+\tau)\,x(t)\,dt.$$

Properties of Cross Correlation function for energy signals.

1. For a real valued signals

$$R_{yx}(\tau) = R_{xy}(-\tau)$$

Proof

$$R_{xy}(\tau) = \int_{-\infty}^{\infty} x(t+\tau)\,y(t)\,dt$$

$$R_{yx}(\tau) = \int_{-\infty}^{\infty} y(t+\tau)\,x(t)\,dt = \int_{-\infty}^{\infty} y(t)\,x(t-\tau)\,dt$$

$$= \int_{-\infty}^{\infty} x(t-\tau)\,y(t)\,dt = R_{xy}(-\tau)$$

For complex valued signals $R_{yx}(\tau) = R_{xy}^*(-\tau)$.

(2) If $R_{xy}(0) = 0$ then $\int_{-\infty}^{\infty} x(t)\, y^*(t)\, dt = 0$. Then the sgls are said to be orthogonal over the entire time interval.

Cross correlation of periodic signals:

The cross correlation between two periodic signals $x(t)$ and $y(t)$ is given by

$$R_{xy}(\tau) = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} x(t)\, y^*(t-\tau)\, dt \quad \text{or} \quad \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} x(t)\, y^*(t-\tau)\, dt.$$

||^{ly}

$$R_{yx}(\tau) = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} y(t)\, x^*(t-\tau)\, dt$$

Properties for periodic signals:

property 1: The fourier transform of cross correlation is equal to multiplication of fourier transform of one signal and complex conjugate of F.T of other signal

$$R_{xy}(\tau) \longleftrightarrow \frac{1}{T_0^2} \sum_{k=-\infty}^{\infty} X_1(kf_0)\, X_2^*(kf_0)\, \delta(f - kf_0).$$

2: If cross correlation is executed at origin $(\tau = 0)$

$\lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} x(t)\, y^*(t)\, dt = 0$ i.e $R_{xy}(0) = 0$. then sgls are said to be orthogonal

3: Cross correlation exhibits conjugate symmetry

$$R_{xy}(\tau) = R_{yx}^*(-\tau)$$

4. The cross correlation is not commutative

$$R_{xy}(\tau) \neq R_{yx}(\tau)$$

Properties for auto correlation function: (4)

Property 1: The autocorrelation is an even fn of τ. That is $R_x(\tau) = R_x(-\tau)$

If $x(t)$ is real valued $\rightarrow$ It has even symmetry

$$R_{xx}(\tau) = \int_{-\infty}^{\infty} x(t)\, x(t-\tau)\, dt \rightarrow R_{xx}(\tau) \int_{-\infty}^{0} x(t)\, x(t+\tau)\, dt$$

$$= \int_{-\infty}^{\infty} x(t+T)\, x(t)\, dt \Rightarrow$$

$t+T = x \Rightarrow dt = dx$.

$$= \int_{-\infty}^{0} f_1(x-T)\, f_1(x)\, dx$$

For complex $\rightarrow$ It has hermitian symmetry. changing the variable x to t, we

$$R_{xx}(\tau) = \int_{-\infty}^{\infty} x(t)\, x^*(t-\tau)\, dt$$

$$= \int_{-\infty}^{\infty} x(t+T)\, x^*(t)\, dt.$$

$$= \int_{-\infty}^{0} f_1(t-T)\, f_1(t)\, dt$$

$$= \phi_{11}(T)$$

i.e $R_{xx}(-\tau) = R^*_{xx}(\tau)$.

Otherwise

$$R_{xx}(\tau) = \int_{-\infty}^{\infty} x(t)\, x^*(t-\tau)\, dt$$

$$R^*_{xx}(\tau) = \int_{-\infty}^{\infty} x^*(t)\, x(t-\tau)\, dt$$

$$\therefore R^*_{xx}(-\tau) = \int_{-\infty}^{\infty} x^*(t)\, x(t+\tau)\, dt$$

$$= R_{xx}(\tau)$$

Property 2:

The autocorrelation fn is bounded by its value at the origin That is

$$R_x(0) \geq [R_x(\tau)] \text{ for any } \tau$$

$\rightarrow$ The largest value occurs at $\tau = 0$ of autocorrelation fn.

proof Consider a finite energy signal $x(t)$.

$$y(t) = \int_{-\infty}^{\infty} |x(t+\tau) - \overset{\rightarrow \text{finite value}}{a}\, x(t)|^2\, dt.$$ Obviously $|y(t)| > 0$

$$y(t) = \int_{-\infty}^{\infty} |x(t+\tau) - a\,x(t)|^2\, dt$$

$$= \int_{-\infty}^{\infty} |x(t+\tau)|^2\, dt + |a|^2 \int_{-\infty}^{\infty} |x(t)|^2\, dt - 2|a| \left| \int_{-\infty}^{\infty} x(t+\tau)\, x(t) \right| dt$$

$$R_{xx}(0) = \int_{-\infty}^{\infty} |x(t)|^2\, dt = \int_{-\infty}^{\infty} |x(t+\tau)|^2\, dt$$

$$y(t) \geq [1+|a|^2]\, R_{xx}(0) - 2|a|\, |R_{xx}(\tau)|$$

For $|a| = 1$

$$2R_{xx}(0) \geq 2|R_{xx}(\tau)|$$ since $|y(t)| > 0$.

$$\Rightarrow R_{xx}(0) > R_{xx}(\tau)$$

other method:

Consider the fn $x(t)$ & $x(t+\tau)$

$$[x(t) + x(t+\tau)]^2 \geq 0$$

$$x^2(t) + x^2(t+\tau) \pm 2x(t)\,x(t+\tau) \geq 0$$

$$x^2(t) + x^2(t+\tau) \geq \pm 2x(t)\,x(t+\tau)$$

Integrating on b.s.

$$\therefore \int_{-\infty}^{\infty} |x(t)|^2\, dt + \int_{-\infty}^{\infty} |x(t+\tau)|^2\, dt \geq 2\int_{-\infty}^{\infty} x(t)\, x(t+\tau)\, dt$$

$$\therefore E + E \geq 2R_{xx}(\tau)$$

$$E \geq R_{xx}(\tau)$$

$$\text{or } R(0) \geq R_{xx}(\tau)$$

Property 3:

The value of autocorrelation fn at $\tau=0$ is equal to energy of the signal

$$E = R_{xx}(0) = \int_{-\infty}^{\infty} |x(t)|^2 dt$$

proof

$$R_{xx}(\tau) = \int_{-\infty}^{\infty} |x(t)|^2 dt$$

$$= \int_{-\infty}^{\infty} x(t)\, x^*(t-\tau) dt$$

$$R_{xx}(0) = \int_{-\infty}^{\infty} x(t) \cdot x^*(t)\, dt \Rightarrow \int_{-\infty}^{\infty} |x(t)|^2 dt = E$$

Autocorrelation of power signals:

$$R_{xx}(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} x(t)\, x^*(t-\tau) dt$$

$$= \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} x(t+\tau)\, x^*(t) dt$$

by putting $\tau=0$

$$R_{xx}(0) = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 dt$$

$$\boxed{R_{xx}(0) = P}$$

$\therefore$ The autocorrelation fn at origin is equal to average power of the signal.

