CLASSIFICATION OF PARTIAL DIFFERENTIAL EQUATIONS OF THE SECOND ORDER

The general second order linear partial differential equation in two independent variables is of the form

$$A(x,y)\frac{\partial^2 u}{\partial x^2} + B(x,y)\frac{\partial^2 u}{\partial x \partial y} + C(x,y)\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = 0$$

Which can be written as

$$Au_{xx} + Bu_{xy} + Cu_{yy} + F(x,y,u,u_x,u_y) = 0 \longrightarrow (1)$$

Where A, B, C, D, E, F are all functions of x and y

A partial differential equation of the form (1) is said to be

i) Elliptic - if $B^2-4AC<0$ at a point in the (x,y) plane. [Laplace eq]

ii) Parabolic - if $B^2-4AC=0$ at a point in the (x,y) plane. [Heat eq]

iii) Hyperbolic - if $B^2-4AC>0$ at a point in the (x,y) plane. [wave eq]

Eg: 1

Consider $u_{xx} + 4u_{xy} + 4u_{yy} - u_x + 2u_y = 0.$

Here $B^2-4AC = 16-16 = 0$

Where $A=1$ $B=4$ $C=4$

Hence it is a parabolic equation

Eg: 2

Consider $x^2u_{xx} + (1-y^2)u_{yy} = 0$; $-\infty < x < \infty, -1<y<1$

Here B^2-4AC, $B=0$ $A=x^2$ $C=(1-y^2)$

$= 0^2 - 4x^2(1-y^2) < 0$, Since $y^2<1$

Hence it is a Elliptic Equation

Eg: 3 $(1+x^2)u_{xx} + (5+2x^2)^2 u_{xy} + (4+x^2)u_{yy} = 0$

Here $B^2-4AC = (5+2x^2)^2 - 4(1+x^2)(4+x^2)$

$= 9>0$ Hence it is Hyperbolic.

Note: The same differential Equation may be elliptic in one region, parabolic in another and hyperbolic in some other region. I

For Eg, the equation $xu_{xx}+u_{yy}=0$ is Elliptic if $x>0$, hyperbolic if $x<0$ and parabolic if $x=0$

Consider the Unit Circle $x^2+y^2=1$

i) It is Elliptic in the inside of Unit Circle.

ii) It is parabolic on the Unit Circle.

iii) It is hyperbolic Outside of Unit Circle.

Eg 1. Classify the following partial differential Equations:

i) $f_{xx}+2f_{xy}+4f_{yy}=0.$

Sol:- Comparing this eq with $Au_{xx}+Bu_{xy}+Cu_{yy}+F(x,y,u,u_x,u_y)=0$

Here $A=1$ $B=2$, $C=4.$

$\therefore B^2-4AC = 4-4\times1\times4=-12<0$

ii) $f_{xx}-2f_{xy}+f_{yy}=0$

Here $A=1$ $B=-2$ $C=1$

$\therefore B^2-4AC = (-2)^2-4\times1\times1$

$= 4-4=0$

So the equation is parabolic.

2. Determine whether the following equation is elliptic or hyperbolic?

$(x+1)u_{xx} - 2(x+2)u_{xy} + (x+3)u_{yy} = 0$

Sol:- Comparing the given equation with

$$Au_{xx} + Bu_{xy} + Cu_{yy} + F(x,y,u,u_x,u_y) = 0$$

Here $A = x+1 \quad B = -2(x+2) \quad C = x+3$

$$\therefore B^2 - 4AC = 4(x+2)^2 - 4(x+1)(x+3)$$

$$= 4[(x^2+4x+4) - (x^2+4x+3)]$$

$$= 4(1) = 4 > 0$$

Hence the eq is hyperbolic at all points of the region.

3. Classify the following equations:

i) $x^2 \frac{\partial^2 u}{\partial x^2} + (1-y^2)\frac{\partial^2 u}{\partial y^2} = 0, \ -\infty < x < \infty, -1 < y < 1$

Comparing the given eq with

$$Au_{xx} + Bu_{xy} + Cu_{yy} + F(x,y,u,u_x,u_y) = 0$$

Here $A = x^2 \quad B = 0 \quad C = 1 - y^2$

$\therefore B^2 - 4AC = 0 - 4x^2(1-y^2) = 4x^2(y^2-1)$

For all x between $-\infty$ to ∞, x^2 is positive

For all y b/w -1 to 1, y^2-1 is negative

$\therefore B^2 - 4AC < 0$ if $-1 < y < 1$, $x \neq 0$

Hence for $-\infty < x < \infty$ $(x \neq 0)$, $-1 < y < 1$

the equation is Elliptic.

Note:

1. For $-\infty < x < \infty$, $x \neq 0$, $y < -1$ or $y > 1$ the equation is hyperbolic
2. For $x = 0$ for all y or for all x, $y = \pm 1$ the eq is parabolic

1. i) Here $A = 1$ $B = 4$ $C = x^2 + 4y^2$

$\therefore$ $B^2 - 4AC = (4)^2 - 4.1(x^2 + 4y^2) = 4(4 - x^2 - 4y^2)$

a) The equation is Elliptic if $B^2 - 4AC < 0$

i.e. if $4 - x^2 - 4y^2 < 0$

i.e. if $x^2 + 4y^2 > 4$ or if $\frac{x^2}{4} + \frac{y^2}{1} > 1$

So, it is elliptic outside the ellipse $\frac{x^2}{4} + \frac{y^2}{1} = 1$

b) The equation is parabolic if $B^2 - 4AC = 0$

i.e if $4 - x^2 - 4y^2 = 0$ or if $\frac{x^2}{4} + \frac{y^2}{1} = 1$

So it is parabolic on the ellipse $\frac{x^2}{4} + \frac{y^2}{1} = 1$

c) The equation is hyperbolic if $B^2 - 4AC > 0$

i.e, if $4 - x^2 - 4y^2 > 0$

i.e, if $4 > x^2 + 4y^2$ or $\frac{x^2}{4} + \frac{y^2}{1} < 1$

So it is hyperbolic inside the ellipse $\frac{x^2}{4} + \frac{y^2}{1} = 1$.

Method of Separation of Variables:

When we have a partial differential equation involving two independent variables say x and y, we seek a solution in the form $X(x) \cdot Y(y)$ and write down (if possible) various types of solutions.

Solved Examples:

1. Solve

$$y^3 \frac{\partial z}{\partial x} + x^2 \frac{\partial z}{\partial y} = 0 \longrightarrow (1)$$

Sol: Let $z = X(x)Y(y)$ be the solution of (1) Then

$$\frac{\partial z}{\partial x} = \frac{dX}{dx} Y ; \quad \frac{\partial z}{\partial y} = X \frac{dY}{dy} \longrightarrow (2)$$

Since z is a solution of (1) [using (2) in (1) we get]

$$\therefore \; y^3 \frac{dX}{dx} Y(y) + x^2 X(x) \frac{dY}{dy} = 0$$

i.e $y^3 X'(x) Y(y) = -x^2 (X)(x) Y'(y)$

or
$$\frac{X'(x)}{x^2 X(x)} = -\frac{Y'(y)}{y^3 Y(y)}$$

In the above, L.H.S is a function of x and R.H.S is a function of y and these are equal for all values of x and y. This is possible if and only if each is equal to the same constant (say) λ. This λ is called separation constant.

$\therefore$ We have

$$\frac{X'(x)}{x^2 X(x)} = \frac{Y'(y)}{y^3 Y(y)} = \lambda \longrightarrow (3)$$

From (3), we get the two ordinary differential eq's

$$X'(x) = \lambda x^2 X(x) \text{ and } Y'(y) = -\lambda y^3 Y(y)$$

Let us solve $X'(x) = \lambda x^2 X(x)$

i.e. $\frac{dX}{dx} = -\lambda x^2 X(x) = 0$

or $\frac{dX}{X} = \lambda x^2\, dx$. Integrating, $\log X = \frac{\lambda x^3}{3} + c$

or $X = Ae^{\lambda x^3/3}$.

Let us Solve $Y'(y) = -\lambda y^3(y)Y \Rightarrow \frac{Y'}{Y} = -\lambda y^3$

Integrating, $\log Y = -\lambda y^4/4 + c$

$\therefore Y = Be^{-\lambda y^4/4}$

Hence the Solution of (1) is given by

$z = X(x)Y(y) = A\, e^{\lambda x^3/3} \cdot Be^{-\lambda y^4/4}$

i.e, $z = c\, e^{\lambda\left(\frac{x^3}{3} - \frac{y^4}{4}\right)}$ Where c is an arbitrary Constant.

2. Solve $\frac{du}{dx} = 2\frac{du}{dt} + u$ Where $u(x,0) = 6e^{-3x}$

or Solve by the method of Separation of Variables

$U_x = 2U_t + u$ Where $u(x,0) = 6e^{-3x}$.

Sol:- We have to find $u(x,t)$ Such that $\frac{du}{dx} = 2\frac{du}{dt} + u \longrightarrow (1)$

Subject to the Condition $u(x,0) = 6e^{-3x} \longrightarrow (2)$

Using the method of Separation of Variables, we seek a Solution of (1) in the form

$u(x,t) = X(x)T(t) \longrightarrow (3)$

If (3) is a Solution of (1),(3) must Satisfy the eq

we have $\frac{du}{dx} = X'(x)T(t)$;

$\frac{du}{dt} = X(x)T'(t) \longrightarrow (4)$

Using (3) and (4) in (1), we get

$$X'(x)T(t) = 2X(x)T'(t) + X(x)T(t)$$

i.e. $$X'(x)T(t) = X(x)\left[2T'(t) + T(t)\right]$$

$$\frac{X'(x)}{X(x)} = \frac{2T'(t) + T(t)}{T(t)}$$

Since L.H.S is a function of x and R.H.S is a function of t, the equality is valid for all x and t if and only if ~~equal~~ each is equal to the same Constant λ for all x and t.

$$\therefore \quad \frac{X'(x)}{X(x)} = \frac{2T'(t) + T(t)}{T(t)} = \lambda$$

i.e. $X'(x) - \lambda X(x) = 0 \Rightarrow X(x) = Ae^{\lambda x}$

and $2T'(t) + T(t) = \lambda T(t)$

i.e $T'(t) + \frac{(1-\lambda)}{2} T(t) = 0$

$\Rightarrow \quad T(t) = Be^{(\lambda-1)t/2}$

Thus $u(x,t) = Ae^{\lambda x} \cdot Be^{(\lambda-1)t/2}$

i.e. $u(x,t) = Ce^{\lambda x} \cdot e^{(\lambda-1)t/2}$.

