UNIT-I

II ORDER LINEAR ODE

- Let $x^2D^2 + xD + y = f(x)$, be a II order ODE,

where,

$\boxed{xD = D}$

$\boxed{x^2D^2 = D(D-1) = D^2 - D}$

$x^3D^3 = D(D-1)(D-2)$

$= (D^2 - D)(D-2)$

$= D^3 - 2D^2 - D^2 + 2D$

$\boxed{x^3D^3 = D^3 - 3D^2 + 2D.}$

$x^4D^4 = D(D-1)(D-2)(D-3).$

Here,

let $x = e^z \Rightarrow z = \log x$.

(or)

$x^n = e^{zn}$.

~~Sol:~~ Q) $(x^2D^2-4xD+6)y = x^2.$

Sol: $(D(D-1)-4D+6)y = e^{2x}.$

$(D^2-D-4D+6)y = e^{2x}.$

$(D^2-5D+6)y = e^{2x}.$

A.E $\Rightarrow m^2-5m+6=0$

~~$\Rightarrow m^2-5m-m+6=0.$~~

~~$\Rightarrow m(m-5)-1$~~

$\Rightarrow m^2-3m-2m+6=0$

$\Rightarrow m = 3,2.$

$$\boxed{y_c = c_1 e^{3x} + c_2 e^{2x}}$$

$$\boxed{y_c = c_1 x^3 + c_2 x^2}$$

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$$y_p = \frac{1}{f(D)} e^{2x}$$

$$= \frac{1}{D^2-5D+6} e^{2x}$$

$D=2$

$$= \frac{1}{4-10+6} e^{2x}$$

Since denominator is 0.

$$\Rightarrow \frac{1}{(D-3)(D-2)'} e^{2x}$$

$$= \frac{x}{(2-3)} e^{2x}$$

$$= -xe^{2x}$$

$$\boxed{y_p = -\log x (x^2)}$$

$\therefore y = y_c + y_p$

$$\Rightarrow \boxed{y = c_1 x^3 + c_2 x^2 - x^2 \log x}$$

Q) $(x^2D^2-4)y = 65\cos(\log x)$

Sol: $(x^2D^2-4)y = 65\cos(\log x)$

$$(D(D-1)-4)y = 65\cos(z).$$

$$(D^2-D-4)y = 65\cos z.$$

$$A.E = m^2-m-4 = 0.$$

$$m = \frac{1+\sqrt{17}}{2}, \frac{1-\sqrt{17}}{2}$$

$$\therefore y_c = e^{\frac{1}{2}z}\left[c_1\cosh\frac{\sqrt{17}}{2}z + c_2\sinh\frac{\sqrt{17}}{2}z\right].$$

$$\boxed{y_c = \sqrt{x}\left[c_1\cosh\left(\frac{\sqrt{17}}{2}\log x\right) + c_2\sinh\left(\frac{\sqrt{17}}{2}\log x\right)\right.}$$

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$$y_p = \frac{1}{D^2 - D - 4} \, 65\cos z$$

$$= (65)\frac{1}{D^2 - D - 4} \cos z.$$

$$D^2 = -1.$$

$$= (65)\frac{1}{-1 - D - 4} \cos z$$

$$= \frac{65}{-D - 5} . \cos z.$$

$$= -\frac{65}{D + 5} \cos z.$$

"Rationalize".

$$= \frac{-65(D-5)}{D^2 - 25} \cos z. \quad \longrightarrow \text{put } D^2 = -1$$

$$= \frac{-65(D-5)}{-1-25} \cos z.$$

$$= \frac{\cancel{+}65(D-5)}{\cancel{+}26} \cos z$$

$$= \frac{65}{26}(D\cos z - 5\cos z)$$

$$= \frac{65}{26}(-\sin z - 5\cos z)$$

$$\boxed{y_p = \frac{65}{26}(-\sin(\log x) - 5\cos(\log x))}$$

$$\boxed{y = y_c + y_p}$$

$$y = \sqrt{x}\left[c_1 \cosh\left(\frac{\sqrt{17}}{2}\log x\right) + c_2 \sinh\left(\frac{\sqrt{17}}{2}\log x\right)\right]$$

