6/3/17

UNIT - III Evaluation of Integrals.

Type I: Evaluation of real integrals in unit circle:

We can evaluate the function (integrals) of the type

$\int_0^{2\pi} f(\cos\theta, \sin\theta)\, d\theta$ where $f(\cos\theta, \sin\theta)$ is a real rational function.

$$\int_0^{2\pi} f(\cos\theta, \sin\theta)\, d\theta \quad (1), \quad |z| = 1 \Rightarrow r = 1 \text{ unit}$$

$$z = re^{i\theta}$$

$$\because r = 1 \Rightarrow z = e^{i\theta}$$

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \qquad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

$$= \frac{1}{2}\left[z + \frac{1}{z}\right] \qquad = \frac{1}{2i}\left[z - \frac{1}{z}\right]$$

$$e^{i\theta} = z$$

$$e^{i\theta} i\, d\theta = dz \quad (\text{diff. w.r.t } \theta)$$

$$d\theta = \frac{dz}{iz}$$

Substituting in eqⁿ (1) we change the integral in terms of $\sin\theta$ & $\cos\theta$ into a func of z.

$$= \oint_C f\left[\frac{1}{2}\left[z + \frac{1}{z}\right], \frac{1}{2i}\left[z - \frac{1}{z}\right]\right] \frac{dz}{iz}$$

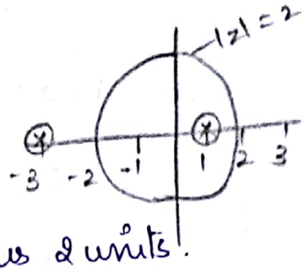
By cauchy's residue theorem,

$$\oint_C f(z)\, dz = 2\pi i \sum r_i \quad \text{where } r_i = \operatorname*{Res}_{z = a_i} f(z), \; i = 1, 2, \ldots n$$

$$\therefore \int_0^{2\pi} f(\cos\theta, \sin\theta)\, d\theta = \oint_C f\left[\frac{1}{2}\left(z + \frac{1}{z}\right), \frac{1}{2i}\left(z - \frac{1}{z}\right)\right] \frac{dz}{iz} = 2\pi i \sum r_i$$

1) Evaluate $\oint_C \frac{1}{(z-1)^2(z+3)} dz$, (i) $|z|=2$, (ii) $|z+1|=3$ by cauchy's residue theorem.

Sol: $f(z) = \frac{1}{(z-1)^2(z+3)}$



$z=1$ is a pole of order '2'.

$z=-3$ is a simple pole.

i) $|z|=2$ is a circle with center at origin & radius 2 units.

Res f(z) at $z=1$:

$$\underset{z=a}{\text{Res}}\, f(z) = \underset{z\to a}{\text{LT}} \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}}\left[(z-a)^m f(z)\right]$$

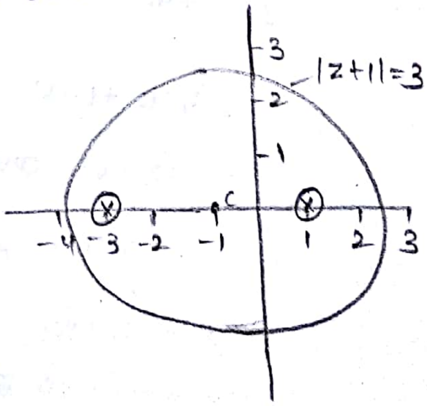
$$= \underset{z\to 1}{\text{LT}} \frac{1}{1!} \frac{d}{dz}\left[(z-1)^2 \frac{1}{(z-1)^2(z+3)}\right]$$

$$= \underset{z\to 1}{\text{LT}} \frac{d}{dz}\left(\frac{1}{z+3}\right) = \underset{z\to 1}{\text{LT}} \frac{-1}{(z+3)^2} = \frac{-1}{16} \quad \text{—} (1)$$

By C-R Theorem,

$$\oint_C f(z)dz = 2\pi i \Sigma R_i = 2\pi i\left(\frac{-1}{16}\right) = \frac{-\pi i}{8}$$

ii) $|z+1|=3$ is a circle with center at $(-1,0)$ and radius 3 units.



Res of f(z) at $z=-3$:

$$\underset{z=a}{\text{Res}}\, f(z) = \underset{z\to a}{\text{LT}} (z-a) f(z)$$

$$\underset{z=-3}{\text{Res}}\, f(z) = \underset{z\to -3}{\text{LT}} \cancel{(z+3)} \frac{1}{(z-1)^2\cancel{(z+3)}} = \frac{1}{16} \quad \text{—} (2)$$

By cauchy's residue theorem,

$$\oint_C f(z)dz = 2\pi i \Sigma R_i = 2\pi i\left(\frac{-1}{16}+\frac{1}{16}\right) = 0 \quad (\text{from } (1) \,\&\, (2)).$$

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② $\int_0^{2\pi} \frac{1}{2+\cos\theta} d\theta$

If limit is 0 to 2π then by default take it as $|z|=1$

Sol: $f(\theta) = \frac{1}{2+\cos\theta}$

we have $|z|=1$

$z = e^{i\theta}$ $(\because r=1)$

$\cos\theta = \frac{1}{2}\left[z+\frac{1}{z}\right]$; $\sin\theta = \frac{1}{2i}\left[z-\frac{1}{z}\right]$

$d\theta = \frac{dz}{iz}$

$$\int_0^{2\pi} \frac{1}{2+\cos\theta} d\theta = \oint_c \frac{1}{2+\frac{1}{2}(z+\frac{1}{z})} \cdot \frac{dz}{iz}$$

$$= \oint_c \frac{1}{2+\frac{1}{2z}(z^2+1)} \cdot \frac{dz}{iz}$$

$$= \oint_c \frac{2\cancel{z}}{4z+z^2+1} \frac{dz}{i\cancel{z}}$$

$$= \frac{2}{i}\oint_c \frac{dz}{4z+z^2+1} \quad \text{--- ①}$$

$\oint_c \frac{1}{z^2+4z+1} dz$; $f(z) = \frac{1}{z^2+4z+1}$

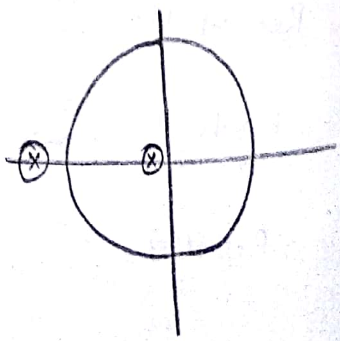
$z^2+4z+1=0$

$z = -2\pm\sqrt{3}$ are simple poles.

$|z|=1 \Rightarrow c(0,0)$ & $r=1$

$z = -2+\sqrt{3} \in |z|=1$

$z = -2-\sqrt{3} \notin |z|=1$



Res of $f(z)$ at $z=-2+\sqrt{3}$:

$$\operatorname*{Res}_{z=a} f(z) = \operatorname*{LT}_{z\to a} (z-a)f(z)$$

$$\operatorname*{Res}_{z=-2+\sqrt{3}} f(z) = \operatorname*{LT}_{z\to -2+\sqrt{3}} \cancel{(z+2-\sqrt{3})} \cdot \frac{1}{\cancel{(z+2-\sqrt{3})}(z+2+\sqrt{3})}$$

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$$= \frac{1}{-2+\sqrt{3}+2+\sqrt{3}} = \frac{1}{2\sqrt{3}}$$

By C-R Theorem, $\oint_c f(z)\,dz = 2\pi i \sum R_i$

$$\frac{2}{i}\oint_c \frac{1}{z^2+4z+1}\,dz = \frac{\cancel{2}}{\cancel{i}}\, 2\pi \cancel{i}\left(\frac{1}{\cancel{2}\sqrt{3}}\right) = \underline{\underline{\frac{2\pi}{\sqrt{3}}}}$$

(3) ~~Evaluate~~ ~~Ex~~ Evaluate $\int_0^{2\pi} \frac{1}{(5-3\cos\theta)^2}\,d\theta$.

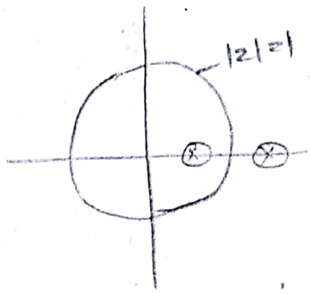
Sol: $z = re^{i\theta}$, $|z| = 1$

$z = e^{i\theta}$

$\cos\theta = \frac{1}{2}\left[z+\frac{1}{z}\right]$

$dz = e^{i\theta}\,d\theta$

$d\theta = \frac{dz}{iz}$.



$$\int_0^{2\pi} \frac{1}{(5-3\cos\theta)^2}\,d\theta = \oint_c \frac{1}{\left[5-\frac{3}{2}\left(z+\frac{1}{z}\right)\right]^2}\frac{dz}{iz}$$

$$= \oint_c \frac{4}{\left(10-3\left(z+\frac{1}{z}\right)\right)^2}\frac{dz}{iz} = \oint_c \frac{4z^{\cancel{2}}}{(10z-3(z^2+1))^2}\frac{dz}{i\cancel{z}}$$

$$= \oint_c \frac{4z}{(-3z^2+10z-3)^2}\frac{dz}{i} = \oint_c \frac{4z}{(3z^2-10z+3)^2}\frac{dz}{i}$$

$$= \frac{4}{i}\oint_c \frac{z}{(z-3)^2\left(z-\frac{1}{3}\right)^2}\,dz$$

$z = 3, \frac{1}{3}$ are double poles.

$$\underset{z=a}{\operatorname{Res}} f(z) = \underset{z\to a}{LT} \frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}\left[(z-a)^m f(z)\right]$$

$$= \underset{z\to\frac{1}{3}}{LT} \frac{1}{1!}\frac{d}{dz}\left[\left(z-\frac{1}{3}\right)^2 \cdot \frac{z}{(z-3)^2\left(z-\frac{1}{3}\right)^2}\right]$$

$$= \underset{z\to\frac{1}{3}}{LT} \frac{d}{dz}\left[\frac{z}{(z-3)^2}\right] = \underset{z\to\frac{1}{3}}{LT}\left[\frac{(z-3)^2\cdot 1 - z\times 2(z-3)}{(z-3)^4}\right]$$

$$= \underset{z\to\frac{1}{3}}{LT} \frac{(z-3)\left[(z-3)-2z\right]}{(z-3)^4} = \underset{z\to\frac{1}{3}}{LT} \frac{-3-z}{(z-3)^3} = \frac{-3-1/3}{\left(\frac{1}{3}-3\right)^3} = \frac{(-9-1)/3}{\left(\frac{1-9}{3}\right)^3}$$

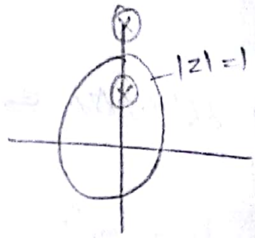
$= 45$

By residue theorem,

$$\oint_c f(z)\,dz = 2\pi i \sum R_i = \frac{4}{i}\cdot 2\pi i\left(\frac{45}{256}\right) = \frac{45\pi}{32}$$

4) Evaluate $\displaystyle\int_0^{2\pi} \frac{d\theta}{(5-3\sin\theta)^2}$, $|z|=1$.

Sol: $|z|=1$, $z=e^{i\theta}$.



$$\sin\theta = \frac{1}{2i}\left[z-\frac{1}{z}\right] \;;\; d\theta = \frac{dz}{iz}$$

$$= \oint_c \frac{dz}{iz\left(5-\frac{3}{2i}\left(z-\frac{1}{z}\right)\right)^2} = \frac{1}{i}\oint_c \frac{dz}{z\left(10i-3\left(z-\frac{1}{z}\right)\right)^2}$$

$$= \frac{1}{i}\oint_c \frac{dz}{z\left(10i-\frac{3(z^2-1)}{z}\right)^2} = \frac{1}{i}\oint_c \frac{z^{\cancel{2}}\,dz}{\cancel{z}\left(10zi-3(z^2-1)\right)^2}$$

$$= \frac{1}{i}\oint_c \frac{z\,dz}{(10zi-3z^2+3)^2} = \frac{1}{i}\oint_c \frac{dz}{\left((z-3i)\left(z-\frac{i}{3}\right)\right)^2}$$

$z=3i, \frac{i}{3}$ are double poles.