For periodic signals:

(1) If $R_{xx}(\tau)$ and $R^*_{xx}(-\tau)$ are complex conjugate of each other then $R(\tau) = R^*(-\tau)$ called as conjugate property.

proof

$$R_{xx}(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} x(t)\, x^*(t-\tau)\, dt$$

$$R^*_{xx}(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} x^*(t)\, x(t-\tau)\, dt$$

$$R^*_{xx}(-\tau) = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} x^*(t)\, x(t+\tau) dt = R(\tau)$$

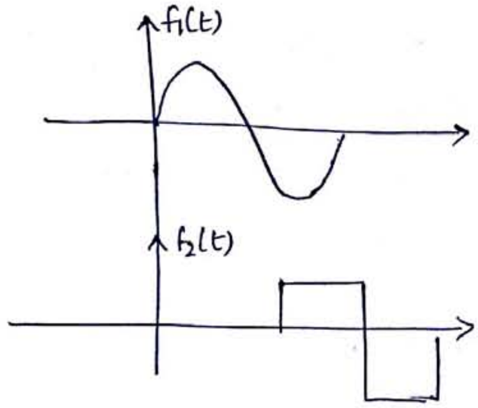
Correlation:

→ correlation means comparison

There are two types of correlations

1. cross correlation
2. Auto correlation.

Cross correlation: It is the measure of similarity between one waveform and time delayed version of the another waveform.

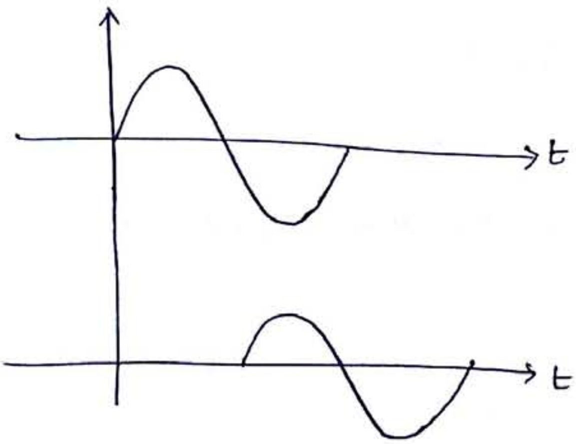


Equation of cross correlation is

$$R_{12}(\tau) = \int_{-\infty}^{\infty} f_1(t)\, f_2^*(t-\tau)\, dt$$

Auto correlation: It is the measure of similarity between one waveform and time delayed version of the same waveform.

Ex:



Equation of auto correlation is

$$R_{11}(\tau) = \int_{-\infty}^{\infty} f_1(t)\, f_1^*(t-\tau)\, dt.$$

APPLICATION OF CORRELATION:

→ It is used in the determination of signal that are contaminated with noise.

CROSS CORRELATION FOR ENERGY SIGNALS:

$$E = \int_{-\infty}^{\infty} f^{2}(t)\, dt \rightarrow \text{cross} \rightarrow \text{two fns}$$

Let $f_1(t)$ & $f_2(t)$ be two energy signals then cross co-relation b/w them is defined as

$$R_{12}(\tau) = \int_{-\infty}^{\infty} f_1(t) f_2^*(t-\tau)\, dt$$

$$R_{12}(\tau) = \int_{-\infty}^{\infty} f_1(t+\tau) f_2^*(t)\, dt$$

Here τ is delay parameter or scanning parameter or scaling parameter

→ From the above two eqns cross correlation function obtained by shifting $f_2(t)$ in +ve direction by an amount τ is equal to the cross correlation function obtained by shifting $f_1(t)$ in -ve direction by an amount τ.

$$R_{21}(\tau) = \int_{-\infty}^{\infty} f_2(t) f_1^*(t-\tau)\, dt$$

$$R_{21}(\tau) = +\int_{-\infty}^{\infty} f_2(t+\tau) f_1^*(t)\, dt$$

By comparing with the convolution integral we can define cross correlation function of $f_1(t)$ & $f_2(t)$ as

$$R_{12}(\tau) = f_1(t) \otimes f_2(-t)$$

→ If $f_1(t)$ & $f_2(t)$ are even fns then cross correlation function becomes equal to convolution.

Differences between convolution & correlation

convolution	correlation
→ $f_1(t) \otimes f_2(t) = \int_{-\infty}^{\infty} f_1(\tau) f_2(t-\tau)\, d\tau$ τ = dummy variable	→ $R_{12}(\tau) = \int_{-\infty}^{\infty} f_1(t) f_2^*(t-\tau)\, dt$ t = dummy variable.
→ convolution is a fn of physical time 't'	→ function of τ
→ It obeys commutation law $f_1(t) \otimes f_2(t) = f_2(t) \otimes f_1(t)$	→ It does not obey commutation law. $R_{12}(\tau) \neq R_{21}(\tau)$
→ It is used to evaluate the response of the system for arbitrary i/p $f_1(t) \otimes f_2(t) = F_1(\omega) F_2(\omega)$	→ It is used to eliminate noise from the signals.