$$+ \frac{65}{26}\left(-\sin(\log x) - 5\cos(\log x)\right)$$

4. $(x^2D^2 - xD + 1)y = x\log x$

5. $(x^2D^2 - 4xD + 6)y = (\log x)^2$

4 Sol: $(x^2D^2 - xD + 1)y = x\log x$

$$(D(D-1) - D+1)y = \cancel{\log x}\; e^z z$$

$$(D^2 - D - D + 1)y = ze^z$$

$$(D^2 - 2D + 1)y = ze^z$$

$$AE \Rightarrow m^2 - 2m + 1 = 0$$

$$\Rightarrow m = 1, 1$$

$$y_c = (c_1 + c_2 z)e^z$$

$$\boxed{y_c = (c_1 + c_2 \log x)x}$$

$$y_p = \frac{1}{D^2 - 2D + 1} e^z z$$

$$= \quad D = D + 1$$

$$= e^z \frac{1}{(D+1)^2 - 2(D+1) + 1} z$$

$$= e^z \cdot \frac{1}{D^2 + \cancel{1} + \cancel{2D} - \cancel{2D} - \cancel{2} + \cancel{1}} z$$

$$= e^z \frac{1}{D^2} z$$

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$$= e^{z} \frac{1}{D^2} z.$$

$$= e^{z} \frac{1}{D} \frac{z^2}{2}.$$

$$= e^{z} \frac{z^3}{6}.$$

$$y_p = \frac{x \cdot (\log x)^3}{6}$$

$$\therefore \boxed{y = y_c + y_p}$$

5) Sol:- $(x^2D^2 - 4xD + 6)y = (\log x)^2.$

$$(D(D-1) - 4D + 6)y = z^2.$$

$$(D^2 - D - 4D + 6)y = z^2$$

$$(D^2 - 5D + 6)y = z^2.$$

$$AE \Rightarrow m^2 - 5m + 6 = 0$$

$$\Rightarrow m^2 - 2m - 3m + 6 = 0$$

$$\Rightarrow m = 2, 3$$

$$y_c = c_1 e^{2z} + c_2 e^{3z}$$

$$\boxed{y_c = c_1 x^2 + c_2 x^3}$$

$$y_p = \frac{1}{f(D)} z^2$$

$$= \frac{1}{D^2 - 5D + 6} z^2.$$

$$= \frac{1}{6\left(1 + \frac{D^2 - 5D}{6}\right)} z^2.$$

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$$= \frac{1}{6}\left[1+\left(\frac{D^2-5D}{6}\right)\right]^{-1} x^2.$$

$$= \frac{1}{6}\left[1-\left(\frac{D^2-5D}{6}\right)+\left[\frac{D^2-5D}{6}\right]^2\right] z^2$$

$$= \frac{1}{6}\left[1-\frac{D^2}{6}+\frac{5D}{6}+\frac{D^4+25D^2-10D^3}{36}\right] z^2$$

$$= \frac{1}{6}\left[1-\frac{D^2}{6}+\frac{5D}{6}+\frac{D^4}{36}+\frac{25}{36}D^2-\frac{10}{36}D^3\right] z^2$$

$$= \frac{1}{6}\left[z^2-\frac{1}{6}(2)+\frac{5}{\cancel{6}_3}(2z)+0+\frac{25}{\cancel{36}_{18}}(z)-0\right].$$

$$= \frac{1}{6}\left[z^2-\frac{1}{3}+\frac{5}{3}z+\frac{25}{18}\right].$$

$$= \frac{1}{6}\left[z^2+\frac{5}{3}z+\frac{19}{18}\right].$$

$$= \frac{1}{6}\left[(\log x)^2+\frac{5}{3}(\log x)+\frac{19}{18}\right].$$

$$\boxed{y = y_p + y_c}$$

$$\boxed{y = c_1 x^2 + c_2 x^3 + \frac{1}{6}\left[(\log x)^2+\frac{5}{3}(\log x)+\frac{19}{18}\right].}$$

$\rightarrow$ Q) $x^3D^3y + 2x^2D^2y + 2y = 10\left(x + \frac{1}{x}\right)$.

Sol: $y(x^3D^3 + 2x^2D^2 + 2) = 10\left(x + \frac{1}{x}\right)$

$y(D^3 - 3D^2 + 2D + 2(D^2 - D) + 2) = 10\left(e^z + \frac{1}{e^z}\right)$.