$z=\frac{i}{3} \in C \;;\; z=3i \notin C$.

$$\operatorname*{Res}_{z=a} f(z) = \lim_{z\to a} \frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}\left[(z-a)^m f(z)\right]$$

$$= \lim_{z\to\frac{i}{3}} \frac{d}{dz}\left[\cancel{\left(z-\frac{i}{3}\right)^2}\cdot\frac{z}{(z-3i)^2\cancel{\left(z-\frac{i}{3}\right)^2}}\right]$$

$$= \lim_{z\to\frac{i}{3}} \frac{d}{dz}\cdot\frac{z}{(z-3i)^2} = \lim_{z\to\frac{i}{3}} \frac{(z-3i)^2\times 1 - z(2(z-3i))}{(z-3i)^4}$$

$$= \lim_{z\to\frac{i}{3}} \frac{(z-3i)\left[z-3i-2z\right]}{(z-3i)^4} = \lim_{z\to\frac{i}{3}} \frac{-(z+3i)}{(z-3i)^3} = \frac{-45}{256}$$

By residue theorem,

$$= \frac{-4}{i}\oint \frac{z}{(z-3i)^2\left(z-\frac{i}{3}\right)^2}dz = \frac{-4}{i}\times 2\pi i \times \frac{-45}{256} = \frac{45\pi}{256}$$

(5) Show that $\int_0^{2\pi} \frac{d\theta}{a+b\cos\theta} = \int_0^{2\pi} \frac{d\theta}{a+b\sin\theta} = \frac{2\pi}{\sqrt{a^2+b^2}}$, $a>b>0$ using ~~remainder~~ residue thm.

Sol: $\cos\theta = \frac{1}{2}(z+\frac{1}{z})$

$d\theta = \frac{dz}{iz}$

$= \frac{1}{iz}\oint_c \left(\frac{dz}{a+\frac{b}{2}(z+\frac{1}{z})}\right) = \frac{1}{iz}\oint_c \frac{dz}{a+\frac{b}{2}(\frac{z^2+1}{z})} = \frac{1}{iz}\oint_c \frac{dz\cdot z}{2az+b(z^2+1)}$

$= \frac{1}{i}\oint_c \frac{2\,dz}{2az+bz^2+b} = \frac{1}{i}\oint_c \frac{2dz}{bz^2+2az+b} = \frac{2}{i}\oint \frac{dz}{bz^2+2az+b}$

$bz^2+2az+b$

$= \frac{-2a \pm\sqrt{4a^2-4b^2}}{2b} = \frac{-a\pm\sqrt{a^2-b^2}}{b} = \frac{-a-\sqrt{a^2-b^2}}{b}, \quad \frac{-a+\sqrt{a^2-b^2}}{b}$

$= \alpha \notin c \qquad = \beta \in c$

$\underset{z=a}{\text{Res}}\, f(z) = \underset{z\to a}{\text{LT}}\,(z-a)f(z)$

$\underset{z=a}{\text{Res}}\, f(z) = \underset{z\to\beta}{\text{LT}}\,\cancel{(z-\beta)}\cdot\frac{1}{(z-\alpha)\cancel{(z-\beta)}} = \frac{1}{\beta-\alpha}$

$= \frac{1}{\frac{-a+\sqrt{a^2-b^2}}{b} - \left[-\frac{a-\sqrt{a^2-b^2}}{a}\right]} = \frac{b}{-a+\sqrt{a^2-b^2}+a+\sqrt{a^2-b^2}} = \frac{b}{2\sqrt{a^2-b^2}}$

By residue Thm,

$\oint_c f(z)\,dz = 2\pi i\,\Sigma R_i$

$= \frac{2}{i}\times 2\pi i \times \frac{b}{2\sqrt{a^2-b^2}} = \frac{2\pi b}{\sqrt{a^2-b^2}}.$

(6) Evaluate $\int_0^{2\pi} \frac{1}{(6-3\cos\theta)^2}\,d\theta$.

Sol: $|z|=1$, $z=e^{i\theta}$

$\cos\theta = \frac{1}{2}\left[z+\frac{1}{z}\right]$, $d\theta = \frac{dz}{iz}$

$= \oint_c \frac{1}{\left(6-\frac{3}{2}(z+\frac{1}{z})\right)^2}\cdot\frac{dz}{iz} = \oint_c \frac{2}{\left(12-3(z+\frac{1}{z})\right)^2}\frac{dz}{iz}$

$$= \oint_c \frac{4z^2}{iz(12z-3(z^2+1))^2} \cdot dz \qquad = \oint_c \frac{4z}{i(12z-3z^2-3)^2}\, dz$$

$$= \frac{1}{i}\oint_c \frac{4z}{(3z^2-12z+3)^2}\, dz \qquad = \frac{4}{i}\oint_c \frac{z}{(3z^2-12z+3)^2}\, dz$$

$3z^2-12z+3$

$$z = \frac{-(-12) \pm \sqrt{144-4(3)(3)}}{2(3)}$$

$$= \frac{4}{i}\oint_c \frac{z}{((z-2-\sqrt{3})(z-2+\sqrt{3}))^2}\, dz.$$

$$z = \frac{12 \pm \sqrt{144-36}}{6}$$

$144 - 36 = 108$

$$z = \frac{12 \pm \sqrt{108}}{6}$$

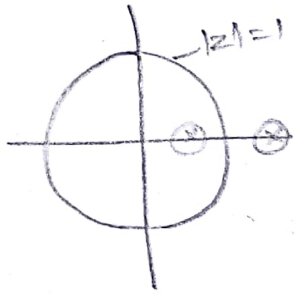
$$z = \frac{12 \pm 6\sqrt{3}}{6} = 2 \pm \sqrt{3}$$

$z = 2+\sqrt{3},\ 2-\sqrt{3}$

$$\operatorname*{Res}_{z=a} f(z) = 2\pi i \sum R_i$$

$z = 2+\sqrt{3} \notin C$

$z = 2-\sqrt{3} \in C$



$z = 2+\sqrt{3},\ 2-\sqrt{3}$ are double poles.

$$\operatorname*{Res}_{z=a} f(z) = \lim_{z\to a} \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} \left((z-a)^m f(z)\right)$$

$$\lim_{z\to 2-\sqrt{3}} \frac{1}{1!} \frac{d}{dz} (z-2+\sqrt{3})^2 \cdot \frac{z}{(z-2-\sqrt{3})^2(z-2+\sqrt{3})^2}$$

$(z-2-\sqrt{3})^2$

$$\lim_{z\to 2-\sqrt{3}} \frac{1}{1!} \frac{d}{dz}\left(\frac{z}{(z-2-\sqrt{3})^2}\right)$$

$$= \lim_{z\to 2-\sqrt{3}} \left[\frac{(z-2-\sqrt{3})^2(1) - z(2(z-2+\sqrt{3}))(1)}{(z-2-\sqrt{3})^4}\right]$$

$$= \left[\underset{z \to 2-\sqrt{3}}{LT} \frac{\cancel{(z-2-\sqrt{3})}\left[(z-2-\sqrt{3}) - z(2)\right]}{(z-2-\sqrt{3})^{\cancel{4}\,3}} \right]$$

$$= \underset{z \to 2-\sqrt{3}}{LT} \left[\frac{z-2-\sqrt{3}-2z}{(z-2-\sqrt{3})^3} \right] = \left[\frac{\cancel{2}-\sqrt{3}-\sqrt{3}-\cancel{2}-2(2-\sqrt{3})}{(\cancel{2}-\sqrt{3}-\cancel{2}-\sqrt{3})^3} \right]$$

$$= \left[\frac{-2\cancel{\sqrt{3}}-4+2\cancel{\sqrt{3}}}{(-2\sqrt{3})^3} \right] = \frac{-4}{-24\sqrt{3}} = \frac{1}{6\sqrt{3}}$$

By residue theorem,

$$\oint_C f(z)\,dz = 2\pi i \sum R_i$$

$$= 2\pi i \left(\frac{4}{i}\frac{1}{6\sqrt{3}} \right) = \cancel{\frac{\pi i}{3\sqrt{3}}} \quad \underline{\underline{\frac{4\pi}{3\sqrt{3}}}}$$

** (7) Evaluate ~~Show that~~ $\displaystyle\int_0^{2\pi} \frac{d\theta}{(a+b\cos\theta)^2} = \frac{\pi a}{(a^2-b^2)^{3/2}}$, $a>b>0$

Sol: $\displaystyle\int_0^{2a} f(x)\,dx = 2\int_0^a f(x)\,dx$ where $f(x)$ is an even function or $f(2a-x) = f(x)$.

$$\therefore \int_0^a f(x)\,dx = \frac{1}{2}\int_0^{2a} f(x)\,dx$$

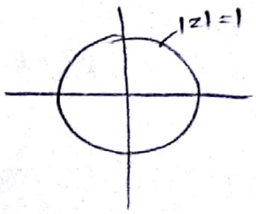
$$= \frac{1}{2}\int_0^{2\pi} \frac{d\theta}{(a+b\cos\theta)^2}$$

$\cos\theta = \frac{1}{2}\left[z+\frac{1}{z}\right]$; $d\theta = \frac{dz}{iz}$.

$$= \frac{1}{2}\int_0^{2\pi} \frac{1}{\left(a+\frac{b}{2}\left(z+\frac{1}{z}\right)\right)^2}\frac{dz}{iz} = \frac{1}{2i}\int_0^{2\pi} \frac{1}{z\left(a+\frac{b}{2}\left(\frac{z^2+1}{z}\right)\right)^2}dz$$

$$= \frac{1}{2i}\int_0^{2\pi} \frac{4z^2\,dz}{z(2az+b(z^2+1))^2} = \frac{1}{2i}\int_0^{2\pi} \frac{4z\,dz}{(2az+b(z^2+1))^2}$$

$$= \frac{1}{2i}\int_0^{2\pi} \frac{4z\,dz}{(2az+bz^2+b)^2} = \frac{2}{i}\int_0^{2\pi} \frac{z\,dz}{(bz^2+2az+b)^2}$$



$$z = \frac{-a-\sqrt{a^2-b^2}}{b} = \alpha \notin C, \quad \frac{-a+\sqrt{a^2-b^2}}{b} \in C = \beta$$

H.W

$$1.\ \int_0^{\pi}\left(\frac{1+2\cos\theta}{5+4\cos\theta}\right)d\theta$$

$z=\alpha, \beta$ are double poles.

$$\underset{z=a}{\operatorname{Res}} f(z) = \underset{z\to a}{Lt} \frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}\left((z-a)^m f(z)\right)$$

$$= \underset{z\to\beta}{Lt} \frac{1}{1!}\frac{d}{dz}\left(\cancel{(z-\beta)^2}\cdot\frac{z}{b^2(z-\alpha)^2\cancel{(z-\beta)^2}}\right)$$

$$= \underset{z\to\beta}{Lt} \frac{1}{b^2}\frac{d}{dz}\left(\frac{z}{(z-\alpha)^2}\right)$$

$$= \underset{z\to\beta}{Lt}\left[\frac{(z-\alpha)^2(1)-z(2(z-\alpha)(1))}{(z-\alpha)^4}\right]\frac{1}{b^2}$$

$$= \underset{z\to\beta}{Lt}\left[\frac{\cancel{(z-\alpha)}[(z-\alpha)-2z]}{(z-\alpha)^{3}}\right]\frac{1}{b^2} = \left(\frac{(\beta-\alpha)-2\beta}{(\beta-\alpha)^3}\right)\frac{1}{b^2}$$

$$= \frac{1}{b^2}\left(\frac{\beta-\alpha-2\beta}{(\beta-\alpha)^3}\right) = \frac{1}{b^2}\left(\frac{-\beta-\alpha}{(\beta-\alpha)^3}\right) = \frac{1}{b^2}\left(\frac{-(\beta+\alpha)}{(\beta-\alpha)^3}\right)$$

$$= \frac{1}{b^2}\left[\frac{\left(\frac{-a+\sqrt{a^2-b^2}}{b} - \frac{a-\sqrt{a^2-b^2}}{b}\right)}{\left[\frac{-a+\sqrt{a^2-b^2}}{b}+\frac{a+\sqrt{a^2-b^2}}{b}\right]^3}\right] = \frac{1}{b^2}\left[\frac{2a/b}{\frac{2\sqrt{a^2-b^2}}{b^3}}\right] = \frac{a}{\sqrt{a^2-b^2}}$$

$$= \frac{a}{4(a^2-b^2)^{3/2}}$$

By residue theorem,

$$\frac{2}{i}\oint_c f(z)\,dz = \frac{2}{i}\times 2\pi i\cdot\frac{a}{4(a^2-b^2)^{3/2}} = \frac{\pi a}{(a^2-b^2)^{3/2}}$$

H.W

(8) Evaluate $\int_0^{\pi}\left(\frac{1+2\cos\theta}{5+4\cos\theta}\right)d\theta$.