PROPERTIES OF CROSS CORRELATION:

Property 1: Proof: The F.T of cross correlation function of two signals is equal to product of FT of one signal and complex conjugate F.T of other signal.

$$F[R_{12}(\tau)] = X_1(f)\, X_2^*(f)$$

Proof

$$X_1(f) = \int_{-\infty}^{\infty} x_1(t)\, e^{-j2\pi ft}\, dt \rightarrow (1)$$

$$X_2(f) = \int_{-\infty}^{\infty} x_2(t)\, e^{-j2\pi ft}\, dt \rightarrow (2)$$

put $t = t-\tau$

$dt = -d\tau$

$t = t - \tau$

$t = t + \infty = \infty$

$t = t - \infty = -\infty$

$\frac{dt}{d\tau} = 0 - \frac{d\tau}{d\tau}$

$\frac{dt}{d\tau} = -1$

$\Rightarrow dt = -d\tau$

$$X_2(f) = \int_{+\infty}^{-\infty} x_2(t-\tau)\, e^{-j2\pi ft} \cdot e^{j2\pi f\tau} (-d\tau)$$

$$X_2(f) = \int_{-\infty}^{\infty} x_2(t-\tau)\, e^{-j2\pi ft} \cdot e^{j2\pi f\tau}\, d\tau$$

Applying conjugate on both sides

$$X_2^*(f) = \int_{-\infty}^{\infty} x_2^*(t-\tau)\, e^{j2\pi ft} \cdot e^{-j2\pi f\tau}\, d\tau \rightarrow (3)$$

Now multiplying the eq(1) & (3) we get

$$X_1(f)\, X_2^*(f) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1(t)\, x_2^*(t-\tau)\, e^{-j2\pi f\tau}\, dt\, d\tau$$

$$= \int_{-\infty}^{\infty} R_{12}(\tau)\, e^{-j2\pi f\tau}\, d\tau \qquad \left[\because R_{12} = \int_{-\infty}^{\infty} x_1(t)\, x_2^*(t-\tau)\, dt\right]$$

$$\boxed{X_1(f)\, X_2^*(f) = F[R_{12}(\tau)]}$$

(2) The cross correlation fn is zero at origin i.e at $\tau = 0$,

$R_{12}(\tau) = R_{12}(0) = 0$

proof: Consider $R_{12}(\tau) = \underset{T\to\infty}{Lt} \frac{1}{2T}\int_{-T}^{T} x_1(t)\, x_2^*(t-\tau)\, dt$

$$R_{12}(0) = \underset{T\to\infty}{Lt} \frac{1}{2T}\int_{-T}^{T} x_1(t)\, x_2^*(t)\, dt$$

$$= 0 \qquad (\because x_1(t) \,\&\, x_2(t) \text{ are periodic})$$

3) The cross correlation fn satisfies conjugate symmetry property i.e $R_{12}(\tau) = R_{21}^*(-\tau)$

proof

$$R_{12}(\tau) = \underset{T\to\infty}{Lt} \frac{1}{2T}\int_{-T}^{T} x_1(t)\, x_2^*(t-\tau)\, dt \to (1)$$

$$R_{21}(\tau) = \underset{T\to\infty}{Lt} \frac{1}{2T}\int_{-T}^{T} x_2(t)\, x_1^*(t-\tau)\, dt \to (2)$$

Putting $\tau = -\tau$

$$R_{21}(-\tau) = \left\{\underset{T\to\infty}{Lt} \frac{1}{2T}\int_{-T}^{T} x_2(t)\, x_1^*(t+\tau)\, dt\right\}$$

Applying conjugate on both sides, we get

$$R_{21}^*(-\tau) = \underset{T\to\infty}{Lt} \frac{1}{2T}\int_{-T}^{T} x_2^*(t)\, x_1(t+\tau)\, dt$$

{Now ~~put~~ substitute $t = t-\tau$
$dt = dt$ }

$$= \underset{T\to\infty}{Lt} \frac{1}{2T}\int_{-T}^{T} x_2^*(t-\tau)\, x_1(t)\, dt$$

$$= \underset{T\to\infty}{Lt} \frac{1}{2T}\int_{-T}^{T} x_1(t)\, x_2^*(t-\tau)\, dt$$

$$= R_{12}(\tau)$$

$$\boxed{\therefore\ R_{21}^*(-\tau) = R_{12}(\tau)}$$

(4) The cross correlation fn does not satisfy commutative property i.e $R_{12}(\tau) \neq R_{21}(\tau)$

proof

$$R_{12}(\tau) = \underset{T\to\infty}{Lt} \frac{1}{2T}\int_{-T}^{T} x_1(t)\, x_2^*(t-\tau)\, dt$$

$$R_{21}(\tau) = \underset{T\to\infty}{Lt} \frac{1}{2T}\int_{-T}^{T} x_2(t)\, x_1^*(t-\tau)\, dt$$

As above eqn are not same

$$R_{12}(\tau) \neq R_{21}(\tau)$$

(5) The cross correlation for energy signals is given by

$$R_{12}(\tau) = \int_{-\infty}^{\infty} x_1(t)\, x_2^*(t-\tau)\, dt$$

(or)

$$R_{21}(\tau) = \int_{-\infty}^{\infty} x_2(t)\, x_1^*(t-\tau)\, dt$$

(8) Consider two energy signals $g_1(t)$ & $g_2(t)$, which may be complex valued. Show that $\int_{-\infty}^{\infty} |g_1(t) - g_2(t)|^2 dt = R_{11}(0) + R_{22}(0) - 2\,Re[R_{12}(0)]$ where $R_{11}(\tau)$ and $R_{22}(\tau)$ are autocorrelation fn of $g_1(t)$ & $g_2(t)$ & $R_{12}(\tau)$ is their cross

PROBLEMS ON AUTOCORRELATION AND CROSS CORRELATION

(1) Determine the cross correlation fn $R_{xy}(\lambda) \rightarrow R_{12}(\lambda)$ of two signal $g_1(t)$ and $g_2(t)$ defined by

$$g_1(t) = \begin{cases} A\cos(2\pi f_1 t + \theta_1) & 0 \le t \le T \\ 0 & \text{elsewhere} \end{cases} \qquad g_2(t) = \begin{cases} A\cos(2\pi f_2 t + \theta_2) & 0 \le t < T \\ 0 & \text{elsewhere} \end{cases}$$

How does varying the frequency difference $|f_1 - f_2|$ affect this cross correlation fn.