$y(D^3 - 3D^2 + \cancel{2D} + 2D^2 - \cancel{2D} + 2)$

$y(D^3 - D^2 + 2) = 10\left[e^z + e^{-z}\right]$.

A.E = $f(m) = 0$

$m^3 - m^2 + 2 = 0$.

$m = -1, 1 \pm i$.

$y_c = c_1 e^{-z} + e^z(c_2 \cos z + c_3 \sin z)$.

$y_c = c_1 x^{-1} + x(c_2 \cos(\log x) + c_3 \sin(\log x))$.

$y_p = \frac{1}{(D^3 - D^2 + 2)} 10(e^z + e^{-z})$.

$= 10\left[\frac{1}{D^3 - D^2 + 2} e^z + \frac{1}{D^3 - D^2 + 2} e^{-z}\right]$

$= 10\left[\frac{1}{1 - 1 + 2} e^z + \frac{1}{-1 - 1 + 2} e^{-z}\right]$

$= 10\left[\frac{1}{2} e^z + \frac{1}{(D+1)(D^2 - 2D + 2)} e^{-z}\right]$

$= 10\left[\frac{e^z}{2} + \frac{z^1 e^{-z}}{1!\,(1+2+2)}\right] = 10\left[\frac{e^z}{2} + \frac{z e^{-z}}{5}\right]$

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$$y_p = 10\left[\frac{x}{2} + \frac{\log x}{5x}\right]$$

$$\boxed{y = y_p + y_c} \rightarrow \boxed{y = c_1 x^{-1} + x\left(c_2 \cos(\log x) + c_3 \sin(\log x)\right) + 10\left[\frac{x}{2} + \frac{\log x}{5x}\right]}$$

2) $x^2D^2 - 3xD + 4y = (1+x)^2$

sol: $(D^2 - D)y - 3Dy + 4y = (1+x)^2$

$$y(D^2 - D - 3D + 4) = (1+e^x)^2$$

$$y(D^2 - 4D + 4) = (1+e^x)^2$$

$$A.E = m^2 - 4m + 4$$

$$m = 2, 2$$

$$y_c = (c_1 + c_2 x)e^{2x} \Rightarrow y_c = (c_1 + c_2 \log x)x^2$$

$$y_p = \frac{1}{f(D)}(1+e^x)^2$$

$$= \frac{1}{D^2 - 4D + 4}(1 + e^{2x} + 2e^x)$$

$4-8+4$

$$= \frac{1}{D^2-4D+4}e^{0x} + \frac{e^{2x}}{D^2-4D+4} + 2\frac{e^x}{D^2-4D+4}$$

$$= \frac{1}{4} + \cancel{0}\,\cancel{D-2}\,\frac{1}{(D-2)^2}e^{2x} + 2\frac{e^x}{1}$$

$$= \frac{1}{4} + \frac{x^2}{2!}e^{2x} + 2e^x$$

$$y_p = \frac{1}{4} + \frac{(\log x)^2}{2}\cdot x^2 + 2x$$

$$\boxed{y = y_c + y_p}$$

$$\boxed{y = (c_1 + c_2 \log x)x^2 + \frac{1}{4} + \frac{(\log x)^2}{2}x^2 + 2x}$$

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→ Legendre's Differential eq^n:

The eq^n of the form,

$$(a+bx)^n \frac{d^n y}{dx^n} + P_1 (a+bx)^{n-1} \frac{d^{n-1} y}{dx^{n-1}} +$$

$$P_2 (a+bx)^{n-2} \frac{d^{n-2} y}{dx^{n-2}} + \cdots + P_n y = Q(n).$$

Here, $P_1, P_2, \ldots P_n$ & Q are function of 'x'.

① → solve. $(1+x)^2 \frac{d^2 y}{dx^2} - 3(1+x)\frac{dy}{dx} + 4y = x^2+x+1$

Let $1+x = u$.

$x = u-1$.