Sol:

$$\cos\theta = \frac{1}{2}\left[z+\frac{1}{z}\right]\ ;\ d\theta = \frac{dz}{iz}.$$

$$= \frac{1}{2}\int_0^{2\pi} \left(\frac{1+2\cos\theta}{5+4\cos\theta}\right) d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} \frac{1+2\left(\frac{1}{2}\left(z+\frac{1}{z}\right)\right)}{5+4\left(\frac{1}{2}\left(z+\frac{1}{z}\right)\right)} \cdot \frac{dz}{iz}$$

$$= \frac{1}{2}\int_0^{2\pi} \frac{1+\left(z+\frac{1}{z}\right)}{5+2\left(z+\frac{1}{z}\right)} \frac{dz}{iz} = \frac{1}{2i}\int_0^{2\pi} \frac{z+(z^2+1)}{5z+(z^2+1)} \frac{dz}{z}.$$

$$= \frac{1}{2i}\int_0^{2\pi} \frac{1}{z}\left[\frac{z+z^2+1}{5z+z^2+1}\right]dz = \frac{1}{2i}\int_0^{2\pi} \frac{1}{z}\left[\frac{z^2+z+1}{2z^2+5z+2}\right]dz.$$

$\cancel{= \frac{1}{2i}\int_0^{2\pi}\frac{1}{z}\Big[}$ $\quad z = \cancel{\frac{-5+\sqrt{21}}{2}} \in C = \alpha$, $\cancel{\frac{-5-\sqrt{21}}{2}} \notin C = \beta$. $\quad (z+2) \notin C$, $(z+1/2) \in C$.

$$= \frac{1}{2i}\int_0^{2\pi} \frac{1}{z}\left[\frac{z^2+z+1}{(z+\frac{\alpha}{2})(z+\frac{\beta}{2})}\right]dz \text{ are simple poles.}$$

$$\operatorname*{Res}_{z=\alpha} f(z) = \lim_{z\to\alpha}(z-1/2)\frac{z^2+z+1}{(z-1/2)(z+\frac{\beta}{2})} = \lim_{z\to\alpha}\frac{z^2+z+1}{(z+\frac{\beta}{2})}$$

$$= \frac{(1/4+1-1/2)}{3/2} = 1/2$$

$$= \frac{1}{4iz}\oint_C \frac{z^2+z+1}{(z+1/2)(z+2)} = \underline{\underline{\frac{1}{8iz}}}.$$

Evaluation of integrals of the type $\int_{-\infty}^{\infty} f(x)\,dx$:

Statement: If $f(z)$ is analytic in the upper half of z-plane except at a finite no. of holes in it and having no poles on the real axis and if $z f(x) \to 0$ as $x \to \infty$ then by contour integration.

$$\int_{-\infty}^{\infty} f(x)\,dx = 2\pi i \sum R^+.$$

Where $\sum R^+$ represents the sum of the residue at poles in the upper half of the z-plane.

Jordon's Lemma:

When $f(z)$ has no singularities on the real axis then

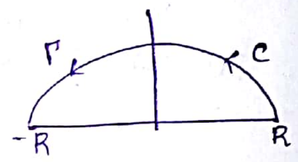
$$\int_C f(z)\,dz = \int_{-R}^{R} f(z)\,dz + \int_{\Gamma} f(z)\,dz$$

$$= 2\pi i \,\Sigma R^+$$

As $z \to \infty$, $R \to \infty$, $\int_{-\infty}^{\infty} f(x)\,dx = \int_{\Gamma} f(z)\,dz \to 0$ as $z \to \infty$.

① Evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{1}{x^4+a^4}\,dx$.

Sol: Let $f(z) = \dfrac{1}{z^4+a^4}$



$$\int_C f(z)\,dz = \int \frac{1}{z^4+a^4}\,dz \quad \text{—— ①}$$

where C is the contour consisting of the semicircle Γ of radius 'R' and bounding diameters $-R$ to R.

From eq^n ① we have,

$$\int_C f(z)\,dz = \int_{-R}^{R} f(z)\,dz + \int_{\Gamma} f(z)\,dz \quad \text{—— ②}$$

By residue theorem, $\int_C f(z)\,dz = 2\pi i \,\Sigma R^+$.

where ΣR^+ is sum of the residues within Γ.

Poles of z are given by

$$z^4 + a^4 = 0$$

$$z^4 = -a^4 \quad \text{—— ③}$$

$$z^4 = a^4(-1)$$

$$z^4 = a^4(\cos\pi + i\sin\pi)$$

$$z^4 = a^4(\cos(2n+1)\pi + i\sin(2n+1)\pi)$$

$$z = a[\cos(2n+1)\pi + i\sin(2n+1)\pi]^{1/4}, \quad n = 0, 1, 2, \ldots$$

$\because [(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta]$

$z = a\left[\cos(2n+1)\frac{\pi}{4} + i\sin(2n+1)\frac{\pi}{4}\right]$

$z = a e^{i(2n+1)\pi/4}$, $n = 0, 1, 2, \ldots$ are simple poles.

$z = ae^{i\pi/4}, ae^{i3\pi/4}, ae^{i5\pi/4}, \ldots$

$z = ae^{i\pi/4}, ae^{i3\pi/4} \in \Gamma$ $\quad (\because \pi/4 = 45^\circ, 3\pi/4 = 135^\circ$ belongs to $\Gamma)$.

Res at $z = e^{i\pi/4}$:

$$\operatorname*{Res}_{z=a} f(z) = \operatorname*{Lt}_{z\to a} (z-a)\cdot f(z)$$

$$\operatorname*{Res}_{z=ae^{i\pi/4}} f(z) = \operatorname*{Lt}_{z\to ae^{i\pi/4}} (z-ae^{i\pi/4})\frac{1}{z^4+a^4}$$

Apply LHR,

$$= \operatorname*{Lt}_{z\to ae^{i\pi/4}} \frac{1}{4z^3} = \operatorname*{Lt}_{z\to ae^{i\pi/4}} \frac{z}{4z^4}$$

$$= \operatorname*{Lt}_{z\to ae^{i\pi/4}} \frac{z}{-4a^4} \quad (\text{from } (3))$$

$$= \frac{-1}{4a^4}\cdot ae^{i\pi/4} = -\frac{e^{i\pi/4}}{4a^3}$$

III'ly

$$\operatorname*{Res}_{z=ae^{i3\pi/4}} f(z) = -\frac{e^{i3\pi/4}}{4a^3}$$

$$\int_{-\infty}^{\infty} f(z)\,dz = \int_C f(z)\,dz = 2\pi i \,\Sigma R^+$$

$$= 2\pi i\left[\frac{-e^{i\pi/4}}{4a^3} + \frac{-e^{i3\pi/4}}{4a^3}\right]$$

$$= \frac{-\pi i}{2a^3}\left[e^{i\pi/4} + e^{i3\pi/4}\right]$$

$$= -\frac{\pi i}{2a^3}\left[\cos\frac{\pi}{4} + i\sin\frac{\pi}{4} + \cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right]$$

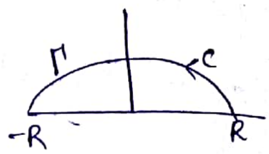
$$= -\frac{\pi i}{2a^3}\left[\cos\frac{\pi}{4} + i\sin\frac{\pi}{4} - \sin\frac{\pi}{4} + i\cos\frac{\pi}{4}\right]$$

$$= \frac{-\pi i}{2a^3}\left[\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}} - \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right] = \frac{-\pi i}{2a^3}\left[\frac{2i}{\sqrt{2}}\right] = \frac{\pi}{\sqrt{2}a^3}.$$

2) Evaluate $\int_{-\infty}^{\infty} \frac{x^2\,dx}{(x^2+a^2)^3}, \; a>0$

Sol: let $f(z) = \frac{x^2}{(x^2+a^2)^3}$

$$\int_C f(z)\,dz = \int \frac{z^2}{(z^2+a^2)^3}\,dz \quad \text{—— (1)}$$



where C is the ~~semicircle~~ contour consisting of semicircle Γ of radius 'R' and bounding diameters −R to R.

From eqn (1) we have,

$$\int_C f(z)\,dz = \int_{-R}^{R} f(z)\,dz + \int_\Gamma f(z)\,dz \quad \text{—— (2)}$$

$z^2+a^2=0$

$z^2=-a^2$

$z^2=i^2a^2$

$z=\pm ai$ are poles of order 3.

$z=ai \in \Gamma$, $z=-ai \notin \Gamma$

Res ~~of~~ f(z) at z = ai:

$$\operatorname{Res}_{z=a} f(z) = \operatorname{Lt}_{z\to a} \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}}\left[(z-a)^m f(z)\right]$$

$$= \operatorname{Lt}_{z\to ai} \frac{1}{2!} \frac{d^2}{dz^2}\left[\cancel{(z-ai)^3} \cdot \frac{z^2}{\cancel{(z-ai)^3}(z+ai)^3}\right]$$

$$= \frac{1}{2} \operatorname{Lt}_{z\to ai} \frac{d^2}{dz^2}\left[\frac{z^2}{(z+ai)^3}\right]$$

$$= \frac{1}{2} \operatorname{Lt}_{z\to ai} \frac{d}{dz}\left[\frac{(z+ai)^3(2z)-(z^2)3(z+ai)^3}{(z+ai)^6}\right]$$

$$= \frac{1}{2} \operatorname{Lt}_{z\to ai} \frac{d}{dz}\left[\frac{(z+ai)^2\left[2z^2+2aiz-3z^2\right]}{(z+ai)^6}\right]$$

$$= \frac{1}{2} \operatorname{Lt}_{z\to ai} \frac{d}{dz}\left[\frac{-z^2+2aiz}{(z+ai)^4}\right]$$

$$= \frac{1}{2} \underset{z\to ai}{Lt} \left[\frac{(z+ai)^4(-2z+2ai) - (z^2+2aiz)\times 4(z+ai)^3}{(z+ai)^8}\right]$$

$$= \frac{1}{2} \underset{z\to ai}{Lt} \left[(z+ai)^3 \left[\frac{-2z^2+2aiz-2aiz+2a^2i^2+4z^2-8aiz}{(z+ai)^8}\right]\right]$$

$$= \frac{1}{2} \underset{z\to ai}{Lt} \frac{2z^2-8aiz-2a^2}{(z+ai)^5} = \frac{1}{\cancel{2}} \underset{z\to ai}{Lt} \frac{\cancel{2}(z^2-4aiz-a^2)}{(z+ai)^5}$$

$$= \frac{(ai)^2-4(ai)(ai)-a^2}{(ai+ai)^5} = \frac{-3(ai)^2-a^2}{(2ai)^5}$$

$$= \frac{3a^2-a^2}{2^5a^5i^5} = \frac{2a^2}{2^5a^5i} = \frac{1}{16a^3i}.$$

By residue theorem,

$$\int_C f(z)\,dz = 2\pi i \sum R^+ = 2\pi \cancel{i}\left(\frac{1}{16a^3\cancel{i}}\right) = \frac{\pi}{8a^3}$$

H.W

$$\int_{-\infty}^{\infty} \frac{x^2}{(x^2+a^2)^2}\,dx.$$

3) Evaluate $\int_{-\infty}^{\infty} \frac{dx}{(x^2+9)(x^2+4)^2}$, $a>0$ using residue theorem.

Sol: $z^2+9=0 \Rightarrow z^2=-9 \Rightarrow z=\pm 3i$ — simple poles.

$z^2+4=0 \Rightarrow z^2=-4 \Rightarrow z=\pm 2i$ — double pole.