Using Condition (2) $u(x,0) = 6e^{-3x}$, we get

$$Ce^{\lambda x} = 6e^{-3x}$$

$\therefore \lambda = -3, \; C = 6$

Hence the required solution is.

$u(x,t) = 6e^{-3x} e^{-2t}$

i.e. $u(x,t) = 6e^{-(3x+2t)}$

3. Solve $\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial y} + 2u$ in the form $u = f(x)g(y)$. Obtain the solution satisfying $u=0$, $\frac{\partial u}{\partial x} = 1+e^{-3y}$ when $x=0$ for all values of y.

(or) Solve $u_{xx} = u_{y} + 2u$ with $u(0,y)=0$ and $\frac{\partial u}{\partial x}(0,y) = 1+e^{-3y}$

Sol:- Let $u = X(x)Y(y)$ be the solution of the given eq.

Then we have.

$$\frac{\partial^2 u}{\partial x^2} = X''(x)Y(y);$$

$$\frac{\partial u}{\partial y} = X(x)Y'(y).$$

Sub these in ~~these~~ given eq, we get

$$X''(x)Y(y) = X(x)Y'(y) + 2X(x)Y(y)$$

$$\therefore [X''(x) - 2X(x)]Y(y) = X(x)Y'(y)$$

$$\Rightarrow \frac{X''(x) - 2X(x)}{X(x)} = \frac{Y'(y)}{Y(y)}$$

In the above, the L.H.S is a function of x and R.H.S is a function of y. The equality is possible for all values of x and y if and only if each is equal to the same constant (say) λ. Hence.

$$\frac{X''(x) - 2X(x)}{X(x)} = \frac{Y'(y)}{Y(y)} = \lambda$$

$$\Rightarrow X''(x) - 2X(x) = \lambda X(x)$$

$$\Rightarrow X''(x) - (\lambda+2)X(x) = 0$$

$$\therefore X(x) = Ae^{\sqrt{\lambda+2}\cdot x} + Be^{-\sqrt{\lambda+2}\cdot x}$$

Also $Y'(y) - \lambda Y(y) = 0$

$\therefore Y(y) = Ce^{\lambda y}$

Thus $u(x,y) = \left(Ae^{\sqrt{(\lambda+2)}x} + Be^{-\sqrt{\lambda+2}\,x}\right)e^{\lambda y} \longrightarrow (1)$

(The Constant C is absorbed in A and B)

One of the Conditions is that $\frac{\partial u}{\partial x} = 1 + e^{-3y}$ for $x=0$ for all y.

Hence in the Solution, we must have e^{0y} and e^{-3y}. Thus the values of λ to be chosen are $\lambda = 0$, $\lambda = -3$. Take $\lambda = 0$.

$$\therefore u = \left(Ae^{\sqrt{2}x} + Be^{-\sqrt{2}x}\right).e^{0y}$$

Since $u(0,y) = 0$ for all y.

$$\therefore A + B = 0$$

Partially differentiating (2) with respect to x,

$$\frac{\partial u}{\partial x} = \sqrt{2}\left(Ae^{\sqrt{2}x} - Be^{-\sqrt{2}x}\right)$$

Since $\frac{\partial u}{\partial x} = 1$ for $x = 0$,

therefore, $\sqrt{2}(A-B) = 1$

$$\therefore A - B = 1/\sqrt{2}$$

$$A + B = 0$$

Solving, we get $A = \frac{1}{2\sqrt{2}}$, $B = \frac{-1}{2\sqrt{2}}$

$\therefore u_1 = \left(\frac{1}{2\sqrt{2}}e^{\sqrt{2}x} - \frac{1}{2\sqrt{2}}e^{-\sqrt{2}x}\right)1 = \frac{1}{\sqrt{2}}\sinh\sqrt{2}x$ is a Part of the Solution.

Consider (1) with $\lambda = -3$

$$u = \left(Ae^{+\sqrt{-1}x} + Be^{-\sqrt{-1}x}\right)e^{-3y}$$

$$\Rightarrow u = \left(A^*\cos x + B^*\sin x\right)e^{-3y}$$

Given $u(0,y) = 0$

$$A^* = 0$$

$$\therefore \; u = B^* \sin x \, e^{-3y}$$

$$\Rightarrow \frac{\partial u}{\partial x} = B^* \cos x \, e^{-3y}$$

Consider $\left(\frac{\partial u}{\partial x}\right)_{x=0} = e^{-3y}$ for all y

$$\therefore \; B^* e^{-3y} = e^{-3y} \text{ for all } y \Rightarrow B^* = 1$$

$$\therefore \; u_2(x,y) = \sin x \cdot e^{-3y}$$

Hence the required Solution is $u(x,y) = \frac{1}{2}\sinh\sqrt{2}x + e^{-3y}\sin x.$

4. Solve by the method of Separation of Variables

$$2x z_x - 3y z_y = 0.$$

Sol:- Given Equation is $2x\frac{\partial z}{\partial x} - 3y\frac{\partial z}{\partial y} = 0 \longrightarrow (1)$

Let $z = X(x).Y(y) \longrightarrow (2)$

be the Solution of (1)

Then $\frac{\partial z}{\partial x} = X'Y, \; \frac{\partial z}{\partial y} = XY'$

Sub in (1), we have $2xX'Y - 3yXY' = 0 \rightarrow (3)$

Separating the Variables, (3) can be written as

$$\frac{2xX'}{X} = \frac{3yY'}{Y}$$

In the above, the Left member is a function of x and right member is a function of y. As the two members are equal, their Common Value cannot change When either variable changes and hence it must be equal to a constant λ. Hence we must have

$$\frac{2xX'}{X} = \frac{3yY'}{Y} = \lambda \longrightarrow (4)$$

From (4) we get the two Ordinary differential eq

$$2xX' - \lambda X = 0 \longrightarrow (5)$$

and

$$3yY' - \lambda Y = 0 \longrightarrow (6)$$

$\therefore$ Solution of (5) is

$$\int \frac{X'}{X} dx = \frac{\lambda}{2} \int \frac{dx}{x}$$

$$\Rightarrow \log X = \frac{\lambda}{2} \log x + \log c_1 = \log x^{\lambda/2} c_1$$

$$\Rightarrow X = c_1 x^{\lambda/2} \longrightarrow (7)$$

Now Solution of (6) is

$$\int \frac{Y'}{Y} dy = \frac{\lambda}{3} \int \frac{dy}{y}$$

$$\Rightarrow \log Y = \frac{\lambda}{3} \log y + \log c_2$$

$$= \log y^{\lambda/3} c_2 \Rightarrow c_2 \cdot y^{\lambda/3} = Y \longrightarrow (8)$$

Hence Sub (7) and (8) in (2) we get

$$z = A\, x^{\lambda/2} y^{\lambda/3} \quad \text{Where } A = c_1 c_2$$

Which is the Complete Solution of (1).

5. Use separation of variables to solve $4u_x + u_y = 3u$ with $u(0,y) = 3e^{-y} - e^{-5y}$.

Sol:- We have to solve $4u_x + u_y = 3u \rightarrow (1)$

Subject to the condition

$$u = 3e^{-y} - e^{-5y} \text{ When } x = 0 \rightarrow (2)$$

Using the method of separation of variables

Let $u = X(x)Y(y)$. Then,

$u_x = X'(x)\,Y(y)$;

$u_y = X(x)\,Y'(y)$

Using these in (1), we get

$$4X'(x)Y(y) + X(x)Y'(y) = 3X(x)Y(y)$$

$$4X'(x)Y(y) = -X(x)\left[Y'(y) - 3Y(y)\right]$$

$$\frac{4X'(x)}{X(x)} = -\frac{Y'(y) - 3Y(y)}{Y(y)}$$

L.H.S is a function of x and R.H.S is a function of y. The equality is possible if and only if each side is equal to the same constant say λ.

$$\therefore \frac{4X'(x)}{X(x)} = -\frac{Y'(y) - 3Y(y)}{Y(y)} = \lambda$$

$$\Rightarrow 4X'(x) - \lambda X(x) = 0 \rightarrow (3)$$

and $Y'(y) - 3Y(y) = -\lambda Y(y)$

or $Y'(y) + (\lambda - 3)Y(y) = 0 \rightarrow (4)$

In view of the Condition ② when $x=0$, e^{-y} and e^{-5y} are to be Solutions of ④

Hence, we have to choose

$$\lambda - 3 = 1,$$

i.e, $\lambda = 4$ and $\lambda - 3 = 5$

i.e, $\lambda = 8$

Taking $\lambda = 4$, ④ becomes

$$Y'(y) + Y(y) = 0$$

$$\Rightarrow \quad Y(y) = Ae^{-y}$$

Then ③ becomes

$$4X'(x) - 4X(x) = 0$$

$$\Rightarrow \quad X'(x) - X(x) = 0$$

$$\therefore \quad X(x) = Be^{x}$$

Thus $u = AB\, e^{x-y} = A^* e^{x-y}$ Where $A^* = A \cdot B$

Taking $\lambda = 8$ in ④, we get

$$Y'(y) + 5Y(y) = 0$$

$$Y = A_1 e^{-5y}$$

Now ③ becomes

$$4X'(x) - 8X(x) = 0$$

$$\Rightarrow X'(x) - 2X(x) = 0$$

$$\therefore \quad X(x) = A_2 e^{2x}$$

Thus $u = A_1 A_2\, e^{2x-5y} = B^* e^{2x-5y}$ Where $B^* = A_1 A_2$

$$u = A^* e^{x-y} + B^* e^{2x-5y}$$

If $x=0$, $u = 3e^{-y} - e^{-5y}$

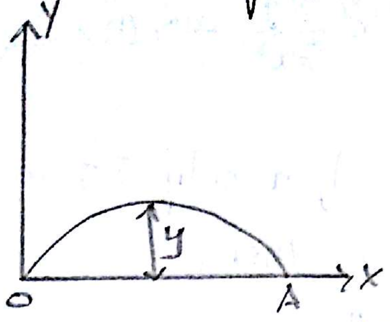
Hence $A^* e^{-y} + B^* e^{-5y} = 3e^{-y} - e^{-5y}$

Thus $A^* = 3$, $B^* = -1$.

$\therefore$ The required solution to the problem is

$u = 3e^{x-y} - e^{2x-5y}$

Vibrations of a stretched string, One dimensional Wave Equation



Consider a uniform elastic string of length l stretched tightly between two pts, and displaced slightly from it Equilibrium position OA.