Diff w.r.t x

$$1 = \frac{du}{dx}$$

$$\boxed{\frac{du}{dx} = 1}$$

$$\therefore \boxed{\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{dy}{du}}.$$

$\Rightarrow u^2 D^2 y - 3u Dy + 4y = (u-1)^2 + u.$

$y(u^2 D^2 - 3uD + 4) = u^2 - u + 1.$

Here $u = e^z$

$\log u = z$

$(D^2 - D - 3D + 4)y = e^{2z} - e^z + e^{0z}$

$(D^2 - 4D + 4)y = e^{2z} - e^z + e^{0z}$

$AE = f(m) = 0$

$m = 2, 2$

$y_c = e^{2z}(c_1 + c_2 z)$

$y_c = u^2(c_1 + c_2 \log u)$

$\boxed{y_c = (1+x)^2 (c_1 + c_2 \log(1+x))}$

$$y_p = \frac{1}{f(D)}(e^{2z} - e^z + e^{0z})$$

$$= \frac{1}{(D^2 - 4D + 4)} e^{2z} - \frac{1}{D^2 - 4D + 4} e^z + \frac{1}{D^2 - 4D + 4} e^{0z}$$

$$= \frac{z^2 e^{2z}}{2!} - e^z + \frac{1}{4}$$

$$= \frac{(\log u)^2 u^2}{2} - u + \frac{1}{4}$$

$$\boxed{y_p = \frac{(\log(1+x))^2 (1+x)^2}{2} - (1+x) + \frac{1}{4}}$$

$y = y_c + y_p$

(1) $\rightarrow \left[(x+1)^2D^2+(x+1)D\right]y = (2x+3)(2x+4)$.

let $x+1=u$.

$x = u-1$

$\frac{dx}{d} \quad 1 = \frac{du}{dx}$

$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx} = \frac{dy}{du}$

$\Rightarrow \left[u^2D^2+uD\right]y = 4(u-1)^2+8(u-1)+6(u-1)+12$

$\Rightarrow \left[u^2D^2+uD\right]y = 4u^2+4-\cancel{8u}+\cancel{8u}-8+6u-6+12$

$\Rightarrow \left[u^2D^2+uD\right]y = 4u^2+6u+2$

$\Rightarrow \cancel{u^x}\left[D^2-D+D\right]y = 4u^2+6u+2$.

$\Rightarrow D^2y = 4u^2+6u+2 \Rightarrow D^2y = 4e^{2z}+6e^{z}+2e^{0z}$

$A\cdot E = m^2=0$

$m=0,0$.

$$y_c = c_1 + c_2 \cancel{} z$$

$$y_c = c_1 + c_2 \log u$$

$$\boxed{y_c = c_1 + c_2 \log(1+x)}$$

$$y_p = \frac{1}{f(D)} 4e^{2z} + 6e^{z} + 2e^{0z}$$

$$= \frac{4e^{2z}}{D^2} + \frac{6e^{z}}{D^2} + \frac{2e^{0z}}{D^2}$$

$$= \frac{4e^{2z}}{4} + \frac{6e^{z}}{1} + \frac{2z^2 e^{0z}}{2!}$$

$$= e^{2z} + 6e^{z} + z^2$$

$$y_p = u^2 + 6u + (\log u)^2$$

$$\boxed{y_p = (1+x)^2 + 6(1+x) + (\log(1+x))^2}$$

$$\boxed{y = y_c + y_p}$$

(5)

$$\rightarrow (x+a)^2 \frac{d^2y}{dx^2} - 4(x+a)\frac{dy}{dx} + 6y = x.$$

let $x + a = u.$

$$x = u - a.$$

& 1 = $\frac{du}{dx}.$

$$\frac{du}{dx} = 1$$

$$\boxed{\frac{dy}{dx} = \frac{dy}{du}}$$

$$\Rightarrow (u^2D^2 - 4uD + 6)y = u - a.$$

$$(D^2 - D - 4D + 6)y = u - a.$$

$$(D^2 - 5D + 6)y = u - a. \Rightarrow (D^2 - 5D + 6)y = e^{z} - ae^{0z}$$

$$A.E = m^2 - 5m + 6 = 0.$$

$$\Rightarrow m = 2, 3.$$

$$y_c = c_1e^{2z} + c_2e^{3z}.$$

$$y_c = c_1u^2 + c_2u^3.$$

$$\boxed{y_c = c_1(x+a)^2 + c_2(x+a)^3}$$

$$y_p = \frac{1}{D^2-5D+6}(e^x - ae^{0x})$$

$$= \frac{e^x}{D^2-5D+6} - \frac{ae^{0x}}{D^2-5D+6}$$

$$= \frac{e^x}{2} - \frac{a}{6}$$

$$= \frac{e^x}{2} - \frac{a}{6}$$

$$= \frac{u}{2} - \frac{a}{6}$$

$$\boxed{y_p = \frac{x+a}{2} - \frac{a}{6}}$$

$$\boxed{y = y_p + y_c}$$

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(5) $\Rightarrow \left((1+x)^2 D^2 - (1+x)D + 1\right)y = \sin 2(\log(x+1))$