$z=2i,\ 3i \in \Gamma$

$z=-2i,-3i \notin \Gamma$

$$\underset{z=2i}{\text{Res}\, f(z)} = \underset{z\to 2i}{Lt} \frac{d}{dz}\left[\cancel{(z-2i)^2}\frac{1}{\cancel{(z-2i)^2}(z+2i)^2(z^2+9)}\right]$$

$$= \underset{z\to 2i}{Lt} \frac{d}{dz}\left[\frac{1}{(z+2i)^2(z^2+9)}\right] = \underset{z\to 2i}{Lt} \frac{d}{dz}\left[(z+2i)^{-2}(z^2+9)^{-1}\right]$$

$$= \underset{z\to 2i}{Lt}\left[(z+2i)^{-2}\times(-1(z^2+9)^{-2}(2z)) + (z^2+9)^{-1}\times(-2)(z+2i)^{-3}\right]$$

$$= \underset{z\to 2i}{Lt}\left[\frac{-2z}{(z+2i)^2(z^2+9)^2} + \frac{-2}{(z^2+9)(z+2i)^3}\right] = \frac{-2(2i)}{(2i+2i)^2((2i)^2+9)^2} - \frac{2}{((2i)^2+9)(2i+2i)^3}$$

$$= \frac{-4i}{-16(25)} - \frac{2}{5(-64i)} = \frac{i}{100} + \frac{1}{160i} = \frac{i}{100} - \frac{i}{160} = \frac{i}{100} - \frac{i}{160} = \frac{3i}{800}$$

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$$\operatorname*{Res}_{z=\alpha} f(z) = \operatorname*{Lt}_{z\to\alpha} (z-\alpha) f(z)$$

$$\operatorname*{Res}_{z=3i} f(z) = \operatorname*{Lt}_{z\to 3i} (z-3i) \frac{1}{(z-3i)(z+3i)(z^2+4)^2} = \frac{1}{(3i+3i)((3i)^2+4)^2} = \frac{1}{6i(25)}$$

$$= \frac{1}{150i} \times \frac{i}{i} = \frac{-i}{150}$$

By residue theorem,

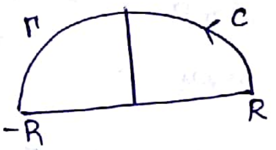
$$\int_C f(z)\,dz = 2\pi i\left[\frac{3i}{800} - \frac{i}{150}\right] = 2\pi i\left[\frac{450i - 800i}{120000}\right]$$

$$= \frac{7\pi}{1200}$$

4) Evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{z^2}{(z^2+9)^2(z^2+4)}\,dz$ by cauchy's ~~integral~~ residue theorem or by contour integration.

Sol: $f(z) = \dfrac{z^2}{(z^2+9)^2(z^2+4)}$

$$\int_C f(z)\,dz = \int \frac{z^2}{(z^2+9)^2(z^2+4)}\,dz.$$



$$\int_C f(z)\,dz = \int_{-R}^{R} f(z)\,dz + \int_\Gamma f(z)\,dz \quad \text{--- (1)}$$

As $z\to\infty$, $R\to\infty$, $\displaystyle\int_\Gamma f(z)\,dz = 0$ --- (2)

$$\int_{-\infty}^{\infty} f(z)\,dz = \int_C f(z)\,dz = 2\pi i \sum R^+.$$

Poles of $f(z)$:

$z^2+9=0$

$z^2=-9$

$z^2=i^2 9$

$z=\pm 3i$ are double poles.

$z^2+4=0 \Rightarrow z=\pm 2i$ are simple poles.

$z = 2i, 3i \in \Gamma$

$z = -2i, -3i \notin \Gamma$

<u>Res of $f(z)$ at $z = 2i$:</u>

$$\operatorname*{Res}_{z=a} f(z) = \operatorname*{Lt}_{z \to a} (z-a) f(z)$$

$$\operatorname*{Res}_{z=2i} f(z) = \operatorname*{Lt}_{z \to 2i} \cancel{(z-2i)} \left[\frac{z^2}{\cancel{(z-2i)}(z+2i)(z^2+9)^2} \right]$$

$$= \frac{(2i)^2}{(2i+2i)((2i)^2+9)^2} = \frac{-4}{4i(25)} = \frac{-1}{25i} \text{ —— (3)}$$

<u>Res $f(z)$ at $z = 3i$:</u>

$$\operatorname*{Res}_{z=a} f(z) = \operatorname*{Lt}_{z \to a} \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} \left[(z-a)^m f(z) \right]$$

$$= \operatorname*{Lt}_{z \to 3i} \frac{1}{1!} \frac{d}{dz} \left[\cancel{(z-3i)^2} \frac{z^2}{(z^2+4)\cancel{(z-3i)^2}(z+3i)^2} \right]$$

$$= \operatorname*{Lt}_{z \to 3i} \frac{d}{dz} \left[\frac{z^2}{(z^2+4)(z+3i)^2} \right]$$

[illegible] $(z+3i)]\ ]$

$$= \underset{z \to 3i}{Lt} \frac{\left[(z^2+4)(z+3i)^2(2z)\right] - z^2\left[(z^2+4)2(z+3i) + (z+3i)^2(2z)\right]}{(z^2+4)^2 (z+3i)^4}$$

$$= \underset{z \to 3i}{Lt} \frac{(z+3i)\left[(z^2+4)(z+3i)(2z)\right] - (z+3i)\left[2z^2(z^2+4) + 2z(z+3i)\right]}{(z^2+4)^2 (z+3i)^4}$$

$$= \underset{z \to 3i}{Lt} \frac{(z+3i)\left[(z^2+4)(z+3i)(2z)\right] - \left[2z^4+8z^2+2z^2+6iz\right]}{(z^2+4)^2 (z+3i)^4}$$

$$= \frac{((3i)^2+4)\left[3i+3i\right](2\times 3i) - \left[2(3i)^4+10(3i)^2+6i(3i)\right]}{((3i)^2+4)^2(3i+3i)^4}$$

$$= \frac{(9i^2+4)\left[(6i)(6i)\right] - \left[2(81i^4)+10(9i^2)+18i^2\right]}{(9i^2+4)^2(6i)^4}$$

$$= \frac{(-9+4)(36i^2) - (2(81)+90i^2+18)}{(-9+4)^2(6i)^4} = \frac{13}{300i}$$

By residue theorem,

$$\int_C f(z)\,dz = 2\pi i\left[\frac{-1}{25i} + \frac{13}{300i}\right] = \frac{\pi}{150}$$

5) Evaluate $\int_{-\infty}^{\infty} \frac{1}{(x^2+a^2)(x^2+b^2)} dx$

Sol: $f(z) = \frac{1}{(z^2+a^2)(z^2+b^2)}$

$$\int_C f(z)dz = \int \frac{1}{(z^2+a^2)(z^2+b^2)}$$

$z^2+a^2=0$	$z^2+b^2=0$
$z^2=-a^2$	$z^2=-b^2$
$z^2=a^2i^2$	$z^2=b^2i^2$
$z=\pm ai$	$z=\pm bi$

$z = ai, bi \in \Gamma$

$z = -ai, -bi \notin \Gamma$

Res $f(z)$ at $z = ai$

$$\underset{z \to ai}{Lt} (z - ai) \frac{1}{(z^2+a^2)(z^2+b^2)}$$

$$\underset{z \to ai}{Lt} \cancel{(z-ai)} \frac{1}{\cancel{(z-ai)}(z+ai)(z^2+b^2)}$$

$$\underset{z \to ai}{Lt} \frac{1}{(z+ai)(z^2+b^2)} = \frac{1}{(2ai)(a^2i^2+b^2)} = \frac{1}{(2ai)(b^2-a^2)}.$$

Res $f(z)$ at $z = bi$

$$\underset{z \to bi}{Lt} \cancel{(z-bi)} \frac{1}{(z^2+a^2)\cancel{(z-bi)}(z+bi)}$$

$$\underset{z \to bi}{Lt} \frac{1}{(z^2+a^2)(z+bi)}$$

$$= \frac{1}{(b^2i^2+a^2)(bi+bi)} = \frac{1}{(2bi)(a^2-b^2)}.$$

By cauchy's residue theorem,

$$\int_c f(z)\,dz = 2\pi i \left[\frac{1}{2ai(b^2-a^2)} + \frac{1}{2bi(a^2-b^2)}\right]$$

$$= \frac{2\pi i}{2i}\left[\frac{1}{a(b^2-a^2)} + \frac{1}{b(a^2-b^2)}\right]$$

$$= \pi\left[\frac{1}{a(b^2-a^2)} + \frac{1}{b(a^2-b^2)}\right] = \frac{\pi}{(b^2-a^2)}\left[\frac{1}{a} - \frac{1}{b}\right]$$

$$= \frac{\pi}{(b^2-a^2)}\left[\frac{b-a}{ab}\right] = \frac{\pi}{ab(b+a)}$$

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Part - B : Bilinear Transformations

The transformation defined by $w = f(z) = \frac{az+b}{cz+d}$ — ①

where a, b, c, d are complex constants and $ad - bc \neq 0$ is called bilinear transformation.

def^n will be asked for 2-3m

$$\frac{dw}{dz} = \frac{(cz+d)(a) - (az+b)(c)}{(cz+d)^2} = \frac{acz+ad-acz-bc}{(cz+d)^2}$$

$$\frac{dw}{dz} = \frac{ad-bc}{(cz+d)^2} \neq 0$$

Hence, the bilinear transformation given by ① is confirmal.

The inverse mapping of the transformation given by ① can be obtained

Inverse mapping:

$z = \frac{-dw+b}{cw-a}$ is also a bilinear transformation (or) Mobius transformation

from eq^n ①, we observe that each point in the z-plane except at $z = -\frac{d}{c}$ maps into a unique point in the w-plane.

Similarly, from eq^n ② each point in w-plane except at $w = \frac{a}{c}$ maps into a unique point in the z-plane. Thus, there is one to one correspondance for all points in the two planes.

→ Every bilinear transformation $w = f(z) = \frac{az+b}{cz+d}$, $ad - bc \neq 0$ is a combination of the following basic transformation:

i) Translation, $w = z + c$

ii) Rotation and magnification, $w = cz$.

iii) Inversion, $w = \frac{1}{z}$.

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Then we can write the B.T., $w = f(z) = \frac{az+b}{cz+d} = \frac{a}{c} + \frac{bc-ad}{c^2} \cdot \frac{1}{z+\frac{d}{c}}$.

By taking $w_1 = z + \frac{d}{c}$, $w_2 = \frac{1}{w_1}$, $w_3 = \frac{bc-ad}{c^2} \cdot w_2$, $w = \frac{a}{c} + w_3$

Thus, by these transformations we successively pass from z plane to w_1 plane, w_1 to w_2 plane, w_2 to w_3 plane, w_3 to w plane.

Since every above transformation maps circles into circles, the bilinear transformation also maps circles into circles.

Properties:

1. A B.T. maps circles into circles i.e. under this transformation, circles in the z-plane are mapped into circles in w-plane.

2. The cross ratio of four points is preserved under this transformation, i.e. z_1, z_2 & $z_3; z_4$ are four points in the z-plane and w_1, w_2 & w_3, w_4 are four points in w-plane then the cross ratio of 4 points is defined as

$$\frac{(z_1-z_2)(z_3-z_4)}{(z_1-z_4)(z_3-z_2)} = \frac{(w_1-w_2)(w_3-w_4)}{(w_1-w_4)(w_2-w_3)}$$

Hence, the unique B.T. that maps 3 given points z_1, z_2, z_3 onto 3 given images ~~(or)~~ w_1, w_2, w_3 is given by

$$\frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)} = \frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)}$$

~~Hence, the~~

The cross ratio property is invariant under a bilinear transformation.

→ **Fixed point (or) invariant point:**

If $w = f(z)$ is a B.T. from z-plane to w-plane then the fixed point of the transformation is given by the eqⁿ $w = f(z) = z$.

Consider the B.T. $w = f(z) = \frac{az+b}{cz+d} = z$

$cz^2 + dz - az - b = 0$

$cz^2 + (d-a)z - b = 0$ is a quadratic in z —— ①

The roots of eq^n ① are called as invarient(or) fixed points.