The PDE giving the transverse vibrations of the string is given by $\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$ where $c^2 = \frac{T}{m}$ (where T is the tension in the string & m is mass per unit length of the string).

It is also called the One dimensional wave Equation.

The boundary conditions which the Eqn $\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$ has to satisfy are:

(i) $y=0$ when $x=0$ } These should be satisfied $\forall t$
(ii) $y=0$ when $x=l$ }

i.e., $\forall$ time t, $y(0,t)=0$ & $y(l,t)=0$.

If the string is made to vibrate by pulling it into a curve $y=f(x)$ & then releasing it; the initial Conditions are

(i) $y=f(x)$ when $t=0$ (ii) $\frac{\partial y}{\partial t}=0$ when $t=0$

(or) $y(x,0)=f(x)$ (or) $\left.\frac{\partial y}{\partial t}\right)_{t=0} = 0$.

Solution of the wave Equation :

The wave eq^n is $\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \rightarrow (1)$

let $y = XT \rightarrow (2)$

Where X is a function of x only & T is a function of t only, be a solution of (1) Then,

$\frac{\partial^2 y}{\partial t^2} = XT''$ & $\frac{\partial^2 y}{\partial x^2} = TX''$

Substitute in Eq^n (1), we have

$XT'' = c^2 TX''$

Separating the variables, we get

$$\frac{X''}{X} = \frac{1}{c^2}\frac{T''}{T} = k \rightarrow (3)$$

$$X'' - kX = 0 \text{ and } T'' - kc^2 T = 0 \rightarrow (4)$$

Solving eq^{n}s (4) we get.

(i) When k is positive & $= p^2$, say

$X = c_1 e^{px} + c_2 e^{-px}$, $T = c_3 e^{cpt} + c_4 e^{-cpt}$

$m^2 = p^2$
$m = \pm p$

(ii) When k is Negative & $= -p^2$

$X = c_1 \cos px + c_2 \sin px$, $T = c_3 \cos cpt + c_4 \sin cpt$

$m^2 = -p^2$
$m = \pm ip$

(iii) When $k = 0$

$X = c_1 x + c_2$, $T = c_3 t + c_4$

Thus the various possible solutions of the wave eq^n (1) are

$y = (c_1 e^{px} + c_2 e^{-px})(c_3 e^{cpt} + c_4 e^{-cpt})$

$y = (c_1 \cos px + c_2 \sin px)(c_3 \cos cpt + c_4 \sin cpt)$

$y = (c_1 x + c_2)(c_3 t + c_4)$

Of these three sol^{n}s, we will choose the one which gives a non-trivial sol^n. on subs of the boundary conditions.

(i) When k is +ve & $= p^2$

$$y = (c_1 e^{px} + c_2 e^{-px})(c_3 e^{cpt} + c_4 e^{-cpt})$$

The bry conditions are $y(0,t)=0$ & $y(l,t)=0$

$$y(0,t) = (c_1+c_2)(c_3 e^{cpt} + c_4 e^{-cpt}) = 0$$

$$\Rightarrow c_1 + c_2 = 0 \Rightarrow c_1 = -c_2$$

$$y(l,t) = (c_1 e^{pl} + c_2 e^{-pl})(c_3 e^{cpt} + c_4 e^{-cpt}) = 0$$

$\because c_1 = -c_2$

$$y = (-c_2 e^{pl} + c_2 e^{-pl})(c_3 e^{cpt} + c_4 e^{-cpt}) = 0$$

$$y = -c_2 (e^{pl} - e^{-pl})(c_3 e^{cpt} + c_4 e^{-cpt}) = 0$$

$\therefore p \neq 0,\ l \neq 0$ & $t \neq 0$ we have $c_2 = 0$

$c_2 = 0 \Rightarrow y(0,t) = 0$ i.e., trivial solution

Case (ii) When $k=0$

$$y = (c_1 x + c_2)(c_3 t + c_4)$$

$$y(0,t) = 0 \Rightarrow y = c_2(c_3 t + c_4) = 0$$

$$\Rightarrow c_2 = 0$$

$$y(l,t) = 0 \Rightarrow y = (c_1 l)(c_3 t + c_4) = 0$$

$\because l \neq 0,\ t=0,\ c_1 = 0 \Rightarrow y = 0$ is also a trivial soln

Case (iii) When k is -ve & $= -p^2$

$$y = (c_1 \cos px + c_2 \sin px)(c_3 \cos cpt + c_4 \sin cpt) \longrightarrow (5)$$

$$y(0,t) = 0 \Rightarrow c_1(c_3 \cos cpt + c_4 \sin cpt) = 0 \longrightarrow (6)$$

$$\& \ y(l,t) = 0 \Rightarrow (c_1 \cos pl + c_2 \sin pl)(c_3 \cos cpt + c_4 \sin cpt) = 0 \longrightarrow (7)$$

From (6)

We have $c_1 = 0$ & thus (7) reduces to

$$c_2 \sin pl \, (c_3 \cos cpt + c_4 \sin cpt) = 0$$

Which is satisfied when $c_2 \neq 0$ & $\sin pl = 0$ or $pl = n\pi$

or $p = \frac{n\pi}{l}$, where $n = 1, 2, 3 \dots$

∴ A sol of the wave Eqⁿ satisfying the boundary Condition is

$$y = c_2 \left(c_3 \cos \frac{n\pi ct}{l} + c_4 \sin \frac{n\pi ct}{l}\right) \sin \frac{n\pi x}{l}$$

$$= \left(a_n \cos \frac{n\pi ct}{l} + b_n \sin \frac{n\pi ct}{l}\right) \sin \frac{n\pi x}{l}$$ on replacing

$c_2 c_3$ by a_n & $c_2 c_4$ by b_n

Adding up the solⁿs for different values of n, we get

$$\boxed{y(x,t) = \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi ct}{l} + b_n \sin \frac{n\pi ct}{l}\right) \sin \frac{n\pi x}{l}}$$ is also a solⁿ → (8)

Now applying the initial Conditions

$y(x,0) = f(x)$ & $\frac{\partial y}{\partial t} = 0$; when $t = 0$, we have

$$f(x) = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi x}{l} \rightarrow (9)$$ represents a

Fourier half range sine series.

for $f(x)$ & $a_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l} \, dx \rightarrow (10)$

At $t = 0$, $\frac{\partial y}{\partial t} = 0$

we have $$\left.\frac{\partial y}{\partial t}\right|_{(x,t)} = \sum_{n=1}^{\infty} \left[-a_n \sin\left(\frac{n\pi ct}{l}\right) \cdot \frac{n\pi c}{l} + b_n \cos\left(\frac{n\pi ct}{l}\right) \cdot \frac{n\pi c}{l}\right] \sin \frac{n\pi x}{l} = 0.$$

At $t=0$

$$\left.\frac{\partial y}{\partial t}\right|_{(x,0)} = \sum_{n=1}^{\infty} \frac{n\pi c}{l} b_n \sin\frac{n\pi x}{l} = 0 \longrightarrow (11)$$

From (11) We have $b_n = 0,\ \forall n$

$\therefore$ (8) reduces to $\boxed{y(x,t) = \sum_{n=1}^{\infty} a_n \cos\frac{n\pi ct}{l} \sin\frac{n\pi x}{l}} \longrightarrow (12)$

Where a_n is given by (10) When $f(x)$
i.e., $y(x,0)$ is known

Problems:-

1. A string is stretched and fastened to two points l apart. Motion is started by displacing the string in the form $y = a\sin(\pi x/l)$ from which it is released at time $t=0$ s.t.t displacement of any pt at a distance x from one End at time t is given by.

$$y(x,t) = a\sin(\pi x/l)\cos(\pi ct/l)$$

Sol The vibration of the string is given by $\frac{\partial^2 y}{\partial t^2} = c^2\frac{\partial^2 y}{\partial x^2} \rightarrow (1)$

As the End pts of the string are fixed, for all time t,

& $y(0,t) = 0 \rightarrow (2)$

$y(l,t) = 0 \rightarrow (3)$

$\therefore$ The initial transverse vel of any pt of the string is zero $\left.\frac{\partial y}{\partial t}\right|_{t=0} = 0 \rightarrow (4)$

Also $y(x,0) = a\sin(\pi x/l) \longrightarrow (5)$

Now, we have to solve (1) Subjects to the boundary conditions 2 & 3 & initial Conditions 4 & 5

$$\frac{\partial^2 y}{\partial t^2} = c^2\frac{\partial^2 y}{\partial x^2}$$

let $y = XT$; $\frac{\partial^2 y}{\partial t^2} = XT''$ & $\frac{\partial^2 y}{\partial x^2} = TX''$

$XT'' = c^2 TX'' \Rightarrow X'' - kX = 0$; $T'' - c^2 kT = 0$

Case (i) : k is +ve & $= p^2$

$$y(x,t) = (c_1 e^{px} + c_2 e^{-px})(c_3 e^{cpt} + c_4 e^{-cpt})$$

Case (ii) : If $k = 0$

$$y(x,t) = (c_1 x + c_2)(c_3 t + c_4)$$

Case (iii) : If k is -ve & $= p^2$

$$y(x,t) = (c_1 \cos px + c_2 \sin px)(c_3 \cos cpt + c_4 \sin cpt)$$

We have to select the one which gives a non-trivial soln on substituting the boundary Conditions

$$y(0,t) = 0 \quad \& \quad y(l,t) = 0$$

Case (i) : $y(x,t) = (c_1 e^{px} + c_2 e^{-px})(c_3 e^{cpt} + c_4 e^{-cpt})$

$$y(0,t) = (c_1 + c_2)(c_3 e^{cpt} + c_4 e^{-cpt}) = 0$$

$$c_1 = -c_2$$

$$y(l,t) = (c_1 e^{pl} + c_2 e^{-pl})(c_3 e^{cpt} + c_4 e^{-cpt}) = 0$$

$$c_2 (e^{-pl} - e^{pl})(c_3 e^{cpt} + c_4 e^{-cpt}) = 0$$

$$c_2 = 0$$

$\Rightarrow y(x,t) = 0$ gives trivial soln.