Sol Cross correlation is given as

$$R_{12}(\tau) = \int_{-\infty}^{\infty} x_1(t)\, x_2^*(t-\tau)\, dt$$

or

$$R_{12}(\tau) = \int_{-\infty}^{\infty} g_1(t)\, g_2^*(t-\lambda)\, dt$$

$$= \int_0^T A\cos(2\pi f_1 t + \theta_1) \cdot A\cos(2\pi f_2(t-\lambda) + \theta_2)\, dt$$

$$= \int_0^T A\cos(2\pi f_1 t + \theta_1) \cdot A\cos(2\pi f_2 t - 2\pi f_2 \lambda + \theta_2)\, dt$$

$\{\because \cos(A-B) + \cos(A+B)]$

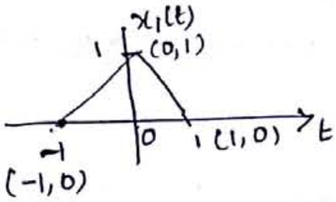
$$= \frac{A^2}{2}\int_0^T \Big\{ \cos(2\pi f_1 t + \theta_1 - 2\pi f_2 t - \theta_2 + 2\pi f_2 \lambda) + \cos(2\pi f_1 t + \theta_1 + 2\pi f_2 t - 2\pi f_2 \lambda + \theta_2) \, dt$$

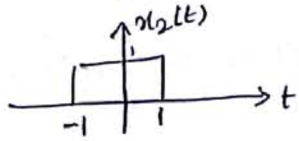
$$= \frac{A^2}{2}\left\{ \left. \frac{\sin(2\pi f_1 t + \theta_1 - 2\pi f_2 t - \theta_2 + 2\pi f_2 \lambda)}{2\pi(f_1 - f_2)} \right]_0^T + \left. \frac{\sin[2\pi(f_1+f_2)t + \theta_1 + \theta_2 - 2\pi f_2 \lambda]}{2\pi(f_1+f_2)} \right\}_0^T \right.$$

$$= \frac{A^2}{2}\left\{ \frac{\sin(2\pi(f_1-f_2)T + \theta_1 - \theta_2 + 2\pi f_2 \lambda) - \sin[\theta_1 - \theta_2 + 2\pi f_2 \lambda)]}{2\pi(f_1 - f_2)} \right.$$

$$\left. + \frac{\sin[(2\pi(f_1+f_2)T + \theta_1 + \theta_2 - 2\pi f_2 \lambda) - \sin[\theta_1 + \theta_2 - 2\pi f_2 \lambda]}{2\pi(f_1+f_2)} \right]$$

2) Find the cross correlation b/w triangular and gate pulse as shown in fig.





Sol

$x_1(t)$ $\quad (x_1, y_1) = (-1,0), (x_2, y_2) = (0,1)$

$y = x_1(t), \; x = t$

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

$$x_1(t) = t + 1$$

$$x_1(t) = \begin{cases} 1+t & \text{for } -1 \leq t \leq 0 \\ 1-t & \text{for } 0 \leq t \leq 1 \end{cases}$$

$x_2(t) = 1$ for $-1 \leq t \leq 1$

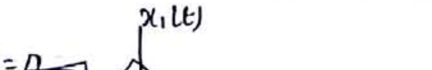
$x_2(t-\tau) = 1$ for $-1 \leq t-\tau \leq 1$

i.e $\tau - 1 \leq t \leq \tau + 1$

$$R_{12}(\tau) = \int_{-\infty}^{\infty} x_1(t)\, x_2(t-\tau)\, dt.$$

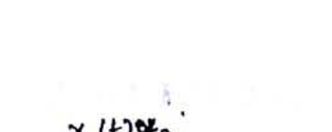
Case i; $\tau < -2$

There is no overlap b/w $x_1(t)$ and $x_2(t)$ $\therefore R_{12}(\tau) = 0$



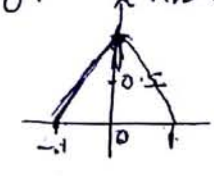
case (ii) $-2 \leq \tau \leq -1$

$$R_{12}(\tau) = \int_{-1}^{0} (t+1)\, dt$$



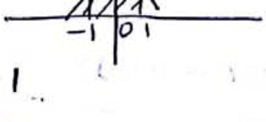
$= \cancel{\left.\frac{t^2}{2}\right|_{-1}^{0} + \left. t \right|_{-1}^{0}}$ $\quad \frac{bh}{2} \Rightarrow \frac{1}{2} \times 1 \times 1 = 0.5$ between -1 to 0.

$= \frac{1}{2} + 1 =$



case (iii)

$R_{12}(\tau) = \frac{1}{2} \times 1 \times 1 +$

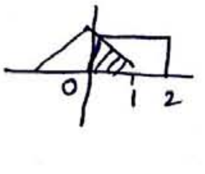


between -1 to 1

$\frac{1}{2} \times 1 \times 1 = 1$

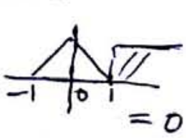
Case (iv) $= \frac{1}{2} \times 1 \times 1$

$= 0.5$



between 0 to 1

Case (v)



$= 0.$

Energy Density Spectrum:

Spectral density: It is the distribution of power or energy of a signal per unit bandwidth as a function of frequency.

Energy and power Signals:

→ Signals with finite energy i.e $0<E<\infty$ and $P=0$ are called energy signals.

e.g: aperiodic signals like pulse

→ Signals with finite average power i.e $0<P<\infty$ and $E=\infty$ are called power sgls i.e periodic signals.

The energy of a signal $x(t)$ is defined as

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt.$$

Parseval's Theorem for energy signal :/ Rayleigh energy theorem:

It defines the energy of a signal in terms of fourier transform.

i.e $$E = \int_{-\infty}^{\infty} |X(f)|^2 df.$$

Proof $x(t) \longleftrightarrow X(f)$. Let $x^*(t)$ be conjugate of $x(t)$ such that

$$x^*(t) \longleftrightarrow X^*(-f)$$

Energy of the signal $x(t)$ is given by

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty} x(t)\, x^*(t)\, dt = \int_{-\infty} x^*(t)\, x(t)\, dt.$$

Replacing $x(t)$ in terms of inverse fourier transform $X(f)$

$$E = \int_{-\infty}^{\infty} x^*(t) \left\{ \int_{-\infty}^{\infty} X(f)\, e^{j\omega t} df \right\} dt$$

Interchanging the order of integration.

$$E = \int_{-\infty}^{\infty} X(f) \left\{ \int_{-\infty}^{\infty} x^*(t)\, e^{j\omega t} dt \right\} df$$

$$E = \int_{-\infty}^{\infty} X(f)\, X^*(f)\ df \Rightarrow \int_{-\infty}^{\infty} |X(f)|^2 df.$$