① find the fixed points for the following.

i) $w = \frac{1+z}{1-z}$

Sol: Given, $w = \frac{1+z}{1-z}$

fixed points are given by equating $w = f(z) = z$.

$$\frac{1+z}{1-z} = z$$

$$z - z^2 = 1 + z$$

$$z^2 - z + z + 1 = 0$$

$$z^2 + 1 = 0$$

$$z^2 = -1 \Rightarrow z^2 = i^2$$

$z = \pm i$ is the fixed point.

Verification: $w = \frac{1+i}{1-i} \times \frac{1+i}{1+i}$ (subs $z = i$).

$$w = \frac{(1+i)^2}{1^2 - i^2} = \frac{2i}{2}$$

$$w = i.$$

Hence verified

ii) $w = \frac{1}{1-z}$

$$\frac{1}{1-z} = z$$

$$1 = z - z^2$$

$$z^2 - z + 1 = 0$$

~~$z = -(-3) \pm \sqrt{\quad}$~~

$$z = \frac{1 \pm i\sqrt{3}}{2}$$

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iii) $w = \dfrac{3iz+1}{z+i}$

$\dfrac{3iz+1}{z+i} = z.$

$3iz+1 = z^2 + iz.$

$z^2 + iz - 3iz - 1 = 0$

$z^2 - 2iz - 1 = 0$

$z = i$ is the fixed point.

H.W

1) $w = \dfrac{3z-4}{z-1}$

2) $w = \dfrac{z-3}{z+1}$

3) $w = \dfrac{1}{z-2}$

4) $w = \dfrac{(2+i)z+1}{z+i}$

H.W

i) $w = \dfrac{3z-4}{z-1}$

$\dfrac{3z-4}{z-1} = z$

$3z-4 = z^2 - z$

$z^2 - z - 3z + 4 = 0$

$z^2 - 4z + 4 = 0$

$z^2 - 2z - 2z + 4 = 0$

$z(z-2) - 2(z-2)$

$(z-2)(z-2)$

$z = 2$ is the fixed point.

ii) $w = \dfrac{z-3}{z+1}$

$\dfrac{z-3}{z+1} = z.$

$z-3 = z^2 + z$

$z^2 + \cancel{z} - \cancel{z} + 3 = 0$

$z^2 + 3 = 0$

$z^2 = -3$

$z = 3i$ is the fixed point.

iii) $w = \dfrac{1}{z-2}$

$\dfrac{1}{z-2} = z$

$1 = z^2 - 2z$

$z^2 - 2z - 1 = 0$

$z = \dfrac{-(-2) \pm \sqrt{4 - 4(1)(-1)}}{2}$

$z = \dfrac{2 \pm \sqrt{8}}{2} = \dfrac{2 \pm 2\sqrt{2}}{2} = 1 \pm \sqrt{2}$

$z = 1+\sqrt{2},\ 1-\sqrt{2}$ is the fixed point.

iv) $w = \dfrac{(2+i)z+1}{z+i}$

$\dfrac{(2+i)z+1}{z+i} = z$

$2z + iz + 1 = z^2 + iz$

$z^2 + iz - iz - 2z - 1 = 0$

$z^2 - 2z - 1 = 0$

$z = 1 \pm \sqrt{2}$

$z = 1+\sqrt{2},\ 1-\sqrt{2}$ are the fixed points.

Q. Find fixed or invarient point for the following:

1. $w = \dfrac{2z-5}{z+4}$

~~R/H~~ Fixed or invarient point is obtained by $w = f(z) = z$.

$$\frac{2z-5}{z+4} = z$$

$$2z-5 = z^2+4z$$

$$z^2+4z-2z+5 = 0$$

$$z^2+2z+5 = 0$$

$$z^2+2z+1+4 = 0$$

$$(z+1)^2-(2i)^2 = 0$$

~~(z+1)(z~~

$= -1 \pm 2i$ are two fixed points.

2. $w = \dfrac{6z-9}{z}$

Fixed point is obtained by

$w = f(z) = z$

$$\frac{6z-9}{z} = z$$

$$6z-9 = z^2$$

$$z^2-6z+9 = 0$$

$$z = \frac{-(-6) \pm \sqrt{36-4(1)(9)}}{2(1)}$$

$z = \frac{6}{2} = 3$ is fixed point.

** 3. $w = \dfrac{2i-6z}{iz-3}$

Fixed point is obtained by

$w = f(z) = z$.

$$\frac{2i-6z}{iz-3} = z.$$

$$2i-6z = iz^2-3z$$

$$iz^2-3z-2i+6z = 0$$

$$iz^2-2i+3z = 0$$

$$iz^2+3z-2i = 0$$

$$z = \frac{-(3) \pm \sqrt{9-4(i)(-2i)}}{2(i)}$$

$$z = \frac{-3 \pm \sqrt{9+8i^2}}{2i} = \frac{-3 \pm \sqrt{1}}{2i}$$

$$z = \frac{-3 \pm 1}{2i} = \frac{-3+1}{2i}, \frac{-3-1}{2i}$$

$$= -\frac{1}{i}, \frac{-4}{2i}$$

$= i$, $2i$ are fixed points.

4. $w = \frac{z-1}{z+1}$

fixed points can be obtained by $w = f(z) = z$.

$\frac{z-1}{z+1} = z$

$z-1 = z^2 + z$.

$z^2 + 1 = 0$

$z = \pm i$ are fixed points.

(5) (1) find the B.T (or) Mobius transformation that maps the points $(0, -i, -1)$ into the points $(i, 1, 0)$

Sol: Let $z_1 = 0$, $z_2 = -i$, $z_3 = -1$, $z_4 = z$.

$w_1 = i$, $w_2 = 1$, $w_3 = 0$, $w_4 = w$

By cross-ratio property,

$$\frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_4)(z_3 - z_2)} = \frac{(w_1 - w_2)(w_3 - w_4)}{(w_1 - w_4)(w_3 - w_2)}$$

$$\frac{[0-(-i)][-1-z]}{(0-z)(-1-(-i))} = \frac{(i-1)(0-w)}{(i-w)(0-1)}$$

$$\frac{-i(1+z)}{-z(i-1)} = \frac{-w(i-1)}{-1(i-w)}$$

$(i + iz)(i - w) = wz(i-1)^2$

$i^2 - iw + i^2 z - iwz = wz(i^2 + 1 - 2i)$

$-1 - iw - z - iwz = wz(-1 + 1 - 2i)$

$-1 - iw - z + iwz = 0$.

$wi(z-1) = z+1$

$w = \frac{z+1}{i(z-1)} = \frac{z+1}{i(z-1)} \times \frac{i}{i}$

$w = \frac{i(z+1)}{i^2(z-1)} = \frac{-i(z+1)}{z-1}$.

★★★ ② find the B.T. that maps $(\cancel{0,i,1})$ $1, i, -1$ to $(0, 1, \infty)$

Sol: Let $z_1 = 1,\ z_2 = i,\ z_3 = -1,\ z_4 = z$

Let $w_1 = 0,\ w_2 = 1,\ w_3 = \infty,\ w_4 = w$

By cross ratio property,

$$\frac{(z_1-z_2)(z_3-z_4)}{(z_1-z_4)(z_3-z_2)} = \frac{(w_1-w_2)(w_3-w_4)}{(w_1-w_4)(w_3-w_2)}$$

$$\frac{(z_1-z_2)(z_3-z_4)}{(z_1-z_4)(z_3-z_2)} = \frac{(w_1-w_2)\cancel{w_3}\left(1-\frac{w_4}{w_3}\right)}{(w_1-w_4)\cancel{w_3}\left(1-\frac{w_2}{w_3}\right)}$$

$$\frac{(1-i)(-1-z)}{(1-z)(-1-i)} = \frac{(0-1)(1-0)}{(0-w)(1-0)} \qquad \left[\because \frac{w_4}{w_3} = \frac{w}{\infty} = 0\right]$$

$$\frac{\cancel{-}(1-i)(z+1)}{\cancel{-}(1-z)(1+i)} = \frac{\cancel{-}1(1-0)}{\cancel{-}w(1-0)} = \frac{1}{w}$$

$$w = \frac{(1-z)(1+i)}{(1-i)(z+1)} \times \frac{(i+1)}{(i+1)} = \frac{(1-z)(1+i)^2}{(z+1)(1^2-i^2)}$$

$$w = \frac{(1-z)(1+i^2+2i)}{(z+1)(2)} = \frac{(1-z)(\cancel{2}i)}{(z+1)\cancel{2}} = \frac{i(1-z)}{(z+1)}$$

③ find the Mobius transformation which maps $(0, i, 1)$ into $(-1, 0, 1)$.

Let the required B.T is $w = \frac{az+b}{cz+d}$

Sol: Let $z_1 = 0,\ z_2 = i,\ z_3 = 1,\ z_4 = z.$

Let $w_1 = -1,\ w_2 = 0,\ w_3 = 1,\ w_4 = w.$

By cross ratio property,

$$\frac{(z_1-z_2)(z_3-z_4)}{(z_1-z_4)(z_3-z_2)} = \frac{(w_1-w_2)(w_3-w_4)}{(w_1-w_4)(w_3-w_2)}$$

$$\frac{(0-i)(1-z)}{(0-z)(1-i)} = \frac{(-1-0)(1-w)}{(-1-w)(1-0)}$$

z	w
$2,1,0$	$1,0,i$
$2,i,-2$	$1,i,-1$
$0,-i,2i$	$5i,\infty,-\frac{i}{3}$
$1,i,-1$	$0,1,\infty$

$$\frac{\cancel{-}i(1-z)}{\cancel{-}z(1-i)} = \frac{(\cancel{-}1)(1-w)}{\cancel{-}(1+w)(1)}$$

$$\frac{i(1-z)}{z(1-i)} = \frac{(1-w)}{(1+w)}$$

$$i(1-z)(1+w) = (1-w)\,z(1-i)$$

$$(i-iz)(1+w) = (1-w)(z-iz)$$

$$i-iw-iz-iwz = z-iz-wz+wiz.$$

$$i-iw-\cancel{iz}-iwz-z+\cancel{iz}+wz-wiz=0$$

$$i-iw-z+wz-2iwz=0$$

$$wz-2iwz-iw = z-i$$

$$w(z-2iz-i) = z-i$$

$$w = \frac{z-i}{(z-2iz+i)} = \frac{\cancel{-}(i-z)}{\cancel{-}(2iz-z\mp i)} = \frac{(i-z)}{(2iz-z\mp i)}$$

4) Find the B.T that maps the points $((1-2i),(2+i),(z+3i))$ into $((2+i),(1+3i),(4))$

Sol: Let $z_1 = 1-2i$, $z_2=2+i$, $z_3=z+3i$, $z_4=z$.