Case (ii) : $k = 0$

$$y(0,t) = c_2 (c_3 t + c_4) = 0 \Rightarrow c_2 = 0$$

$$y(l,t) = c_1 l (c_3 t + c_4) = 0 \Rightarrow c_1 = 0$$

$\therefore y(x,t) = 0$ gives trivial sol

<u>Case (iii)</u> : k is -ve & $= -p^2$

$$y(x,t) = (c_1 \cos px + c_2 \sin px)(c_3 \cos cpt + c_4 \sin cpt)$$

$$y(0,t) = c_1 (c_3 \cos cpt + c_4 \sin cpt) = 0$$

$$\therefore c_1 = 0$$

$$y(x,t) = c_2 \sin px (c_3 \cos cpt + c_4 \sin cpt) \longrightarrow ⑥$$

$$\frac{\partial y}{\partial t} = c_2 \sin px (-c_3 cp \sin cpt + c_4 cp \cos cpt)$$

$$\left.\frac{\partial y}{\partial t}\right|_{t=0} = c_2 c_4 \sin px \cdot cp = 0$$

$$\therefore c_2 c_4 cp = 0$$

If $c_2 = 0$, ⑥ gives trivial sol. $y(x,t) = 0$

$$\therefore c_4 = 0$$

$$\therefore y(x,t) = c_2 c_3 \sin px \cos cpt \longrightarrow ⑦$$

we have $y(l,t) = c_2 c_3 \sin pl \cos cpt = 0$ for all t

$\therefore$ c_2 & $c_3 \neq 0$, we have $\sin pl = 0$ or $pl = n\pi$ or $p = \frac{n\pi}{l}$

$$y(x,t) = c_2 c_3 \sin \frac{n\pi x}{l} \cos \frac{n\pi ct}{l} \longrightarrow ⑧$$

Imposing the last initial condition

$$y(x,0) = c_2 c_3 \sin \frac{n\pi x}{l} = a \sin \frac{\pi x}{l}$$

which will be satisfied by taking $c_2 c_3 = a$ & $n = 1$

$\boxed{\therefore y(x,t) = a \sin \frac{\pi x}{l} \cos \frac{\pi ct}{l}}$ Which is required soln.

2. A tightly stretched string of length l with fixed ends is initially in equilibrium position. It is set vibrating by giving each pt a vel $v_0 \sin^3 \pi x/l$. Find the displacement $y(x,t)$.

Sol The Eq^n of the vibrating string is $\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \rightarrow$ ①

The bndry conditions are $y(0,t)=0$, $y(l,t)=0 \rightarrow$ ②

The Initial Conditions are $y(x,0)=0 \rightarrow$ ③

& $\left.\frac{\partial y}{\partial t}\right|_{t=0} = V_0 \sin^3 \pi x/l \rightarrow$ ④

The sol of ① are given by

$y(x,t) = (c_1 e^{px} + c_2 e^{-px})(c_3 e^{cpt} + c_4 e^{-cpt})$

$y(x,t) = (c_1 x + c_2)(c_3 t + c_4)$

$y(x,t) = (c_1 \cos px + c_2 \sin px)(c_3 \cos cpt + c_4 \sin cpt)$

Case (i) : When k is +ve & $= p^2$

$y(0,t) = (c_1 + c_2)(c_3 e^{cpt} + c_4 e^{-cpt}) = 0$

$c_1 = -c_2$.

$y(l,t) = c_2 (e^{-px} - e^{px})(c_3 e^{cpt} + c_4 e^{-cpt}) = 0$

$c_2 = 0$

$\therefore y(x,t) = 0$ gives trivial sol.

Case (ii) : $k = 0$

$y(0,t) = c_2(c_3 t + c_4) = 0 \Rightarrow c_2 = 0$

$y(l,t) = c_1 l (c_3 t + c_4) = 0 \Rightarrow c_1 = 0$

$\therefore y(x,t) = 0$ gives trivial sol.

Case (iii) : When k is -ve & $= -p^2$

$y(x,t) = (c_1 \cos px + c_2 \sin px)(c_3 \cos cpt + c_4 \sin cpt)$

$y(0,t) = c_1 (c_3 \cos cpt + c_4 \sin cpt) = 0$

$\therefore c_1 = 0$

$y(l,t) = c_2 \sin pl (c_3 \cos cpt + c_4 \sin cpt) = 0$

$c_2 \neq 0$, $\sin pl = 0$ or $p = \frac{n\pi}{l}$.

$$y(x,t) = c_2 \sin\frac{n\pi x}{l}\left(c_3\cos\frac{n\pi ct}{l} + c_4 \sin\frac{n\pi ct}{l}\right)$$

From the initial conditions $y(x,0)=0$

$$y(x,0) = c_2 c_3 \sin\frac{n\pi x}{l} = 0 \;\forall x$$

i.e., $c_2 c_3 = 0$

$$\Rightarrow y(x,t) = c_2 c_4 \sin\frac{n\pi x}{l}\sin\frac{n\pi ct}{l} = b_n \sin\frac{n\pi x}{l}\sin\frac{n\pi ct}{l}$$

where $b_n = c_2 c_4$

Adding all such sol, the general sol of ① is

$$y(x,t) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}\sin\frac{n\pi ct}{l} \rightarrow ⑤$$

$$\frac{dy}{dt} = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}\cdot\cos\frac{n\pi ct}{l}\left(\frac{n\pi c}{l}\right)$$

$$\left.\frac{\partial y}{\partial t}\right|_{t=0} = \sum b_n\left(\frac{n\pi c}{l}\right)\sin\frac{n\pi x}{l}$$

The given initial condition is $\left.\frac{\partial y}{\partial t}\right|_{t=0} = V_0 \sin^3\frac{\pi x}{l}$

$$\therefore V_0 \sin^3\frac{\pi x}{l} = \left.\frac{\partial y}{\partial t}\right|_{t=0} = \sum \frac{n\pi c}{l} b_n \sin\frac{n\pi x}{l}$$

$(\sin 3\theta = 3\sin\theta - 4\sin^3\theta)$

$$\frac{V_0}{4}\left[3\sin\frac{\pi x}{l} - \sin\frac{3\pi x}{l}\right] = \sum \frac{n\pi c}{l} b_n \sin\frac{n\pi x}{l}$$

$$= \frac{\pi c}{l} b_1 \sin\frac{\pi x}{l} + \frac{2\pi c}{l} b_2 \sin\frac{2\pi x}{l} + \frac{3\pi c}{l} b_3 \sin\frac{3\pi x}{l} + \cdots$$

Equating coeff's on bothsides we get

$$\frac{3V_0}{4} = \frac{\pi c}{l} b_1, \quad 0 = \frac{2\pi c}{l} b_2\,; \quad -\frac{V_0}{4} = \frac{3\pi c}{l} b_3 \cdots$$

$$b_1 = \frac{3lV_0}{4\pi c}, \; b_2 = 0\,; \; b_3 = \frac{-V_0 l}{12\pi c}, \quad b_2 = b_4 = b_5 = b_6 = 0.$$

Subs in ⑤. the desired sol is

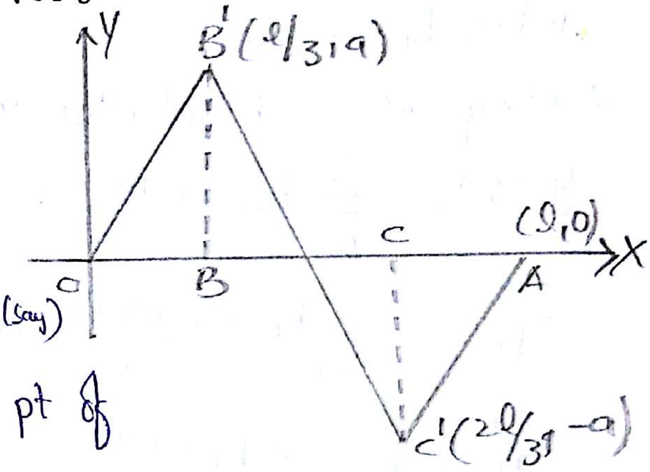
$$y = \frac{3lV_0}{2\times 2\pi c}\sin\frac{\pi x}{l}\sin\frac{\pi ct}{l} - \frac{V_0 l}{12\pi c}\sin\frac{3\pi x}{l}\sin\frac{3\pi ct}{l}.$$

$$= \frac{lV_0}{12\pi c}\left[9\sin\frac{\pi x}{l}\sin\frac{\pi ct}{l} - \sin\frac{3\pi x}{l}\sin\frac{3\pi ct}{l}\right].$$

3. The pts of trisection of a string are pulled aside through the same distance on opposite sides of the position of Equilibrium and the string is released from rest. Derive an Expression for the displacement of the string at subsequent time & show that the mid-pt of the string always remains at rest.

Sol Let B & C be the pts of the trisection of the string OA $(=l)$

Initially the string is held in the form $OB'C'A$ where $BB' = CC' = a$ (say)



The displacement $y(x,t)$ of any pt of the string is $\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \rightarrow ①$

So, the bndry conditions are $y(0,t) = 0 \rightarrow ②$

$$y(l,t) = 0 \rightarrow ③$$

$$\left.\frac{\partial y}{\partial t}\right|_{t=0} = 0 \rightarrow ④$$

The remaining condition is that at $t=0$, the string rests in the form of the line $OB'C'A$. The Eqn of OB' is $y = \frac{3a}{l} x$

The Eqn of $B'C'$ is $y - a = \frac{-2a}{l/3}(x - l/3)$

$$y = a - \frac{6a}{l}(x - l/3)$$

$$= a - \frac{6ax}{l} + 2a = 3a - \frac{6ax}{l}$$

$$y = \frac{3a}{l}(l - 2x)$$

Eqn of $C'A$ is $y = \frac{\cancel{+}a}{\cancel{+}l/3}(x - l) = \frac{3a}{l}(x - l)$

Hence the initial Condition is

$$y(x,0) = \frac{3ax}{l}, \quad 0 \le x \le l/3$$
$$= \frac{3a}{l}(l-2x), \quad l/3 \le x \le \frac{2l}{3} \quad \rightarrow (5)$$
$$= \frac{3a}{l}(l-x), \quad \frac{2l}{3} \le x \le l$$

The sol of $\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$ are given by

$$y = (c_1 e^{px} + c_2 e^{-px})(c_3 e^{cpt} + c_4 e^{-cpt})$$
$$y = (c_1 x + c_2)(c_3 t + c_4)$$
$$y = (c_1 \cos px + c_2 \sin px)(c_3 \cos cpt + c_4 \sin cpt)$$