Energy Spectral density: It is the distribution of energy of signal in frequency domain. which is also called as energy density spectrum (ESD or ED) given by

$$\psi(f) = |X(f)|^2 \quad \rightarrow (1)$$

Let $x(t)$ and $y(t)$ be the input and output of a linear system. i.e $x(t) \longleftrightarrow X(f)$ and $y(t) \longleftrightarrow Y(f)$ and $H(f)$ be system transfer function.

$$Y(f) = H(f)X(f) \longrightarrow (2)$$

Using Eq (1) we can write as

$$\psi_x(f) = |X(f)|^2$$

$$\psi_y(f) = |Y(f)|^2.$$

$$\psi_y(f) = |Y(f)|^2 = |H(f)|^2 |X(f)|^2 = |H(f)|^2 \psi_x(f).$$

$$\boxed{\psi_y(f) = |H(f)|^2 \psi_x(f)}$$

ESD of the output is the product of ESD of input and square of the magnitude of transfer function.

Power Density Spectrum:

Average power is defined as

$$P = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 dt$$

But power P is defined as

$$P = \overline{x^2(t)} = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 dt.$$

Parseval's Power Theorem: It defines the power of a signals in terms of its fourier series coefficients.

$$P = \sum_{n=-\infty}^{\infty} |F_n|^2$$

Proof : Consider a function $x(t)$

$$|x(t)|^2 = x(t)\, x^*(t) \quad \rightarrow \text{conjugate of } x(t). \quad \rightarrow (1)$$

Average power of $x(t)$ for one cycle is

$$P = \frac{1}{T}\int_{-T/2}^{T/2} |x(t)|^2 dt = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\, x^*(t)\, dt \quad \rightarrow (2)$$

we have exponential Fourier series

$$x(t) = \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0 t} \quad \rightarrow (3)$$

Replacing $x(t)$ of eq(2) by eq(3) we get

$$P = \frac{1}{T}\int_{-T/2}^{T/2} \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0 t} x^*(t)\, dt$$

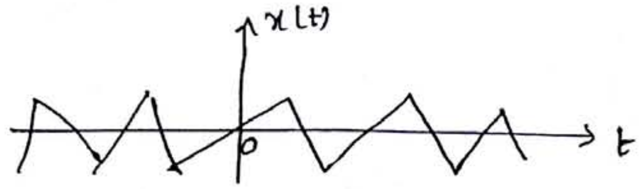
$$= \frac{1}{T}\sum_{n=-\infty}^{\infty} F_n \int_{-T/2}^{T/2} x^*(t)\, e^{jn\omega_0 t} dt = \sum_{n=-\infty}^{\infty} F_n \cdot \frac{1}{T}\int_{-T/2}^{T/2} x^*(t) e^{jn\omega_0 t} dt$$

$$= \sum_{n=-\infty}^{\infty} F_n \cdot F_n^* = \sum_{n=-\infty}^{\infty} |F_n|^2 = P.$$

↳ parsevals power theorem.

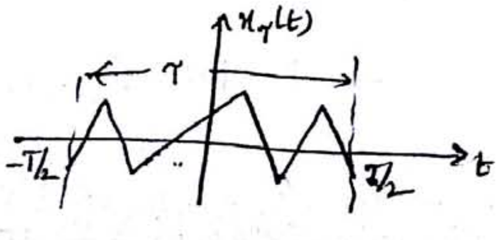
Power Spectral Density (PSD):

The distribution of average power of the signal in frequency domain is called power spectral density or power density spectrum (PSD or PD)



Let us assume that signal is zero outside the interval $|\tau/2)$.

$$x_\tau(t) = \begin{cases} x(t) & |t| < \tau/2 \\ 0 & \text{elsewhere} \end{cases}$$



The signal $x_T(t)$ is of finite duration T and hence it energy signal with energy E given by

$$E = \int_{-\infty}^{\infty} |x_T(t)|^2 dt = \int_{-\infty}^{\infty} |X_T(f)|^2 df$$

where $x_T(t) \longleftrightarrow X_T(f)$.

As $x(t)$ over interval $(-T/2, T/2)$ is same as $x_T(t)$ over the interval $(-\infty, \infty)$

$$\int_{-\infty}^{\infty} |x_T(t)|^2 dt = \int_{-T/2}^{T/2} |x(t)|^2 dt$$

$$\therefore \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 = \frac{1}{T} \int_{-\infty}^{\infty} |X_T(f)|^2 df \rightarrow (1)$$

If $T \rightarrow \infty$, the left hand side of eq(1) represents average power P

$$P = \int_{-\infty}^{\infty} \lim_{T \rightarrow \infty} \frac{|X_T(f)|^2 df}{T}$$

If $T \rightarrow \infty$, $\frac{|X_T(f)|^2}{T}$ approaches finite value denoted by $S(f)$ or $S(\omega)$

$$S(f) = \lim_{T \rightarrow \infty} \frac{|X_T(f)|^2}{T}$$

Average power

$$P = \overline{x^2(t)} = \int_{-\infty}^{\infty} S(f)\, df = \frac{1}{2\pi} \int_{-\infty} S(\omega)\, d\omega .$$

→ The PSD of periodic function is given by

$$S(f) = \sum_{n=-\infty}^{\infty} |F_n|^2 \delta(f - n f_0)$$

$$\text{||}^{ly} \quad S(\omega) = 2\pi \sum_{n=-\infty}^{\infty} |F_n|^2 \delta(\omega - n\omega_0) .$$

I/p and o/p relation of linear system in terms of PSD is given by

$$S_y(f) = |H(f)|^2 \cdot S_x(f) .$$

Comparsion of ESD & PSD

ESD	PSD
(1) It gives the distribution of energy of a signal in frequency	(1) It gives the distribution of power of signal in frequency domain
(2) It is given by $\psi(f) = \lvert X(f)\rvert^2$	(2) It is given by $S(f) = \lim_{T\to\infty} \frac{\lvert X_T(f)\rvert^2}{T}$
(3) Total energy is given by $E = \int_{-\infty}^{\infty} \psi(f)\,df$	(3) Total power is given by $P = \int_{-\infty}^{\infty} S(f)\,df$
(4) The autocorrelation for an energy signal and its ESD form a Fourier transform pair $R(\tau) \longleftrightarrow \psi(f)$	(4) The autocorrelation for a power signal and its PSD form a Fourier transform pair. $R(\tau) \longleftrightarrow S(f)$