let $w_1 = 2+i$, $w_2=1+3i$, $w_3=4$, $w_4=w$

By cross ratio property,

$$\frac{(z_1-z_2)(z_3-z_4)}{(z_1-z_4)(z_3-z_2)} = \frac{(w_1-w_2)(w_3-w_4)}{(w_1-w_4)(w_3-w_2)}$$

$$\frac{(1-2i-2-i)(\cancel{z}+3i-\cancel{z})}{(1-2i-z)(2+i-z-3i)} = \frac{(2+i-1-3i)(4-w)}{(2+i-w)(4-1-3i)}$$

$$\frac{(-1-3i)(3i)}{(1-2i-z)(2-z-2i)} = \frac{(1-2i)(4-w)}{(2+i-w)(3-3i)}$$

$$\frac{(-3i-9i^2)}{(1-2i-z)(2-z-2i)} = \frac{(4-8i-w+2iw)}{(2+i-w)(3-3i)}$$

$$\frac{(-3i+9)}{(1-2i-z)(2-z-2i)} = \frac{(4-w-8i+2iw)}{(2+i-w)(3-3i)} \Rightarrow \frac{(-3i+9)}{(1-2i-z)(2-z-2i)} = \frac{(4-w-8i+2iw)}{(6+3i-3w-6i-3 \ldots +3 \ldots}$$

$$\frac{(-3i+9)}{(1-2i-z)(2-z-2i)} = \frac{(4-w-8i+2iw)}{(9-3i-3w+3iw)}$$

$(1-2i-z)(2-z-2i)$

$= (2-4i-2z-z+2iz+z^2-2i+4i^2+4 \ldots$

$= (2-6i-3z+6iz+z^2-4)$

$= (z^2+6iz-6i-3z-2)$

$$\frac{(-3i+9)}{(z^2+6iz-6i-3z-2)} = \frac{(4-w-8i+2iw)}{(9-3i-3w+3iw)}$$

$$\frac{(4-w)}{w-2-i} \times \frac{-3(i-3)}{-6} = \frac{-(i+3)(2+3i-z)}{-2(z-1+2i)}$$

$$\frac{(-3+i)(4-w)}{6(w-2-i)} = \frac{(-i+3)(2+3i-z)}{2(z-1+2i)}$$

$$(4-w)[-6z+6-12i+2iz-2i+4] = (w-2-i)[-12i+18+6iz+36+54-18z]$$

$$w = \frac{-10z-3iz+7+16i}{-4z+2i+7}$$

---

⑤ $(2,1,0)$ to $(1,0,i)$

Sol: Let $z_1=2$, $z_2=1$, $z_3=0$, $z_4=z$

Let $w_1=1$, $w_2=0$, $w_3=i$, $w_4=w$

By cross ratio property,

$$\frac{(z_1-z_2)(z_3-z_4)}{(z_1-z_4)(z_3-z_2)} = \frac{(w_1-w_2)(w_3-w_4)}{(w_1-w_4)(w_3-w_2)}$$

$$\frac{(2-1)(0-z)}{(2-z)(0-1)} = \frac{(1-0)(i-w)}{(1-w)(i-0)}$$

$$\frac{\cancel{-}z}{(2-z)(\cancel{-}1)} = \frac{(i-w)}{i(1-w)}$$

$$z\,(i(1-w)) = (i-w)(2-z)$$

$$iz - iwz = 2i - 2w - iz + wz$$

$$2w - wz - iwz = 2i - iz - iz$$

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$w(2-z-iz) = 2i-2iz.$

$$w = \frac{2i(1-z)}{(2-z-iz)}$$

⑥ $(2,i,-2)$ to $(1,i,-1)$

Sol: Let $z_1=2$, $z_2=i$, $z_3=-2$, $z_4=z$

Let $w_1=1$, $w_2=i$, $w_3=-1$, $w_4=w$

By cross ratio property,

$$\frac{(z_1-z_2)(z_3-z_4)}{(z_1-z_4)(z_3-z_2)} = \frac{(w_1-w_2)(w_3-w_4)}{(w_1-w_4)(w_3-w_2)}$$

$$\frac{(2-i)(-2-z)}{(2+2)(-2-i)} = \frac{(1-i)(-1-w)}{(1-w)(-1-i)}$$

$$\frac{(2-i)(2+z)}{4(2+i)} = \frac{(1-i)(1+w)}{(1-w)(1+i)} \Rightarrow \cancel{(2-i)(2+z)(1+i) = 4(2+i)(1-i)}$$

$(2-i)(2+z)(1-w)(1+i) = (8+4i)(1-i)(1+w)$

$(4+2z-2i-iz)(1-w+i-iw) = (8-8i+4i-4i^2)(1+w)$

$(2-i+2i-i^2)(2+z) = (8+4i)(1-i)$

$(3+i)(2+z) = (8-i+4i-4i^2)$

$(6+3z+2i+iz) = (12+3i)$

$z(3+i) = 12+3i-6-2i$

$$z = \frac{6+i}{(3+i)}$$

⑦ $(0,-i,2i)$ to $(5i,\infty,-\frac{i}{3})$

Sol: Let $z_1=0$, $z_2=-i$, $z_3=2i$, $z_4=z$.

Let $w_1=5i$, $w_2=\infty$, $w_3=-\frac{i}{3}$, $w_4=w$.

By cross ratio property.

$$\frac{(z_1-z_2)(z_3-z_4)}{(z_1-z_4)(z_3-z_2)} = \frac{(w_1-w_2)(w_3-w_4)}{(w_1-w_4)(w_3-w_2)}$$

$$\frac{(0+i)(2i-z)}{(0-z)(2i+i)} = \frac{\cancel{w_2}\left(\frac{w_1}{w_2}-1\right)(w_3-w_4)}{(w_1-w_4)\,\cancel{w_2}\left(\frac{w_3}{w_2}-1\right)}$$

$$\frac{(1)(2i-z)}{-3iz} = \frac{(0-1)(\frac{-i}{3}-w)}{(5i-w)(0-1)}$$

$$\frac{2i^2-iz}{-3iz} = \frac{-(\frac{-i}{3}-w)}{(5i-w)} \Rightarrow \frac{-2-iz}{-3iz} = \frac{\frac{i}{3}+w}{-w+5i}$$

$$(-2-iz)(-w+5i) = -3iz(\frac{i}{3}+w)$$

$$+2w-10i+izw-5i^2z = -i^2z-3iwz.$$

$$+2w-10i+iwz+5z = z-3iwz.$$

$$+2w+iwz+3iwz = z-5z+10i$$

$$+2w+4iwz = -4z-10i$$

$$2w(+1+2iz) = -4z-10i$$

$$w = \frac{-4z-10i}{2(2iz+1)}$$

⑧ $(1,i,-1)$ to $(0,1,\infty)$

Sol: let $z_1=1$, $z_2=i$, $z_3=-1$, $z_4=z$

let $w_1=0$, $w_2=1$, $w_3=\infty$, $w_4=w$

By cross ratio property,

$$\frac{(z_1-z_2)(z_3-z_4)}{(z_1-z_4)(z_3-z_2)} = \frac{(w_1-w_2)(w_3-w_4)}{(w_1-w_4)(w_3-w_2)}$$

$$\frac{(1-i)(-1-z)}{(1-z)(-1-i)} = \frac{(0-1)\cancel{w_3}(1-\frac{w_4}{w_3})}{(0-w)\cancel{w_3}(1-\frac{w_2}{w_3})}$$

$$\frac{\cancel{-}(1-i)(1+z)}{\cancel{-}(1-z)(1+i)} = \frac{(0-1)(1-0)}{(0-w)(1-0)}$$

$$\frac{1+z-i-iz}{1-z+i-iz} = \frac{\cancel{-}1}{\cancel{-}w}$$

~~1+z-i-iz.~~

$$w = \frac{1-z+i-iz}{1+z-i-iz} = \frac{1-z+i(1-z)}{1+z-i(1-z)}$$

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(6) $(2, i, -2)$ to $(1, i, -1)$

Sol: Let $z_1 = 2,\ z_2 = i,\ z_3 = -2,\ z_4 = z.$

Let $w_1 = 1,\ w_2 = i,\ w_3 = -1,\ w_4 = w$

By cross ratio property,

$$\frac{(z_1-z_2)(z_3-z_4)}{(z_1-z_4)(z_3-z_2)} = \frac{(w_1-w_2)(w_3-w_4)}{(w_1-w_4)(w_3-w_2)}$$

$$\frac{(2-i)(-2-z)}{(2-z)(-2-i)} = \frac{(1-i)(-1-w)}{(1-w)(-1-i)}$$

$$\frac{(2-i)(2+z)}{(2-z)(2+i)} = \frac{(1-i)(1+w)}{(1-w)(1+i)}$$

$$\frac{4-2i+2z-iz}{4-2z+2i-iz} = \frac{1-i+w-iw}{1-w+i-iw}$$

$$\underbrace{(4-2i+2z-iz)(1-w+i-iw)}_{①} = \underbrace{(4-2z+2i-iz)(1-i+w-iw)}_{②}$$

① = $4-4w+4i-4iw-2i+2iw-2i^2+2i^2w+2z-2wz+2iz-2iwz-iz+iwz-i^2z+i^2wz$

$4-4w+4i-4iw-2i+2iw+2-2w+2z-2wz+2iz-2iwz-iz+iwz+z-wz.$

$6-6w+2i-2iw+3z-3wz+iz-iwz$ —— ③.

② = $4-4i+4w-4iw-2z+2iz-2wz-2iwz+2i-2i^2+2iw-2i^2w-iz+i^2z-iwz+i^2wz.$

$4-4i+4w-4iw-2z+2iz-2wz-2iwz+2i+2+2iw+2w-iz-z-iwz-wz.$

$6-2i+6w-2iw-3z-iz-3wz-3iwz$ —— ④

③ = ④

$\cancel{6}-6w+2i-\cancel{2iw}+3z-3\cancel{w}z+iz-iwz = \cancel{6}-2i+6w-\cancel{2iw}-3z-iz-3\cancel{w}z-3iwz$

$2i-6w+3z+iz-iwz+2i-6w+3z+iz+3iwz = 0$

$4i-12w+6z+2iz+2iwz = 0$

$2iwz-12w = -4i-6z-2iz$

$2w(iz-6) = -4i-6z-2iz$

$$w = \frac{-4i-6z-2iz}{2(iz-6)}$$

(or).

$$\frac{(1-w)}{(1+w)} = \frac{(2-z)}{(2+z)} \frac{(1-i)}{(1+i)} \times \frac{(2+i)}{(2-i)}$$

$$= \frac{(2-z)}{(2+z)} \cdot \frac{(1-i)}{(1+i)} \times \frac{(1-i)}{(1-i)} \times \frac{(2+i)}{(2-i)} \times \frac{(2+i)}{(2+i)}$$

$$= \frac{(2-z)}{(2+z)} \cdot \frac{(1-i)^2}{(1^2-i^2)} \times \frac{(2+i)^2}{(2^2-i^2)}$$

$$= \frac{(2-z)}{(2+z)} \cdot \frac{(-3i-4i^2)}{5} \Rightarrow \frac{(2-z)}{(2+z)} \times \frac{(4-3i)}{5}$$

By componendo & dividendo,

$$\frac{a}{b} = \frac{c}{d} = \frac{(a+b)}{(a-b)} = \frac{c+d}{c-d}$$

$$\frac{(1-w)+(1+w)}{(1-w)-(1+w)} = \frac{(2-z)(4-3i)+(2+z)5}{(2-z)(4-3i)-(2+z)5}$$

$$\frac{1-w+1+w}{1-w-1-w} = \frac{8-6i-4z+3iz+10+5z}{8-6i-4z+3iz-10-5z}$$

$$\frac{2}{-2w} = \frac{18-6i+z+3iz}{-2-6i-9z+3iz}$$

$$-w = \frac{-[9z-3iz+2+6i]}{18-6i+z+3iz}$$

$$w = \frac{(3-i)z+2(1+3i)}{(1+3i)z+3(6-2i)}$$

9) Find the B.T. which maps the points $(0,-i,-1)$ into $(i,1,0)$. Find the image of the line $y=mx$ under this transformation.

Sol: let $w = \dfrac{az+b}{cz+d}$ be the required B.T.

$z_1=0,\ z_2=-i,\ z_3=-1,\ z_4=z.$

$w_1=i,\ w_2=1,\ w_3=0,\ w_4=w.$

By cross ratio property,

$$\frac{(z_1-z_2)(z_3-z_4)}{(z_1-z_4)(z_3-z_2)} = \frac{(w_1-w_2)(w_3-w_4)}{(w_1-w_4)(w_3-w_2)}.$$

$$\frac{(0+i)(-1-z)}{(0-z)(-1+i)} = \frac{(i-1)(0-w)}{(i-w)(0-1)}$$

$$\cancel{-}\frac{(0+i)(1+z)}{(0\cancel{-}z)(i-1)} = \frac{(i-1)(\cancel{-}w)}{(i-w)(\cancel{-}1)}$$

$$\frac{(0+i)(1+z)}{z(i-1)} = \frac{(i-1)w}{(i-w)} \Rightarrow \frac{(0+i)(1+z)}{iz-z} = \frac{iw-w}{i-w}$$

$(\cancel{1}+z+i+iz)(i-w) = (iz-z)(iw-w)$

$i+iz+i^2+i^2z-w+wz-iw-wiz = i^2wz-iwz-iwz+wz.$

$i+iz-1-z-w-wz-iw-i\cancel{wz} = -wz-iwz-i\cancel{wz}+wz$

$i+iz-1-z-w-wz-iw+wz+iwz-wz=0.$

$i+iz-1-z-w-wz-iw+wz+iwz-wz=0.$

$i+iz-1-z-w(1-z-i+\cancel{z}+iz-\cancel{z})=0$

$$\cancel{w=\frac{i+iz-1-z}{(1-z-i+iz)}}$$

$$w=\frac{(z+1)}{i(z-1)}\times\frac{i}{i} = \frac{i(z+1)}{i^2(z-1)} = \frac{i(z+1)}{(1-z)}$$

Image: $w=\frac{i(z+1)}{(1-z)}$

$w(1-z)=i(z+1)$

$w-wz-iz=i$

$z=\frac{w-i}{w+i}$

we have $z=x+iy$, $w=u+iv$

$$x+iy=\frac{(u+iv)-i}{(u+iv)+i} = \frac{[u+i(v-1)]}{[u+i(v+1)]}\times\frac{[u-i(v+1)]}{[u-i(v+1)]}$$

$$= \frac{u^2 - iu(v+1) + iu(v-1) - i^2(v^2-1)}{u^2 - i^2(1+v)^2}$$

$$= \frac{u^2 - i\cancel{uv} - iu + i\cancel{uv} - iu + v^2 - 1}{u^2 + (1+v)^2}$$

$$x+iy = \frac{(u^2+v^2-1) + i(-2u)}{u^2+(1+v)^2}$$

$$x = \frac{u^2+v^2-1}{u^2+(1+v)^2} \quad ; \quad y = \frac{-2u}{u^2+(1+v)^2}$$

In z-plane, $y = mx$.

$$\frac{-2u}{\cancel{u^2+(1+v)^2}} = \frac{m(u^2+v^2-1)}{\cancel{u^2+(1+v)^2}}.$$

$$m(u^2+v^2) + 2u - m = 0.$$

$\therefore$ $y = mx$ in z-plane is mapped to circle in w-plane.