The non-trivial sol is given by

$$y = (c_1 \cos px + c_2 \sin px)(c_3 \cos cpt + c_4 \sin cpt)$$

we have $y(0,t) = 0$

$$y = c_1(c_3 \cos cpt + c_4 \sin cpt) = 0 \Rightarrow c_1 = 0$$

$y(l,t) = 0$; $y = c_2 \sin pl (c_3 \cos cpt + c_4 \sin cpt) = 0$

$c_2 \ne 0$, $\sin pl = 0$ or $p = \frac{n\pi}{l}$

$$\therefore y(x,t) = c_2 \sin \frac{n\pi x}{l}\left(c_3 \cos \frac{n\pi ct}{l} + c_4 \sin \frac{n\pi ct}{l}\right)$$

The given initial condition is $\left.\frac{\partial y}{\partial t}\right|_{t=0} = 0$

$$\frac{\partial y}{\partial t} = c_2 \sin \frac{n\pi x}{l}\left(-c_3\left(\frac{n\pi c}{l}\right) \sin \frac{n\pi ct}{l} + c_4\left(\frac{n\pi c}{l}\right) \cos \frac{n\pi ct}{l}\right)$$

$$\left.\frac{\partial y}{\partial t}\right|_{t=0} = c_2 \sin \frac{n\pi x}{l} \cdot c_4\left(\frac{n\pi c}{l}\right) = c_2 c_4 \left(\frac{n\pi c}{l}\right) \sin \frac{n\pi x}{l}$$
$$= c_2 c_4 \left(\frac{n\pi c}{l}\right) \sin \frac{n\pi x}{l} = 0 \cdot \Rightarrow c_2 c_4 = 0 \ \forall n$$

$$y(x,t) = c_2 c_3 \sin \frac{n\pi x}{l} \cdot \cos \frac{n\pi ct}{l}.$$

(or) $y(x,t) = b_n \sin\left(\frac{n\pi x}{l}\right) \cos\left(\frac{n\pi ct}{l}\right)$ $[\because b_n = c_2 c_3]$

The general solution is .

$$= \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right) \cos\left(\frac{n\pi c t}{l}\right) \longrightarrow (6)$$

putting, t=0, we have.

$$y(x,0) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right) \longrightarrow (7)$$

In order that the Condition (5) may be satisfied 5 & 7 must be Same This requires the Expansion of $y(x,0)$ into a fourier half range sine series in the interval $(0,l)$

$$b_n = \frac{2}{l}\left[\int_0^{l/3} \frac{3ax}{l} \sin\frac{n\pi x}{l}\,dx + \int_{l/3}^{2l/3} \frac{3a}{l}(l-2x)\sin\frac{n\pi x}{l}\,dx + \int_{2l/3}^{3l/3} \frac{3a}{l}(x-l)\sin\frac{n\pi x}{l}\,dx\right]$$

$$= \frac{6a}{l^2}\left\{\left[\frac{-x\cos\frac{n\pi x}{l}}{n\pi/l} + \frac{\sin\frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)^2}\right]_0^{l/3} + \left[\frac{-(l-2x)\cos\frac{n\pi x}{l}}{n\pi/l} - \frac{2\sin\frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)^2}\right]_{l/3}^{2l/3} + \left[\frac{-(x-l)\cos\frac{n\pi x}{l}}{n\pi/l} + \frac{\sin\frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)^2}\right]_{2l/3}^{l}\right\}$$

$$= \frac{6a}{l^2}\left\{\cancel{\frac{-l^2}{3n\pi}\cos\frac{n\pi}{3}} + \frac{l^2}{n^2\pi^2}\sin\frac{n\pi}{3} + \cancel{\frac{l^2}{3n\pi}\cos\frac{2n\pi}{3}} - \frac{2l^2}{n^2\pi^2}\sin\frac{2n\pi}{3} + \cancel{\frac{l^2}{3n\pi}\cos\frac{n\pi}{3}} + \frac{2l^2}{n^2\pi^2}\sin\frac{n\pi}{3} + \frac{l^2}{n^2\pi^2}\sin n\pi - \cancel{\frac{l^2}{3n\pi}\cos\frac{2n\pi}{3}} - \frac{l^2}{n^2\pi^2}\sin\frac{2n\pi}{3}\right]$$

$$= \frac{6a}{l^2}\left[\frac{3l^2}{n^2\pi^2}\sin\frac{n\pi}{3} - \frac{3l^2}{n^2\pi^2}\sin\frac{2n\pi}{3}\right]$$

$$= \frac{6a}{l^2} \cdot \frac{3l^2}{n^2\pi^2}\left[\sin\frac{n\pi}{3} - \sin\frac{2n\pi}{3}\right] = \frac{18a}{n^2\pi^2}\left[\sin\frac{n\pi}{3} - \sin\frac{2n\pi}{3}\right]$$

$$= \frac{18a}{n^2\pi^2}\left[\sin\frac{n\pi}{3} - \sin\left(n\pi - \frac{n\pi}{3}\right)\right]$$ $\left[\because \sin\left(n\pi - \frac{n\pi}{3}\right) = \sin n\pi \cos\frac{n\pi}{3} - \cos n\pi \sin\frac{n\pi}{3}\right]$

$$= \frac{18a}{n^2\pi^2}\left[\sin\frac{n\pi}{3} + (-1)^n \sin\frac{n\pi}{3}\right]$$

$$= \frac{18a}{n^2\pi^2} \sin\frac{n\pi}{3}\left[1 + (-1)^n\right]$$

$b_n = 0$ if n is odd

$b_n = \frac{36a}{n^2\pi^2} \sin\frac{n\pi}{3}$, when n is Even

Eqn. (6) gives

$$y(x,t) = \sum_{n=2,4}^{\infty} \frac{36a}{n^2\pi^2} \sin\frac{n\pi}{3} \sin\frac{n\pi x}{l} \cos\frac{n\pi ct}{l}.$$

$$y(x,t) = \frac{9a}{\pi^2} \sum_{m=1}^{\infty} \frac{1}{m^2} \sin\frac{2m\pi}{3} \sin\frac{2m\pi x}{l} \cos\frac{2m\pi ct}{l} \longrightarrow (7)$$

putting $x = l/2$ in (7), we find that $y(l/2,t) = 0$.

$\sin m\pi = 0$ for all integral values of m.

Hence the mid pt of the string is always at rest.

4. A tightly stretched Flexible string has its Ends fixed at $x=0$ and $x=l$. At time $t=0$, the string is given a shape defined by $F(x) = \mu x(l-x)$, where μ is a Constant & then released. Find the displacement of any pt x of the string at any time $t>0$.

Sol The wave Eqn is $\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \longrightarrow (1)$

The boundary conditions are $y(0,t) = 0$ & $y(l,t) = 0$ $\longrightarrow (2)$.

The Initial Conditions are $y(x,0) = F(x) = \mu x(l-x) \rightarrow (3)$

$\frac{\partial y}{\partial t}\Big|_{t=0} = 0 \rightarrow (4)$

The sols of (1) are given by

$y(x,t) = (c_1 e^{px} + c_2 e^{-px})(c_3 e^{cpt} + c_4 e^{-cpt})$ $\because$ if k is +ve & $= p^2$

$y(x,t) = (c_1 x + c_2)(c_3 t + c_4)$ $\because$ if $k=0$

$y(x,t) = (c_1 \cos px + c_2 \sin px)(c_3 \cos cpt + c_4 \sin cpt)$

$\because$ if k is -ve & $= -p^2$

From the above we will select the sol which gives a non-trivial sol.

Case (i) : When k is +ve & $= p^2$.

Applying the bndry conditions. $y(0,t) = 0$ & $y(l,t) = 0$

$y(0,t) = (c_1 + c_2)(c_3 e^{cpt} + c_4 e^{-cpt}) = 0$

$c_1 = -c_2$

$y(l,t) = c_2(e^{-pl} - e^{pl})(c_3 e^{cpt} + c_4 e^{-cpt}) = 0$

$c_2 = 0$

$\Rightarrow y(x,t) = 0$ which is a trivial sol.

Case (ii) : When $k=0$

$y(x,t) = (c_1 x + c_2)(c_3 t + c_4)$

$y(0,t) = c_2(c_3 t + c_4) = 0 \Rightarrow c_2 = 0$

$y(l,t) = c_1 l(c_3 t + c_4) = 0 \Rightarrow c_1 = 0$

$\therefore y(x,t) = 0$ gives a trivial sol.

Case (iii) : When k is -ve & $= -p^2$

$y(x,t) = (c_1 \cos px + c_2 \sin px)(c_3 \cos cpt + c_4 \sin cpt)$

$y(0,t) = c_1(c_3 \cos cpt + c_4 \sin cpt) = 0 \Rightarrow c_1 = 0.$

$y(l,t) = c_2 \sin pl \,(c_3 \cos cpt + c_4 \sin cpt) = 0$

$c_2 \neq 0, \; \sin pl = 0 \Rightarrow pl = n\pi$ (or) $p = \frac{n\pi}{l}$

$y(x,t) = c_2 \sin \frac{n\pi x}{l}\left(c_3 \cos \frac{n\pi ct}{l} + c_4 \sin \frac{n\pi ct}{l}\right) \rightarrow (5)$

Diff (5) w.r.t 't' & then substituting the initial

Condition $\frac{\partial y}{\partial t}\Big|_{t=0} = 0$

$$\frac{\partial y}{\partial t} = c_2 \sin \frac{n\pi x}{l}\left(-c_3\left(\frac{n\pi c}{l}\right)\sin \frac{n\pi ct}{l} + c_4\left(\frac{n\pi c}{l}\right)\cos \frac{n\pi ct}{l}\right)$$

$$\frac{\partial y}{\partial t}\Big|_{t=0} = c_2 c_4 \left(\frac{n\pi c}{l}\right)\sin \frac{n\pi x}{l} = 0$$