RELATION BETWEEN AUTOCORRELATION AND SPECTRAL DENSITIES

1. The autocorrelation function $R(\tau)$ of an energy signal and its energy spectral density (ESD), $\psi(f)$ forms a fourier transform pair,

$$R(\tau) \longleftrightarrow \psi(f)$$

proof: Cross correlation of two energy signals $x(t)$ and $y(t)$ is given as

$$R_{xy}(\tau) = \int_{-\infty}^{\infty} X(f)\, Y^*(f)\, e^{j\omega\tau}\, df.$$

If both functions are same, then auto correlation is given by

$$R(\tau) = \int_{-\infty}^{\infty} X(f)\, X^*(f)\, e^{j\omega\tau}\, df = \int_{-\infty}^{\infty} \lvert X(f)\rvert^2 e^{j\omega\tau}\, df$$

$$= F^{-1}\left[\lvert F(f)\rvert^2\right]$$

But $\lvert F(f)\rvert^2 = \psi(f)$

$$\therefore\ R(\tau) = F^{-1}[\psi(f)] \Rightarrow F[R(\tau)] = \psi(f)$$

$$R(\tau) \longleftrightarrow \psi(f)$$

2) The autocorrelation function $R(\tau)$ and power spectral density (PSD), $S(f)$ of a power signal form a fourier transform pair.

$$R(\tau) \longleftrightarrow S(f)$$

Proof

Autocorrelation function of power $x(t)$ in terms of fourier series coefficients is given as

$$R(\tau) = \sum_{n=-\infty}^{\infty} \underbrace{x_n x_{-n}}_{\text{exponential fourier series coefficients}} e^{jn\omega_0\tau}$$

$$R(\tau) = \sum_{n=-\infty}^{\infty} |x_n|^2 e^{jn\omega_0\tau}$$

Taking Fourier transform.

$$F[R(\tau)] = \int_{-\infty}^{\infty} \left[\sum_{n=-\infty}^{\infty} |x_n|^2 e^{jn\omega_0\tau}\right] e^{-j\omega\tau} d\tau$$

Interchanging the order of integration & summation, we get

$$F[R(\tau)] = \sum_{n=-\infty}^{\infty} |x_n|^2 \int_{-\infty}^{\infty} e^{-j\tau(\omega - n\omega_0)} d\tau$$

$$= 2\pi \sum_{n=-\infty}^{\infty} |x_n|^2 \cdot \delta(\omega - n\omega_0) = \sum_{n=-\infty}^{\infty} |x_n|^2 \delta(f - nf_0).$$

The RHS is the PSD $S(\omega)$ or $S(f)$ of periodic function $x(t)$

$$\therefore F[R(\tau)] = S(f)$$

$$R(\tau) = F^{-1}[S(f)]$$

$$\therefore R(\tau) \longleftrightarrow S(f).$$

RELATION BETWEEN CONVOLUTION AND CORRELATION:

(1) In correlation, physical time 't' plays the role of dummy variable & it appears after solving the integral but in convolution delay parameter τ plays the role of dummy variable.

2) Correlation $R_{xy}(\tau)$ is a function of delay parameter τ, whereas convolution is a function of time.

3) Correlation can be obtained by convolving $x(t)$ & $y^*(t)$

4) Convolution does not depend on which function is being shifted whereas correlation does i.e convolution is commutative.

(7) The convolution and correlation are identical for even signals.

Proof For two signals $x_1(t)$ and $x_2(t)$

The definition for correlation is given by

$$R_{12} = \int_{-\infty}^{\infty} x_1(t)\, x_2(t-\tau)\, dt \longrightarrow ①$$

$$R_{21} = \int_{-\infty}^{\infty} x_2(t)\, x_1(t-\tau)\, dt \longrightarrow ②$$

The definition for convolution

$$x_1(t) * x_2(t) = \int_{-\infty}^{\infty} x_1(\tau)\, x_2(t-\tau)\, d\tau \longrightarrow ③$$

$$= \int_{-\infty}^{\infty} x_2(\tau)\, x_1(t-\tau)\, d\tau \longrightarrow ④$$

Taking eq ① & eq ③

$$R_{12} = \int_{-\infty}^{\infty} x_1(t)\, x_2(t-\tau)\, dt$$

Replacing dummy variable 't' by 'p' we get

$$= \int_{-\infty}^{\infty} x_1(p)\, x_2(p-\tau)\, dp$$

Since it is a even signal i.e $x(t) = x(-t)$

$$= \int_{-\infty}^{\infty} x_1(p)\, x_2(-(\tau-p))\, dp = S$$

$$S = \int_{-\infty}^{\infty} x_1(p)\, x_2(\tau-p)\, dp$$

Replace dummy variable τ by t

$$S = \int_{-\infty}^{\infty} x_1(p)\, x_2(t-p)\, dp$$

Again replacing the variable p by τ

$$S = \int_{-\infty}^{\infty} x_1(\tau)\, x_2(t-\tau)\, d\tau$$

$$= x_1(t) * x_2(t).$$

Extension of Autocorrelation Property.

(4) Fourier transform of auto correlation yields π times the energy density spectrum of $f_1(t)$ $\quad |F(\omega)|^2 = \int_{-\infty}^{\infty} R_{11}(T) e^{-j\omega t} dT = \pi S_1(\omega)$

proof

From definition of FT, FT of $R_{11}(T)$ is

$$\int_{-\infty}^{\infty} R_{11}(T) e^{-j\omega T} dT = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_1(t) \cdot f_1(t-T) e^{-j\omega T} \, dt \, dT$$

$$= \int_{-\infty}^{\infty} f_1(t) e^{-j\omega t} dt \int_{-\infty}^{\infty} f_1(t-T) e^{j\omega(t-T)} dT$$

putting $t-T = -x \Rightarrow dT = dx$

$$= F_1(\omega) \int_{-\infty}^{\infty} f_1(-x) e^{-j\omega x} dx$$

$$= F_1(\omega) F_1(-\omega)$$

$$= |F_1(\omega)|^2 = \pi S_1(\omega).$$

(5) Fourier transform of R_{12} (cross correlation fn) is $F_1(\omega)F_2(-\omega) \longleftrightarrow F_1(\omega)F_2(-\omega)$

proof Fourier transform of $f_1(t)$ & $f_2(t)$ are

$f_1(t) \Leftrightarrow F_1(\omega)$, $f_2(t) \leftrightarrow F_2(\omega)$ $\,||^{ly}\,$ $f_1(-t) \longleftrightarrow F_1(-\omega)$ & $f_2(-t) \longleftrightarrow F_2(-\omega)$

$$R_{12}(T) = f_1(t) * f_2(-t).$$

∴ Fourier transform of $f_1(t) * f_2(-t)$ is $F_1(\omega) F_2(-\omega)$.