★★

2 Show that the transformation $w = \frac{i(1-z)}{1+z}$ transforms $|z| = 1$ circle into the real axis of w-plane and the interior of the circle $|z| < 1$ into the upper half of z-plane.

Sol: Given: $w = \frac{i(1-z)}{1+z}$

$$w(1+z) = i - iz.$$

$$w + wz = i - iz$$

$$wz + iz = i - w$$

$$z = \frac{i-w}{w+i}$$

we have $z = x+iy$, $w = u+iv$

$$x+iy = \frac{i-(u+iv)}{(u+iv)+i} = \frac{-u+i(1-v)}{u+i(1+v)} \quad \text{--- (1)}$$

$$x+iy = \frac{[-u+i(1-v)]}{u+i(1+v)} \times \frac{u-i(1+v)}{u-i(1+v)}$$

$$= \frac{-u^2 + iu(1+v) - i^2(1-v^2) + iu(1-v)}{u^2 - i^2(1+v)^2} \Rightarrow \frac{-u^2 + iu + iuv + 1 - v^2 + iu - iuv}{u^2 + (1+v)^2}$$

$$= \frac{-u^2 - v^2 + 1 + i(2u)}{u^2 + (1+v)^2}$$

$$x = \frac{-u^2 - v^2 + 1}{u^2 + (1+v)^2} \quad ; \quad y = \frac{2u}{u^2 + (1+v)^2}$$

i) $|z| = 1$ is circle with center at origin & radius 1 unit.

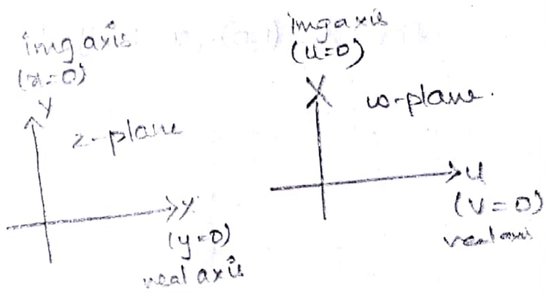
$$\left| \frac{-u + i(1-v)}{u + i(1+v)} \right| = 1 \qquad \text{from } ①.$$

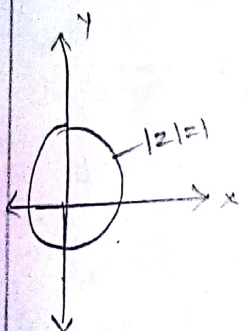
$$|-u + i(1-v)| = |u + i(1+v)|$$

$$\sqrt{u^2 + (1-v)^2} = \sqrt{u^2 + (1+v)^2} \qquad [\because |x+iy| = \sqrt{x^2+y^2}]$$

$$u^2 + 1 + v^2 - 2v = u^2 + 1 + v^2 + 2v$$

$$-4v = 0 \Rightarrow \underline{\underline{v = 0}}.$$





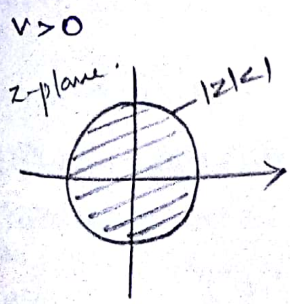
$\therefore$ $|z| = 1$ i.e. circle in z-plane is mapped to $v=0$ (i.e) real axis of w-plane.

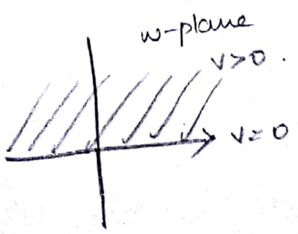
ii) $|z| < 1$

$-4v < 0$

$4v > 0$

$v > 0$





$\therefore$ $|z| < 1$, (i.e) interior of the circle is mapped to $v > 0$ (i.e) upper half of the w-plane.

③ Find and plot the images of the triangular region with vertices (0,0), (0,1) & (1,0) under the transformation $w = (1-i)z + 3$

Sol: We have $z = x+iy$, $w = u+iv$

$$u+iv = (1-i)(x+iy)+3$$

$$u+iv = x+iy-ix-i^2y+3$$

$$u+iv = (x+y+3)+i(y-x)$$

$$u = x+y+3 \quad ; \quad v = y-x$$

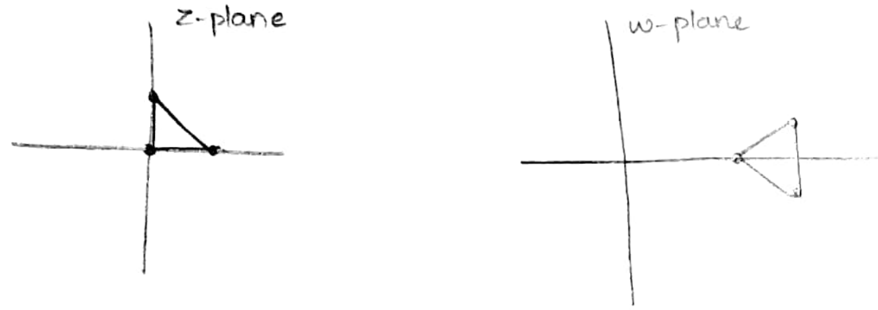
For $z_0 = (0,0)$, $x=0$, $y=0$. ; $u=3$, $v=0$. ∴ $w_0(3,0)$

for $z_1 = (0,1)$, $x=0$, $y=1$; $u=4$, $v=1$ ∴ $w_1(4,1)$

for $z_2 = (1,0)$, $x=1$, $y=0$; $u=4$, $v=-1$ ∴ $w_2(4,-1)$.

z-plane

w-plane



H.w:

② $w = z+(2-i)$

$$u+iv = x+iy+2-i$$

$$= (x+2)+i(y-1)$$

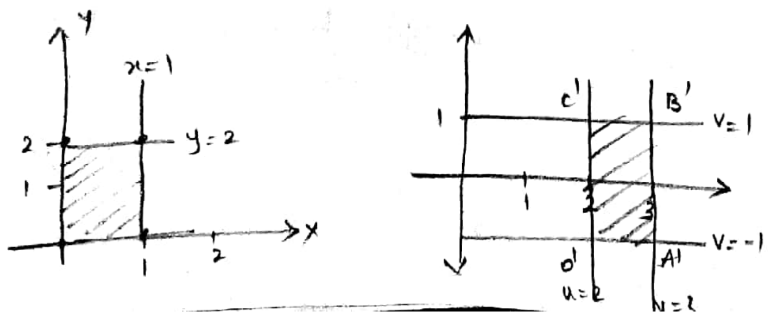
$$u = x+2, \quad v = y-1$$

$x=0$, $u=2$

$x=1$, $u=3$

$y=0$, $V=-1$

$y=2$, $V=1$



① Find the images of
i) $x = y+1$
ii) the line joining $A(1+i)$ to $B(2+3i)$ in
(1,1) (2,3)
z plane under
$w = \frac{i}{z}$

② What is the region of the w-plane into which the rectangular region in z-plane bounded by the lines $x=0$, $y=0$; $x=1$, $y=2$ is mapped under the transf. $w = z+(2-i)$

∴ The rectangular region OABC in z-plane is mapped to rectangular region O'A'B'C' in w-plane.

4) Determine the image of the region under the transformation $w = \frac{1}{z}$, $0 < y < 2$.

Sol: Let the B.T. be $w = \frac{1}{z}$.

We have, $z = x+iy$, $w = u+iv$

$$w = \frac{1}{z}$$

$$z = \frac{1}{w}$$

$$x+iy = \frac{1}{u+iv} \times \frac{u-iv}{u-iv}$$

$$= \frac{u-iv}{u^2+v^2}$$

$$x = \frac{u}{u^2+v^2}, \quad y = \frac{-v}{u^2+v^2}$$

In z plane, $0 < y < 2$.

i) $y > 0$

$$\frac{-v}{u^2+v^2} > 0$$

$-v > 0$

$v < 0$ i.e. lower half of w-plane.

ii) $y < 2$

$$\frac{-v}{u^2+v^2} < 2$$

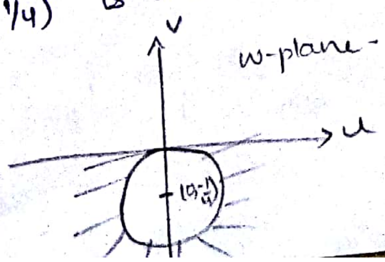
$$-v < 2u^2+2v^2$$

$$2u^2+2v^2+v > 0$$

$$u^2+v^2+\frac{v}{2} > 0$$

$$u^2+\left(v^2+2.v.\frac{1}{4}+\frac{1}{16}\right)-\frac{1}{16} > 0$$

$\left(u^2\right)+\left(v+\frac{1}{4}\right)^2 > \left(1/4\right)^2$ is a circle with center $\left(0,-\frac{1}{4}\right)$ & $r = \frac{1}{4}$ units.



∴ The region $0<y<2$ in z-plane is mapped to exterior of the circle $u^2+v^2+\frac{v}{2}$ in the lower half of the w-plane.

Show that $w=\frac{z-i}{1-iz}$ maps (i) interior of $|z|=1$ onto lower half of w-plane ii) upper half of z-plane onto interior of $|w|=1$

⑤ Under the transformation $w=\frac{1}{z}$ find the image of circle $|z-2i|=2$.

Sol:

$$w=\frac{1}{z}$$

$$z=\frac{1}{w}$$

$$x+iy=\frac{1}{u+iv}$$

$$x=\frac{u}{u^2+v^2}\ ,\quad y=\frac{-v}{u^2+v^2}$$

$|z-2i|=2$ is circle with center $(0,2)$ and radius $=2$ units.

$$|(x+iy)-2i|=2.$$

$$|x+i(y-2)|=2. \qquad \left[|x+iy|=\sqrt{x^2+y^2}\right]$$

$$x^2+(y-2)^2=4$$

$$\frac{u^2}{(u^2+v^2)}+\left(\frac{-v}{u^2+v^2}-2\right)^2=4$$

$$\frac{u^2}{u^2+v^2}+\frac{(-v-2u^2-2v^2)^2}{(u^2+v^2)^2}=4$$

$$u^2+(-v-2u^2-2v^2)^2=4(u^2+v^2)^2$$

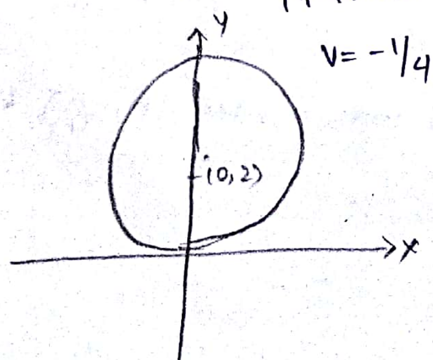
$$u^2+v^2+4u^4+4v^4+4vu^2+8u^2v^2+4v^3=4u^4+4v^4+8u^2v^2$$

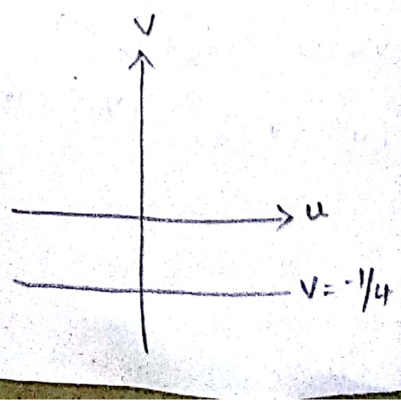
$$(u^2+v^2)+4v(u^2+v^2)=0$$

$$(u^2+v^2)(1+4v)=0$$

$$1+4v=0$$

$$v=-1/4$$





$\therefore$ The circle $|z-2i|=2$ in z-plane is mapped to $v=-1/4$ in w-plane.