If $c_2 = 0$, $y(x,t)$ gives a trivial sol

$\therefore c_4 = 0$

$y(x,t) = c_2 c_3 \sin \frac{n\pi x}{l} \cos \frac{n\pi ct}{l} \rightarrow (6)$

put $c_2 c_3 = b_n$

$y(x,t) = b_n \sin \frac{n\pi x}{l} \cos \frac{n\pi ct}{l} \rightarrow (7)$

The most general sol is

$$y(x,t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \cos \frac{n\pi ct}{l} \rightarrow (8)$$

put $t = 0$

$$y(x,0) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \rightarrow (9)$$

which is half range sine series. But given initial Condition $y(x,0) = F(x) = \mu x(l-x)$

$$b_n = \frac{2}{l}\int_0^l F(x) \sin \frac{n\pi x}{l}\,dx$$

$$= \frac{2}{l}\int_0^l (\mu l x - \mu x^2) \sin \frac{n\pi x}{l}\,dx$$

$$= \frac{2\mu}{l}\left[-(lx-x^2)\frac{\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}} + (l-2x)\frac{\sin\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}} + (-2)\frac{\cos\frac{n\pi x}{l}}{\frac{n^3\pi^3}{l^3}}\right]_0^l$$

$$= \frac{2\mu}{l}\left[(l-l^2)\frac{l^2}{n^2\pi^2}\sin n\pi - \frac{2l^3}{n^3\pi^3}\cos n\pi + \frac{2l^3}{n^3\pi^3}\right]$$

$$= \frac{-4\mu}{l}\frac{l^3}{n^3\pi^3}\left[\cos n\pi - \cos 0\right] = \frac{-4\mu l^2}{n^3\pi^3}\left[(-1)^n - 1\right]$$

$b_n = 0$. when n is even; $b_n = \frac{8\mu l^2}{n^3\pi^3}$ if n is odd.

$$b_{2n-1} = \frac{8kl^2}{(2n-1)^3\pi^3} \quad \forall n$$

Subs in (9)

$$y(x,t) = \sum_{n=1}^{\infty} \frac{8kl^2}{(2n-1)^3\pi^3} \sin\frac{(2n-1)\pi x}{l} \cos\frac{(2n-1)\pi ct}{l}.$$

5. A tightly stretched string with fixed End pts $x=0$ & $x=l$ is initially at rest in its Equilibrium position of it is vibrating by giving to Each of its pts a velocity $\lambda x(l-x)$, find the displacement of the string at any distance x from one End at any time t.

Sol The wave Eq^n is given by $\frac{\partial^2 y}{\partial t^2} = c^2\frac{\partial^2 y}{\partial x^2}$ $\rightarrow$ (1)

The bndry conditions are given by

$y(0,t)=0$ & $y(l,t)=0$ for all time t $\rightarrow$ (2)

The initial Conditions are given by

$y(x,0) = f(x) = 0$ & $\rightarrow$ (3)

$\frac{\partial y}{\partial t}\Big|_{t=0} = \lambda x(l-x)$ $\rightarrow$ (4)

$y(x,t) = C_2 \sin\frac{n\pi x}{l}\left(C_3\cos\frac{n\pi ct}{l} + C_4\sin\frac{n\pi ct}{l}\right)$ $\rightarrow$ (5)

$y(x,0) = C_2 C_3 \sin \frac{n\pi x}{l} \Rightarrow C_3 = 0$

$\therefore y(x,t) = C_2 C_4 \sin \frac{n\pi x}{l} \sin \frac{n\pi ct}{l}$

The general sol is given by

$$y(x,t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \sin \frac{n\pi ct}{l} \quad \text{---}$$

$$\frac{\partial y}{\partial t} = \sum_{n=1}^{\infty} b_n \left(\frac{n\pi c}{l}\right) \sin \frac{n\pi x}{l} \cos \frac{n\pi ct}{l}$$

$$\left.\frac{\partial y}{\partial t}\right|_{t=0} = \sum_{n=1}^{\infty} \left(\frac{n\pi c}{l}\right) \sin \frac{n\pi x}{l}$$

$= \lambda x(l-x)$ which is a half range sine series.

$$b_n\left(\frac{n\pi c}{l}\right) = \frac{2}{l}\int_0^l g(x) \sin \frac{n\pi x}{l}\, dx$$

$$= \frac{2\lambda}{l}\int_0^l (lx - x^2) \sin \frac{n\pi x}{l}\, dx$$

$$b_n\left(\frac{n\pi c}{l}\right) = \frac{2\lambda}{l}\left[-(lx-x^2)\frac{\cos \frac{n\pi x}{l}}{\frac{n\pi}{l}} + (l-2x)\frac{\sin \frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}} + (-2)\frac{\cos \frac{n\pi x}{l}}{\frac{n^3\pi^3}{l^3}}\right]_0^l.$$

$$\cancel{b_n} \;\; b_n\left(\frac{n\pi c}{l}\right) = \frac{2\lambda}{l}\left[\frac{-2l^3}{n^3\pi^3}\cos n\pi + \frac{2l^3}{n^3\pi^3}\cos 0\right]$$

$$= \frac{-4\lambda l^2}{n^3\pi^3}\left[\cos n\pi - \cos 0\right]$$

$$= \frac{-4\lambda l^2}{n^3\pi^3}\left[(-1)^n - 1\right]$$

$$b_n = \frac{-4\lambda l^2}{n^3\pi^3}\,\frac{l}{n\pi c}\left[(-1)^n - 1\right]$$

$$= \frac{-4\lambda l^3}{n^4\pi^4 c}\left[(-1)^n - 1\right]$$

$b_n = 0$ when n is Even

$b_n = \dfrac{8\lambda l^3}{n^4\pi^4 c}$ when n is odd

$b_{2n-1} = \dfrac{8\lambda l^3}{(2n-1)^4\pi^4 c}$ for all n

subs in (6)

$$y(x,t) = \sum_{n=1}^{\infty} \frac{8\lambda l^3}{(2n-1)^4\pi^4 c} \sin\frac{(2n-1)\pi x}{l} \sin\frac{(2n-1)\pi ct}{l}.$$

One Dimensional Heat Equation:-

One dimensional heat Eqn is given by

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$$, where α^2 is the thermal diffusivity of the substance.

$\alpha^2 = K/s\rho$

K - thermal conductivity

S - specific heat

ρ - density

Solution of one-dimension heat eqn by seperation of Variables:

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2} \quad \text{---} (1)$$

Let $u(x,t) = XT$ --- (2) be a solution of eqn (1)

where X is purely a function of x &

T is purely a function of t.

We have $\frac{\partial^2 u}{\partial x^2} = TX''$ & $\frac{\partial u}{\partial t} = XT'$

$$XT' = \alpha^2 TX''$$

$$X\frac{dT}{dt} = \alpha^2 T \frac{d^2X}{dx^2}$$

$$\frac{\frac{1}{\alpha^2}\frac{dT}{dt}}{T} = \frac{\frac{d^2X}{dx^2}}{X} = K \quad \text{---} (3)$$

case(i) :- When K is +ve & $= p^2$

$$\frac{dT}{dt} = p^2\alpha^2 T \quad \text{i.e} \quad \frac{dT}{T} = p^2\alpha^2 dt$$

Int, $\log T = p^2\alpha^2 t + \log c_3$

$$\log \frac{T}{c_3} = \alpha^2 p^2 t \quad \text{or} \quad T = c_3 e^{\alpha^2 p^2 t}$$

Again $\frac{d^2X}{dx^2} = p^2X$

$m^2 = p^2$ or $m = \pm p$

$X = c_1 e^{px} + c_2 e^{-px}$

$\therefore u(x,t) = (c_1 e^{px} + c_2 e^{-px}) c_3 e^{\alpha^2 p^2 t}$ —— (4)

case (ii) :- When $k = 0$

$\frac{dT}{dt} = 0 \implies T = c_3$

$\frac{d^2X}{dx^2} = 0$ or $X = c_1 x + c_2$

$\therefore u(x,t) = (c_1 x + c_2) c_3$ —— (5)

case (iii) :- When k is $-$ve & $= -p^2$

$\frac{dT}{dt} = -\alpha^2 p^2 T$ or $\frac{dT}{T} = -\alpha^2 p^2 dt$

$\log T = -\alpha^2 p^2 t + \log c_3$

$\log \frac{T}{c_3} = -\alpha^2 p^2 t$ or $T = c_3 e^{-\alpha^2 p^2 t}$

$\frac{d^2X}{dx^2} = -p^2 X$

$m^2 = -p^2$

$X = c_1 \cos px + c_2 \sin px$

$\therefore u(x,t) = (c_1 \cos px + c_2 \sin px) c_3 e^{-\alpha^2 p^2 t}$ —— (6)

(4), (5) & (6) are three independent sols of (1)

Out of these (3) sols, we have to choose the sol which gives a non-trivial sol on the substitution of the bndry conditions.

steady - state conditions & zero homogenous boundary conditions:

Suppose a bar AB of length l is heated at both ends by a constant temp $T_1°$ & $T_2°$. Then the temperature in the bar remains constant after sometime. Hence temp $u(x,t)$ is a function of x alone since there is no change in temp in the bar with respect to time. If the temp at any point no longer varies with time is called steady state conditions prevails

Hence $\frac{\partial u}{\partial t}=0$ & the heat eqn reduces to $\frac{\partial^2 u}{\partial x^2}=0$

i.e $\frac{d^2u}{dx^2}=0$ —— ①

$\therefore\ u(x,0) = ax+b$

1. Find the temp $u(x,t)$ in a bar which is perfectly insulated laterally, whose ends are kept at temp $0°C$ & whose initial temp in ($°c$) is $f(x)=x(10-x)$ given that its length is 10cm. constant cross section of area $1 cm^2$, density $10.6\ gm/cm^3$, thermal conductivity 1.04 cal/cm·deg and specific heat 0.056 cal/gm deg.