$$\therefore \quad R_{12}(T) = f_1(t) * f_2(-t) \longleftrightarrow F_1(\omega) \cdot F_2(-\omega).$$

(6) Graphically $R_{12}(T)$ is same as $R_{21}(T)$ where it is folded back about the vertical axis at $T=0$. $R_{12}(T) = R_{21}(-T)$.

proof

$$R_{12}(T) = \int_{-\infty}^{\infty} f_1(t) \cdot f_2(t-T) \, dt = \int_{-\infty}^{\infty} f_1(t+T) f_2(t) \, dt$$

$$R_{21}(T) = \int_{-\infty}^{\infty} f_2(t) \cdot f_1(t-T) \, dt$$

$$R_{21}(-T) = \int_{-\infty}^{\infty} f_2(t) \cdot f_1(t-(-T)) \, dt = \int_{-\infty}^{\infty} f_2(t) f_1(t+T) \, dt = R_{12}(T)$$

ESD(Energy Spectral Density):

→ Gives the distribution of energy of the signal in the frequency domain.

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 d\omega$$

we know

$\omega = 2\pi f$

$d\omega = 2\pi df$

$$\therefore E = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(f)|^2 \cdot 2\pi df = \int_{-\infty}^{\infty} |X(f)|^2 df \rightarrow (1)$$

coz $|X(2\pi f)|$ is written as $X(f)$.

where $|X(f)| \rightarrow$ amplitude Spectrum.

If we denote $|X(f)|^2$ by $\psi(f)$

$\therefore$ ESD : $\psi(f) = |X(f)|^2 \rightarrow (2)$

putting (2) in eq (1)

$$E = \int_{-\infty}^{\infty} \psi(f) df.$$

E → total energy of the sgl.

$\int_{-\infty}^{\infty} \psi(f) df$ → total area under the curve $\psi(f)$.

→ $\psi(f)$ represents → Energy spectral density of sgl $x(t)$ in joules per hertz.

Effect of Systems on ESD

Let the ESD of $x(t)$ be $\psi_x(f)$ and $y(t)$ be $\psi_y(f)$. The signal $x(t)$ is applied at the input of LTI system and $y(t)$ is obtained at the output.

→ Let LTI system to be an ideal filter which has pass band from f_L to f_H. i.e only signal will be passed without any effect from f_L to f_H.

We know that $E = \int_{-\infty}^{\infty} \psi(f)\,df$

Energy at the output will be

$$E_y = \int_{-\infty}^{\infty} \psi_y(f)\,df.$$

→ If $\psi_y(f)$ is symmetric for positive & negative values of 'f', then

$$E_y = 2\int_{0}^{\infty} \psi_y(f)\,df.$$

$$E_y = 2\int_{f_L}^{f_H} \psi_y(f)\,df$$

$$= 2\int_{f_L}^{f_H} |Y(f)|^2\,df.$$

We know $Y(\omega) = H(\omega)\,X(\omega)$.

$$E_y = 2\int_{f_L}^{f_H} |H(f)X(f)|^2\,df$$

$$= 2\int_{f_L}^{f_H} |H(f)|^2\,|X(f)|^2\,df$$

$$= 2\int_{f_L}^{f_H} |H(f)|^2\,\psi_x(f)\,df$$

→ The filter passes all the frequencies b/w f_L & f_H i.e $H(f) = 1$ for $f_L \leq f \leq f_H$.

$$E_y = 2\int_{f_L}^{f_H} \psi_x(f)\,df$$

energy of the o/p sgl. in terms of ESD of i/p sgl.

RELATION BETWEEN CORRELATION AND CONVOLUTION:

The convolution of $f_1(t)$ and $f_2(-t)$ by $\rho_{12}(t)$, we have

$$\rho_{12}(t) = f_1(t) * f_2(-t)$$

$$= \int_{-\infty}^{\infty} f_1(\tau)\, f_2(\tau - t)\, d\tau$$

The dummy variable τ in the above integral may be replaced by another variable x.

$$\rho_{12}(t) = \int_{-\infty}^{\infty} f_1(x)\, f_2(x-t)\, dx.$$

changing the variable from t to τ, we get

$$\rho_{12}(\tau) = \int_{-\infty}^{\infty} f_1(x)\, f_2(x-\tau)\, dx$$

$$= \phi_{12}(\tau)$$

Hence $\phi_{12}(\tau) = f_1(t) * f_2(-t)\big|_{t=\tau} = \rho_{12}(\tau)$

$||^{ly}$ $\phi_{21}(\tau) = f_1(-t) * f_2(t)\big|_{t=\tau} = \rho_{21}(\tau).$

and $\phi_{11}(\tau) = f_1(t) * f_1(-t)\big|_{t=\tau} = \rho_{11}(\tau)$

DETECTION OF PERIODIC SIGNALS IN THE PRESENCE OF NOISE BY CORRELATION:

→ Now we consider that periodic signals are affected by noise that finds the applications in the detection of radar and sonar signals, periodic component in brain waves and cyclical component in ocean wave analysis.

→ If $s(t)$ is periodic signal and $n(t)$ represents the noise signal, then

$$\lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} s(t)\, n(t-\tau)\, dt = 0 \quad \text{for all } \tau.$$

$$\underbrace{\phi_{sn}(\tau)}_{\text{crosscorrelation fn}} = 0$$

Detection by Autocorrelation:

Let s(t) be a periodic signal mixed with noise signal n(t). Then the received sgl f(t) is $[s(t)+n(t)]$.