(6) Show that under the transformation $w=\frac{1}{z}$, the image of the lines $y=x-1$ and $y=0$ are the circle $u^2+v^2-u-v=0$ and the line $v=0$.

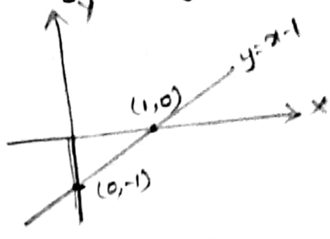
Sol: $w=\frac{1}{z}$

$y=x-1 \Rightarrow x-y=1$

$\frac{x}{1}+\frac{y}{-1}=1$ is a st. line passing through $(1,0)$ & $(0,-1)$

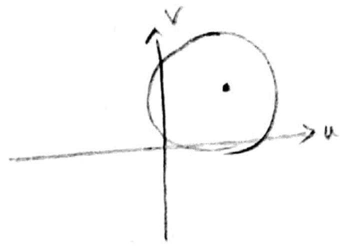
$z=\frac{1}{w}$

$x=\frac{u}{u^2+v^2}$, $y=\frac{-v}{u^2+v^2}$



$y=x-1$

$\frac{-v}{u^2+v^2}=\frac{u}{u^2+v^2}-1$



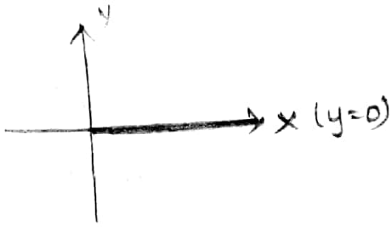
$\frac{-v}{u^2+v^2}=\frac{u-u^2-v^2}{u^2+v^2}$

$-v=u-u^2-v^2$

$u^2+v^2-u-v=0$

Hence proved.

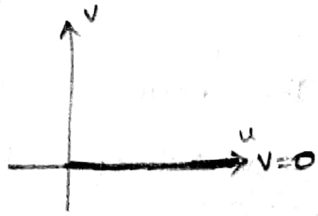
ii) $y=0$



$\frac{-v}{u^2+v^2}=0$

$-v=0$

$v=0.$



Hence proved.

(7) S.T the image of hyperbola $x^2-y^2=1$ under the transformation $w=\frac{1}{z}$ is $r^2=\cos 2\theta$.

Sol: $w=\frac{1}{z}$, $z=re^{i\theta}$, $w=Re^{i\phi}$

$w=\frac{1}{z} \Rightarrow Re^{i\phi}=\frac{1}{r}e^{-i\theta}$

$R = \frac{1}{r}$, $\phi = -\theta$.

$x = r\cos\theta$, $y = r\sin\theta$

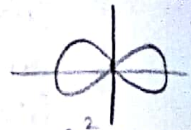
Given : $x^2 - y^2 = 1 \Rightarrow r^2(\cos^2\theta - \sin^2\theta) = 1$

$\because r = \frac{1}{R}$, $\theta = -\phi$

$\frac{1}{R^2}(\cos^2(-\phi) - \sin^2(-\phi)) = 1$

$\frac{1}{R^2}[\cos^2\phi - \sin^2\phi] = 1 \Rightarrow \cos^2\phi - \sin^2\phi = R^2$

Lemniscate



$r^2 = \cos 2\theta$

$\cos 2\phi = R^2$ $\quad [\because \cos^2\phi - \sin^2\phi = 1]$

$r^2 = \cos 2\theta$

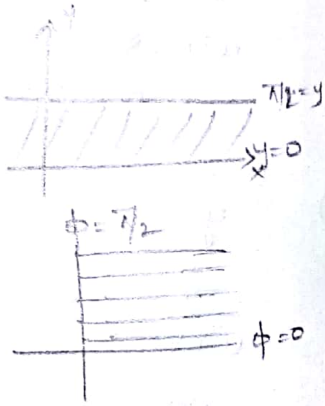
$\therefore$ The image of $x^2 - y^2 = 1$ in z-plane is curve $R^2 = \cos 2\phi$ in w-plane.

3M

(8) Find the image of the region in the z-plane between the lines $y=0$ & $y = \frac{\pi}{2}$ under the trans. $w = e^z$.

Sol: Given:

The trans is $w = e^z$.



$u + iv = e^{x+iy} = e^x(\cos y + i\sin y)$

$u = e^x\cos y$, $v = e^x \sin y$

$u^2 + v^2 = e^{2x}$, $w = Re^{i\phi}$

$Re^{i\phi} = e^x . e^{iy} \Rightarrow R = e^x$ & $\phi = y$

when $y = 0$, $\phi = 0$; $y = \pi/2$, $\phi = \pi/2$

The infinite strip betn $y=0$ & $y = \frac{\pi}{2}$ is mapped into the first quadrant of w-plane.

(7) Find the image of domain in z-plane to the left of line $x = -3$ under the trans. $w = z^2$.

Sol: The trans. is $w = z^2$.

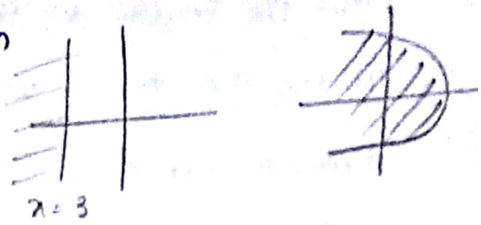
$u + iv = (x+iy)^2 = (x^2 - y^2) + i2xy$

$= u = x^2 - y^2$, $v = 2xy$

$v^2 = 4x^2y^2 \Rightarrow v^2 = -4x(u-x^2)$

If $x=-3$, $v^2 = -36(u-9)$ is a parabola with vertex $(\alpha, \beta) = (9,0)$ &

focus $(\alpha + a, \beta) = (9-9, 0) = (0,0)$ & $LL' = 36$.



The image of the domain left to the line $x=-3$ is the interior of parab $v^2 = -36(u-9)$ in w-plane.

(10) Show that the trans. $w = \frac{2z+3}{z-4}$ transforms the circle $x^2+y^2-4x=0$ into the st. line $4u+3=0$.

Sol: $w = \frac{2z+3}{z-4} \Rightarrow wz - 4w = 2z+3$

$wz - 2z = 3+4w \Rightarrow z = \frac{3+4w}{w-2}$

$\bar{z} = \frac{3+4\bar{w}}{\bar{w}-2}$

Given $x^2+y^2-4x = z\bar{z} - 2(z+\bar{z})$ $\quad [\because z\bar{z} = x^2+y^2$ & $x = \frac{z+\bar{z}}{2}]$

$$\frac{(3+4w)(3+4\bar{w})}{(w-2)(\bar{w}-2)} - 2\left(\frac{3+4w}{w-2} + \frac{3+4\bar{w}}{\bar{w}-2}\right) = 0$$

$9 + 12w + 12\bar{w} + 16w\bar{w} - 2(\bar{w}-2)(3+4w) - 2(3+4\bar{w})(w-2) = 0$

$9 + 12w + 12\bar{w} + 16w\bar{w} - 6\bar{w} - 8w\bar{w} + 12 + 16w - 6w + 12 - 8w\bar{w} + 16\bar{w} = 0$

$33 + 22w + 22\bar{w} = 0$ $\quad [\because w + \bar{w} = u+iv+u-iv = 2u]$

$33 + 22 \cdot 2u = 4u + 3 = 0$

** (11) Find the image of $c<y<d$ under the trans $w = e^z$.

Sol: $c<y<d$

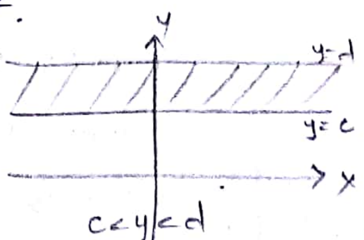
let $z = re^{i\theta}$, $w = Re^{i\phi}$, $w = e^z$, $u+iv = e^{x+iy}$

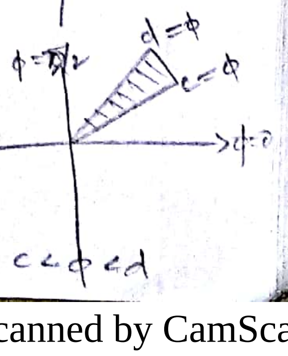
$u = e^x \cos\theta$, $y = e^x \sin\theta$

$Re^{i\phi} = e^x \cdot e^{iy} \Rightarrow R = e^x$, $\phi = y$

$y = c \Rightarrow \phi = c$

$y = d \Rightarrow \phi = d$





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(12) Find the image of the triangle with vertices i, $1+i$ & 1 in z-plane under the trans. $w = 3z+4-2i$.

Sol:

$w = 3z+4-2i$

$u+iv = 3(x+iy)+4-2i$

$= 3x+3iy+4-2i$

$= (3x+4)+i(3y-2)$

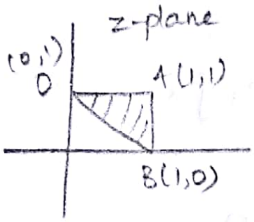
$u = 3x+4$; $v = 3y-2$.

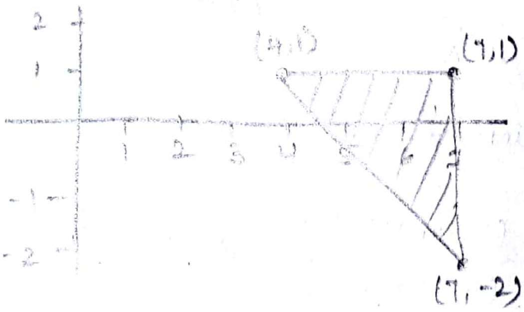
$z = i$, $z=(0,1)$ $\qquad u = 3x+4 = 3(0)+4 = 4$

$x=0$, $y=0$. $\qquad v = 3y-2 = 3(1)-2 = 1 \Rightarrow w_0 = (4,1)$

$z = 1+i$, $z_A(1,1)$, $x_A=1$, $y_A=1$, $u=7$, $v=1$, $w_{A'} = (7,1)$

$z = 1$, $z_B(1,0)$, $x=1$, $y=0$, $u=\cancel{3}\,7$, $v=-2$, $w_{B'} = (\cancel{3}\,7,-2)$.





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(13) S.T. trans. $w = \frac{5-4z}{4z-2}$ transforms the circle $|z|=1$ into a circle of radius units in w-plane & find the center of circle.

Sol: Let the B.T. be $w = \frac{5-4z}{4z-2}$.

$w(4z-2) = 5-4z$

$4wz+4z = 5+2w$

$z = \frac{5+2w}{4(w+1)}$

$x+iy = \frac{5+2(u+iv)}{4(u+iv+1)} = \frac{(2u+5)+i(2v)}{4(u+1)+i(4v)}$

$|z| = 1$

$$\left|\frac{(2u+5)+i2v}{4(u+1)+i(4v)}\right| = 1 \Rightarrow |(2u+5)+i(2v)| = |4(u+1)+i4v|$$

$$4u^2+25+20u+4v^2 = 16(u^2+v^2+1+2u)$$

$$4u^2+4v^2+4u-3=0$$

$$u^2+v^2+u-\frac{3}{4}=0$$

$$\left(u^2+2.u.\frac{1}{2}+\frac{1}{4}\right)+v^2-\frac{3}{4}-\frac{1}{4}=0$$

$$\left(u+\frac{1}{2}\right)^2+v^2=1$$

$|z|=1$ i.e. circle in z-plane is mapped to circle with centre $(\frac{-1}{2},0)$ & $r=1$ unit in w-plane.