Sol The heat eqn is $\frac{\partial u}{\partial t}=\alpha^2\frac{\partial^2 u}{\partial x^2}$ —— ①

The boundary conditions are $u(0,t) = 0 = T_1$ — (2)
$u(l,t) = 0 = T_2$ — (3)

The initial temp is $u(x,0) = f(x) = x(10-x)$ — (4)

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$$

$$\frac{\frac{dT}{dt}}{T} = \alpha^2 \frac{\frac{d^2X}{dx^2}}{X} = K$$

case (i) :- K is +ve. $k = p^2$

$$u(x,t) = (c_1 e^{px} + c_2 e^{-px})\, c_3 e^{\alpha^2 p^2 t} \quad — (5)$$

case (ii) :- $K = 0$

$$u(x,t) = (c_1 x + c_2)\, c_3 \quad — (6)$$

case (iii) :- K is -ve $k = -p^2$

$$u(x,t) = (c_1 \cos px + c_2 \sin px)\, c_3 e^{-\alpha^2 p^2 t} \quad — (7)$$

(5), (6) & (7) are three independent sols of (1). Out of which we have to select the one which gives a non-trivial sol.

case (i) :- K is +ve $k = +p^2$

$$u(x,t) = (c_1 e^{px} + c_2 e^{-px})\, c_3 e^{\alpha^2 p^2 t}$$

Sub the boundary condition

$$u(0,t) = (c_1 + c_2)\, c_3 e^{\alpha^2 p^2 t} = 0$$

$$c_1 = -c_2$$

$$u(l,t) = c_2 (e^{-px} - e^{px})\, c_3 e^{\alpha^2 p^2 t} = 0$$

$$c_2 = 0$$

$\therefore$ $u(x,t) = 0$, a trivial sol

Case (ii) :- $K=0$

$u(x,t) = (c_1x+c_2)c_3$

$u(0,t) = c_2c_3 = 0$

or

$c_2 = 0$ (or) $c_3 = 0$

$u(x,t) = 0$, a trivial sol

case (iii) :- K is -ve $k = -p^2$

$u(x,t) = (c_1\cos px + c_2\sin px)\, c_3 e^{-\alpha^2p^2t}$

$u(0,t) = c_1c_3e^{-\alpha^2p^2t} = 0$

$c_1 = 0$

$u(x,t) = c_2\sin px\, c_3e^{-\alpha^2p^2t}$

$u(l,t) = c_2\sin pl\, c_3e^{-\alpha^2p^2t} = 0$

$c_2 \neq 0$, $\sin pl = 0$ (or) $p = \frac{n\pi}{l}$

$\therefore u(x,t) = c_2c_3\sin\frac{n\pi}{l}x\, e^{-\alpha^2p^2t}$

$\therefore u(x,t) = b_n\sin\frac{n\pi}{l}x\, e^{-\frac{\alpha^2n^2\pi^2t}{l^2}}$ —— (8)

The most general solution is

$$u(x,t) = \sum_{n=1}^{\infty} b_n\sin\frac{n\pi x}{l} e^{-n^2\pi^2\alpha^2t/l^2} \quad \text{—— (9)}$$

put $t=0$

$$u(x,0) = \sum_{n=1}^{\infty} b_n\sin\frac{n\pi x}{l} \quad \text{—— (10)}$$

$= f(x) = x(10-x)$ which is a half-range sine series

$b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}dx$

$= \frac{2}{l}\int_0^{l_0}(10x-x^2)\sin\frac{n\pi x}{l}dx$

$10x-x^2$	sin
$10-2x$	$-\cos x$
-2	$-\sin$
	cos

$$= \frac{2}{l}\left[-(10x-x^2)\frac{\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}} + (10-2x)\frac{\sin\frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)^2} - 2\frac{\cos\frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)^3}\right]_0^{l}$$

given $l = 10$

$$= \frac{1}{5}\left[-(10x-x^2)\frac{10}{n\pi}\cos\frac{n\pi x}{10} + (10-2x)\frac{10^2}{n^2\pi^2}\sin\frac{n\pi x}{10} - \frac{2\times 10^3}{n^3\pi^3}\cos\frac{n\pi x}{10}\right]_0^{10}$$

$$= \frac{1}{5}\left[\frac{-2\times 10^3}{n^3\pi^3}\cos n\pi + \frac{2\times 10^3}{n^3\pi^3}\cos 0\right]$$

$$= \frac{-2\times 10^3}{5n^3\pi^3}\left[(-1)^n - 1\right]$$

$b_n = 0$, when n is even

$= \frac{4\times 10^3}{5n^3\pi^3}$, when n is odd

$$b_{2n-1} = \frac{800}{(2n-1)^3\pi^3} \quad \forall n$$

$$\therefore u(x,t) = \sum_{n=1}^{\infty} \frac{800}{(2n-1)^3\pi^3} \sin\frac{n\pi x}{10}\, e^{-(2n-1)^2\pi^2\alpha^2 t/100}$$

Given $\rho = 10.6\ gm/cm^2$; $k = 1.04$ cal/cm deg ; $s = 0.056$ cal/gm dg

$$\alpha^2 = \frac{k}{s\rho} = \frac{1.04}{0.056\times 10.6} = 1.752$$

$$\therefore u(x,t) = \sum_{n=1}^{\infty} \frac{800}{(2n-1)^3\pi^3} \sin\frac{n\pi x}{10}\, e^{-0.0175(2n-1)^2\pi^2\alpha^2 t}$$

2. An insulated rod of length l has its end A & B maintained at $0°c$ & $100°c$ respectively until steady state conditions prevail. If the end B is suddenly reduced to $0°c$ & kept so while that of A is maintained, find the temperature at a distance x from A & at time t.

sol The heat eqn is given by

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2} \quad \text{—} (1)$$

with $u(0,t) = 0 = T_1$ — (2) & $u(l,t) = 100 = T_2$ — (3)

Under steady conditions $\frac{\partial u}{\partial t} = 0$

$\therefore$ (1) reduces to $\frac{\partial^2 u}{\partial x^2} = 0$

Its sol is $u = ax + b$

From (2) $u(0,t) = 0$

$0 = b$

$\therefore u = ax$

From (3) $u(l,t) = al + b = al = 100$

$$a = \frac{100}{l}$$

$$\therefore u(x,0) = \frac{100x}{l}$$

Hence initial temp at any point x at $t=0$ is

$$u(x,0) = \frac{100}{l}x \quad \text{—} (4)$$

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$$

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}\, e^{-n^2\pi^2\alpha^2 t/l^2} \quad — ⑤$$

put $t=0$

$$u(x,0) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \quad — ⑥$$

which is a Half range cosine series (HRSS)

$$= \frac{100x}{l}$$

$$\therefore b_n = \frac{2}{l}\int_0^l \frac{100x}{l} \sin\frac{n\pi x}{l}\, dx$$

$$= \frac{200}{l^2}\left[\frac{-x\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}} + \frac{\sin\frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)^2}\right]_0^l$$

$$= \frac{200}{l^2}\left[\frac{-l^2}{n\pi}\cos n\pi\right]$$

$$= \frac{-200}{n\pi}(-1)^n$$

$$b_n = \frac{(-1)^{n+1}200}{n\pi}$$

$$\therefore u(x,t) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}\cdot 200}{n\pi} \sin\frac{n\pi x}{l}\, e^{-n^2\pi^2\alpha^2 t/l^2}$$

Steady-state conditions and non-zero (non-homogenous) boundary conditions:

Suppose a bar AB of length l is heated at both ends by a constant temp T_1° & T_2° resp until steady state condition is reached. Then the temp at beginning A & end B of the bar are suddenly changed to $T_3°$ & $T_4°$ resp & maintained. However, if the ends of the bar are kept at non-zero temp, then the equation is given by $u(x,t) = X(x) + V(x,t)$

where $X(x)$ is the subsequent steady state solution (i.e independent of t) & $V(x,t)$ is the transcent sol (which decs with inc of time).

1. A bar of 10cm long with insulated sides has its ends A & B maintained at temp 50°c & 100°c resp, until steady state conditions prevail. The temp A is suddenly raised to 90°c & at the same time that at B is lowered to 60°c. Find the temp distribution in the bar at time t. Show that the temp at the middle point of the bar remains unaltered for all time, regard-less of the material of the bar.

Sol: The heat eqn is $\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$ — ①.

S/t $u(0,t) = T_1 = 50$ — ②

$u(10,t) = T_2 = 100$ — ③

Under steady-state conditions $\frac{\partial u}{\partial t} = 0$

$\therefore$ from ① $\frac{\partial^2 u}{\partial x^2} = 0$

Its sol is $u(x) = C_1 x + C_2$ — (4)

$u(0) = C_2 = 50$

$\therefore C_2 = 50$

$u(10) = C_1 l + C_2 = 100$

$C_1(10) = 100 - 50 = 50$

$C_1 = 5$

$\therefore u(x,0) = 5x + 50$

The problem for the subsequent flow is

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$$

S/t $u(0,t) = 90 = T_1$ — (5)

$u(10,t) = 60 = T_2$ — (6)

$u(x,0) = 5x + 50$ — (7)

Since the boundary values are non-zero we split $u(x,t)$ into two parts as

$u(x,t) = X(x) + V(x,t)$ — (8)

where $X(x)$ is a sol of (1) satisfying (5) & (6). and is called steady state solution whereas $u_t(x,t)$ given by is called the transcent solution which decreases with increase of time

Now $X(x) = C_3 x + C_4$

$X(0) = C_4 = T_1 = 90$

$\therefore C_4 = 90$

$X(l) = C_3 l + C_4 = 60$

$C_3(10) + 90 = 60$

$C_3 = -3$

$\therefore X(x,0) = -3x + 90$ — (9)

The boundary conditions for $V(x,t)$ are

$V(x,t) = u(x,t) - X(x)$

$V(0,t) = u(0,t) - X(0)$

$= 90 - 90 = 0$ — (10)

$V(l,t) = u(l,t) - X(l)$

$= 60 - 60 = 0$ — (11)

$V(x,0) = u(x,0) - X(x,0)$

$= 5x + 50 - (-3x + 90)$

$= 8x - 40$ — (12)

To check V satisfies the heat eqn $V_t = \alpha^2 V_{xx}$

$V_t(x,t) = u_t(x,t)$

$V_{xx} = u_{xx} - X''(x)$

$= u_{xx} \quad (\because X''(x) = 0)$

$\therefore V(x,t)$ satisfies the heat eqn

$$\frac{\partial V}{\partial t} = \alpha^2 \frac{\partial^2 V}{\partial x^2}$$

Let $V(x,t) = X(x)\,T(t)$

case(i):- $k>0$ $q = p^2$

$$XT' = \alpha^2 T X''$$

$$\frac{\alpha^2 \frac{d^2X}{dx^2}}{X} = \frac{\frac{dT}{dt}}{T} = k$$

$\frac{dT}{T} = \alpha^2 p^2 dt$, Int $\log T = p^2\alpha^2 t + \log c_3$

$$T = c_3 e^{\alpha^2 p^2 t}$$

$$\frac{d^2X}{dx^2} = p^2 X$$

$$X = c_1 e^{px} + c_2 e^{-px}$$

$$V(x,t) = (c_1 e^{px} + c_2 e^{-px})\, c_3 e^{\alpha^2 p^2 t} \quad \text{— (a)}$$

case(ii) :- $k=0$

$$\frac{dT}{dt} = 0 \Rightarrow T = c_3$$

$\frac{d^2X}{dx^2} = 0$ or $X = c_1 x + c_2$

$$\therefore V(x,t) = (c_1 x + c_2)\, c_3 \quad \text{— (b)}$$

case(iii) :- when k is $-ve$ $q = -p^2$

$\frac{dT}{dt} = -\alpha^2 p^2 T$ or $\frac{dT}{T} = -\alpha^2 p^2 dt$

$$\log T = -\alpha^2 p^2 t + \log c_3 .$$

$$T = c_3 e^{-\alpha^2 p^2 t}$$

$$\frac{d^2X}{dx^2} = -p^2 X$$

$$X = C_1 \cos px + C_2 \sin px$$

$$V(x,t) = (C_1 \cos px + C_2 \sin px)\, C_3 e^{-\alpha^2 p^2 t} \quad \text{—} \; (c)$$