→ Let $\bar{\varphi}_{ff}(\tau)$, $\bar{\varphi}_{ss}(\tau)$, $\bar{\varphi}_{nn}(\tau)$ denote the autocorrelation functions of f(t), s(t) and n(t) respectively.

$$\bar{\phi}_{ff}(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} f(t)\, f(t-\tau)\, dt$$

$$= \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} [s(t)+n(t)][s(t-\tau)+n(t-\tau)]\, dt$$

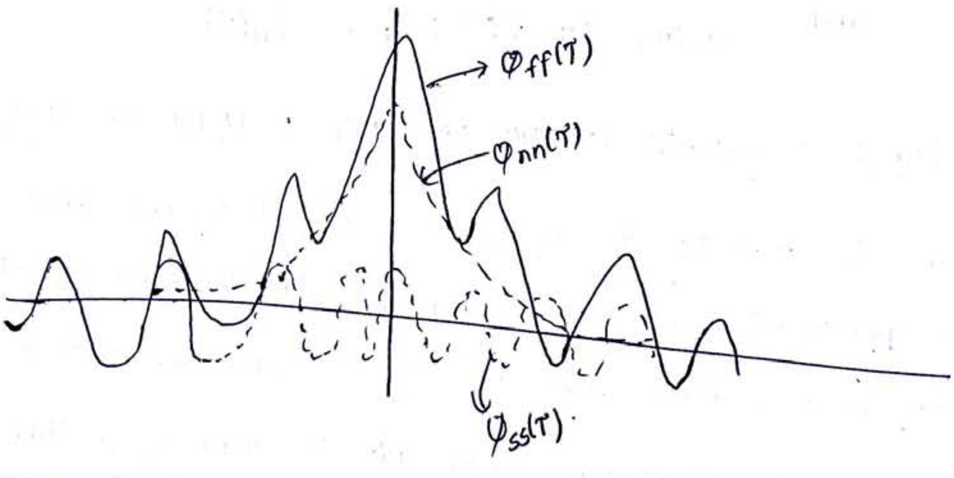
$$= \bar{\phi}_{ss}(\tau) + \bar{\phi}_{nn}(\tau) + \bar{\phi}_{sn}(\tau) + \bar{\phi}_{ns}(\tau)$$

(∵ s(t) & n(t) are uncorrelated)

$$\bar{\phi}_{sn}(\tau) = \bar{\phi}_{ns}(\tau) = 0$$

$$\therefore \quad \bar{\phi}_{ff}(\tau) = \bar{\phi}_{ss}(\tau) + \bar{\phi}_{nn}(\tau).$$

exhibit a periodic nature at larger values of τ.



→ It follows that f(t) contains a periodic signal of frequency displayed by $\bar{\phi}_{ff}(\tau)$.

→ If $\bar{\phi}_{ff}(\tau)$ does exhibit such a periodic nature it is possible to separate $\bar{\phi}_{ss}(\tau)$ and $\bar{\phi}_{nn}(\tau)$ {non periodic component}.

EXTRACTION OF SIGNAL FROM NOISE BY FILTERING:

→ A signal masked by noise can be detected either by correlation techniques or filtering.

→ Correlation technique in time domain and filtering in frequency domain.

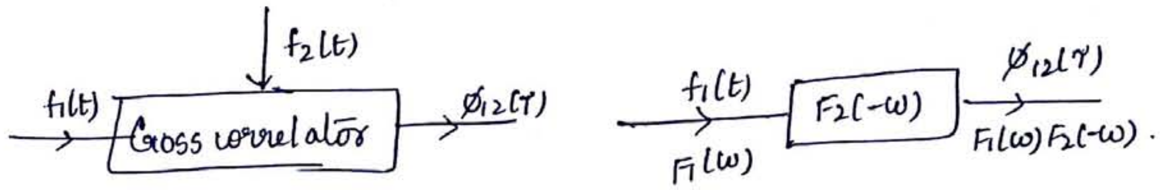


Fig: Crosscorrelation in time & frequency domain.

The impulse response $h(t)$ of a system with a transfer function $F_2(-\omega)$ is given by

$$h(t) = F^{-1}[F_2(-\omega)]$$

But $f_2(t) \longleftrightarrow F_2(\omega)$

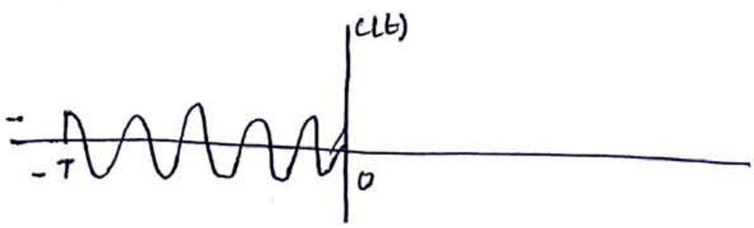
and $f_2(-t) \longleftrightarrow F_2(-\omega)$

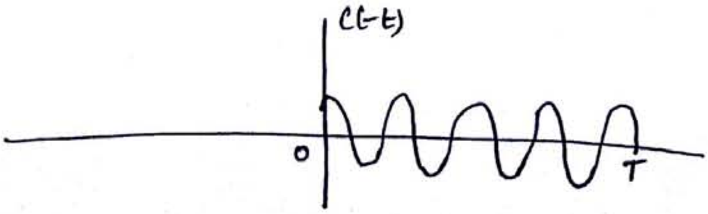
$\therefore$ hence $h(t) = f_2(-t)$.

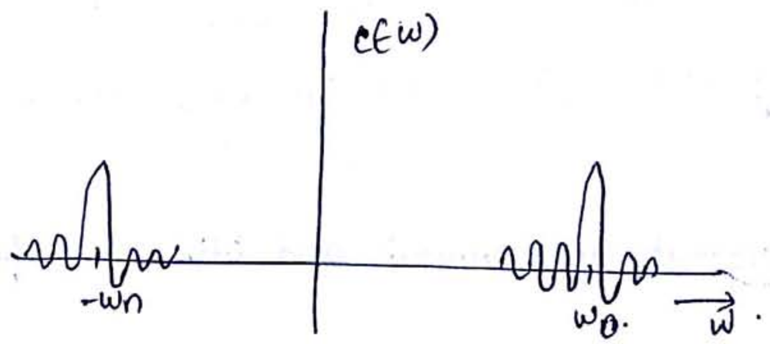
→ The received signal $f(t)$ is

$$f(t) = s(t) + n(t)$$

→ We are filtering out all of the noise signal and extracting the desired periodic signal $s(t)$ by a filter which allows only the frequency components present in $s(t)$ to pass through.






c(ω)
-ωn
ωo.
ω