out of (a), (b) & (c) we have to select the one which gives non-trivial sol. On substituting the boundary conditions

case (i) :- $V(x,t) = (C_1 e^{px} + C_2 e^{-px})\, C_3 e^{\alpha^2 p^2 t}$

$$V(0,t) = (C_1 + C_2)\, C_3 e^{\alpha^2 p^2 t} = 0$$

$$C_3 = 0$$

$\therefore V(0,t) = 0$, trivial sol

case (ii) :- $V(x,t) = (C_1 x + C_2)\, C_3$

$$V(0,t) = C_2 C_3$$

$$C_2 = 0$$

$\therefore V(0,t) = 0$, trivial sol

case (iii) :- $V(x,t) = (C_1 \cos px + C_2 \sin px)\, C_3 e^{-\alpha^2 p^2 t}$

$$V(0,t) = C_1 C_3 e^{\alpha^2 p^2 t} = 0$$

$C_3 \neq 0$, $C_1 = 0$

$$V(x,t) = C_2 \sin px \, C_3 e^{-\alpha^2 p^2 t}$$

$$V(l,t) = C_2 C_3 \sin pl \; e^{-\alpha^2 p t} = 0$$

$C_2 \neq 0$, $C_3 \neq 0$, $\sin pl = 0$ or $p = \frac{n\pi}{l}$

$$V(x,t) = b_n C_2 C_3 \sin \frac{px}{l} e^{-\alpha^2 p^2 t}$$

$$V(x,t) = b_n \sin \frac{n\pi x}{l} e^{-n^2 \pi^2 \alpha^2 t / l^2} \quad \text{——} \; (13)$$

put $t = 0$, $V(x,0) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$ —— (14)

But $V(x,0) = 8x-40$ from (12)

Expanding in half range sine series

$$\therefore V(x,0) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{l} \quad \text{—— (15)}$$

From (14) & (15)

$$B_n = b_n = \frac{2}{l} \int_0^l (8x-40) \sin \frac{n\pi x}{l} dx$$

$$= \frac{2}{l} \int_0^l (8x-40) \sin \frac{n\pi x}{l} dx$$

$$= \frac{2}{l} \left[-(8x-40) \frac{\cos \frac{n\pi x}{l}}{\frac{n\pi}{l}} + 8 \frac{\sin \frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)^2} \right]_0^l$$

$l = 10$

$$= \frac{1}{5} \left[-(8x-40) \frac{10}{n\pi} \cos \frac{n\pi x}{10} + \frac{8(10)^2}{n^2\pi^2} \sin \frac{n\pi x}{10} \right]_0^{10}$$

$$= \frac{1}{5} \left[-\frac{40l}{n\pi} \cos n\pi + \frac{40l}{n\pi} \cos 0 \right]$$

$$= \frac{-48l}{n\pi} \left[\cos n\pi - \cos 0 \right] = \frac{-48l}{n\pi} \left[(-1)^n - 1 \right]$$

$b_n = 0$, when n is even

$b_n = \frac{16l}{n\pi}$, when n is odd

$$= V(x,t) = \sum_{n=1}^{\infty} \frac{16l}{n\pi} \sin \frac{n\pi x}{l} e^{-n^2\pi^2\alpha^2 t/l^2}$$

$$u(x,t) = X(x) + V(x,t)$$

$$= -3x + 90 + \frac{16l}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin \frac{n\pi x}{l} e^{-n^2\pi^2\alpha^2 t/l^2}$$

At $t = 5$,

$$u(x,t) = -3x + 90 + \frac{80}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin \frac{n\pi x}{5} e^{-n^2\pi^2\alpha^2 t/25}$$

Bar with insulated ends:

If there is no heat flow through the ends of the bar then two ends are said to be insulated thermally, then the corresponding boundary conditions are

If the end $x=0$ is insulated then

$$\left.\frac{\partial u}{\partial x}\right)_{x=0} = u_x(0,t) = 0$$

If the end $x=l$ is insulated then

$$\left.\frac{\partial u}{\partial x}\right)_{x=l} = u_x(l,t) = 0$$

3. The temperature at one end of a bar of length $l$cm with insulated sides is kept at $0°$ & the other end is kept at $100°c$ until steady state conditions prevail. The two ends are then suddenly insulated so that the temp gradient is zero at each end thereafter. Find the temp distribution.

Sol: In the steady-state condition, we have

$$\frac{d^2x}{dx^2} = 0$$

& its sol is $u = c_1x + c_2$

The boundary conditions are

$$u(0) = 0 \quad \& \quad u(l) = 100$$

$$u(0) = c_2 = 0$$

$$u(l) = c_1 l + c_2 = 100$$

$$C_1 = \frac{100}{l}$$

$$\therefore u = \frac{100}{l} x$$

This will be initial temp for the subsequent heat flow after the ends are thermally insulated. Hence the problem reduces to

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2} \quad \text{--- (1)}$$

If the ends are insulated, no heat flows through the ends. " Hence the boundary conditions are

$$\left.\frac{\partial u}{\partial x}\right)_{x=0} = 0 \quad \text{--- (2)}$$

$$\left.\frac{\partial u}{\partial x}\right)_{x=l} = 0 \quad \text{--- (3)} \quad \text{for all } t$$

Also the initial condition is

$$u(x,0) = \frac{100}{l} x, \quad 0 < x < l \quad \text{--- (4)}$$

The sols of (1) by the method of separation of variables are

$$u(x,t) = (C_1 e^{px} + C_2 e^{-px}) C_3 e^{\alpha^2 p^2 t}$$

$$u(x,t) = (C_1 x + C_2) C_3$$

$$u(x,t) = (C_1 \cos px + C_2 \sin px) C_3 e^{-\alpha^2 p^2 t}$$

Out of the above 3 sols we have to select the one which gives a non trivial sol subject to the ~~bdry~~ boundary conditions

② & ③

case (i) :- When k is +ve & $= p^2$

$$u(x,t) = (c_1 e^{px} + c_2 e^{-px})\, c_3 e^{\alpha^2 p^2 t}$$

$$\frac{\partial u}{\partial x} = (c_1 p e^{px} - c_2 p e^{-px})\, c_3 e^{\alpha^2 p^2 t}$$

$$\left.\frac{\partial u}{\partial x}\right)_{x=0} = p(c_1 - c_2)\, c_3 e^{\alpha^2 p^2 t} = 0$$

$$c_3 = 0$$

$\therefore u(x,t) = 0$, a trivial sol

case (ii) :- $k = 0$

$$u(x,t) = (c_1 x + c_2)\, c_3$$

$$\frac{\partial u}{\partial x} = c_1 c_3$$

$$\left.\frac{\partial u}{\partial x}\right)_{x=0} = c_1 c_3 = 0$$

$$c_3 = 0$$

$\therefore u(x,t) = 0$, trivial sol

case (iii) :- When k is −ve & $= -p^2$

$$u(x,t) = (c_1 \cos px + c_2 \sin px)\, c_3 e^{-\alpha^2 p^2 t}$$

$$\frac{\partial u}{\partial x} = (-c_1 p \sin px + c_2 p \cos px)\, c_3 e^{-\alpha^2 p^2 t}$$

$$\left.\frac{\partial u}{\partial x}\right)_{x=0} = c_2 c_3\, p e^{-\alpha^2 p^2 t} = 0$$

$$\therefore c_2 c_3 = 0$$

$$\therefore u(x,t) = c_1 c_3 \cos px\, e^{-\alpha^2 p^2 t} \quad \text{— ⑤}$$

$$\frac{\partial u}{\partial x} = -c_1 c_3\, p \sin px\, e^{-\alpha^2 p^2 t}$$

$$\left.\frac{\partial u}{\partial x}\right)_{x=l} = -c_1c_3\, p \sin pl\, e^{-\alpha^2p^2t} = 0$$

$$c_1c_3 \neq 0 \;, \sin pl = 0 \text{ or } p = \frac{n\pi}{l}$$

$$\therefore u(x,t) = c_1c_3 \cos\frac{n\pi x}{l}\, e^{-\alpha^2n^2\pi^2t/l^2} \quad \text{--- (6)}$$

The most general sol is

$$u(x,t) = \sum_{n=0}^{\infty} A_n \cos\frac{n\pi x}{l}\, e^{-\alpha^2\pi^2\alpha^2t/l^2}$$

$$u(x,t) = A_0 + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{l}\, e^{-n^2\pi^2\alpha^2t/l^2} \quad \text{--- (7)}$$

put $t=0$

$$u(x,0) = A_0 + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{l} \quad \text{--- (8)}$$

But $u(x,0) = \frac{100}{l}x$ from (4)

Expanding in a half range cosine series we get

$$u(x,0) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} \quad \text{--- (9)}$$

From (8) & (9)

$$a_0 = \frac{2}{l}\int_0^l f(x)\,dx = \frac{2}{l}\int_0^l \frac{100x}{l}\,dx = \frac{200}{l^2}\left[\frac{x^2}{2}\right]_0^l$$

$$= \frac{100}{l^2}[l^2] = 100$$

$$\therefore \boxed{a_0 = 100}$$

$$\frac{a_0}{2} = 50$$

$$\therefore \boxed{A_0 = 50}$$

$$A_n = a_n = \frac{2}{l}\int_0^l \frac{100x}{l}\cos\frac{n\pi x}{l}\,dx$$

$$= \frac{200}{l^2}\left[x\,\frac{\sin\frac{n\pi x}{l}}{\frac{n\pi}{l}} + \frac{\cos\frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)^2}\right]_0^l$$

$$= \frac{200}{l^2}\left\{\frac{l^2}{n^2\pi^2}\cos n\pi - \frac{l^2}{n^2\pi^2}\cos 0\right]$$

$$= \frac{200}{n^2\pi^2}\left[(-1)^n - 1\right]$$

$a_n = 0$, when n is even

$= \frac{-400}{n^2\pi^2}$, when n is odd

Sub in eqn (7)

$$u(x,t) = 50 - \frac{400}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\cos\frac{(2n-1)\pi x}{l}\,e^{-(2n-1)^2\pi^2\alpha^2 t/l^2}$$