Let, $1+x = u$

$x = u-1$

$\cancel{d}\ \frac{du}{dx} = 1$

$$\boxed{\frac{dy}{dx} = \frac{dy}{du}}$$

$(u^2D^2 - uD + 1)y = \sin 2(\log u)$.

$\cancel{u^2D}\ (D^2 - D - D + 1)y = \sin 2z$

$(D^2 - 2D + 1)y = \sin 2z$.

$A \cdot E \Rightarrow m = 1, 1$

$$y_c = (c_1 + c_2 z)e^z$$
$$= (c_1 + c_2 \log u)u.$$
$$\boxed{y_c = (c_1 + c_2 \log(x+1))(x+1)}$$

$$y_p = \frac{1}{D^2 - 2D + 1}(\sin 2z).$$

$$D^2 = -4.$$

$$= \frac{1}{-4 - 2D + 1} \sin 2z.$$

$$= -\left(\frac{1}{+2D+3} \sin 2z\right).$$

$$= -\left(\frac{2D-3}{4D^2-9} \sin 2z\right).$$

$$D^2 = -4.$$

$$= +2\left(\frac{2D-3}{+25} \sin 2z\right)$$

$$= \frac{2}{25}\left[4\cos 2z - 3\sin 2z\right]$$

$$= \frac{2}{25}\left[4\cos 2(\log u) - 3\sin 2(\log u)\right]$$

$$\boxed{y_p = \frac{2}{25}\left[4\cos 2(\log(x+1)) - 3\sin 2(\log(x+1))\right]}$$

$$y = y_p + y_c.$$

Ordinary point:

$x=0$ is called an ordinary point of differential eq^n if P & Q donot become infinite and expand in the form of power series, that point is called an ordinary point.

Singular point:

$x=0$ is not an ordinary point then it is said to be singular point.

w.k.T II order differential eq^n is:

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + Q(x)y = 0.$$

(or)

$$y'' + py' + Qy = 0$$

where p, Q are functions of x.

Singular point

Regular Singular point

(i) $xP(x) \neq \infty$ at $x=0$

(ii) $x^2 Q(x) \neq \infty$ at $x=0$

Irregular singular point

(i) $xP(x) = \infty$ at $x=0$.

(ii) $x^2 Q(x) = \infty$ at $x=0$

$\rightarrow xy'' + xy' + xy = 0.$

divide x on both sides

$y'' + y' + y = 0.$

This is of form

$y'' + Py' + Qy = 0.$

$\Rightarrow P = 1, Q = 1.$

for $x = 0$, $P = 1$, $Q = 1$ $(P \neq \infty)$, $(Q \neq \infty)$. $\therefore$ It is ordinary point

$\rightarrow xy'' + xy' + y = 0$.

divide x on b.s

$$y'' + y' + \frac{1}{x} y = 0$$

This is of form

$$y'' + Py' + Qy = 0$$

at $x = 0$

~~where~~ $P = 1$ $(P \neq \alpha)$

$$Q = \frac{1}{0} = \infty \quad (Q = \infty).$$

it is singular point.

8) $xy'' + xy' + y = 0$

divide x on b s.

$$y'' + y' + \frac{1}{x} y = 0$$

This is of form

$$y'' + Py' + Qy = 0$$

for

$x = 0$, $P = 1$

$$Q = \frac{1}{x}$$

It is singular point.

$$x[P(x)] = x(1) = x$$

$x = 0$. $x[P(x) = 0]$ $(\neq \infty)$.

$$x^2(Q(x)) = x^2\left(\frac{1}{x}\right) = x$$

$x = 0$, $x^2[Q(x)] = 0$ $(\neq \infty)$.

$\therefore$ It is regular singular point.

→ ② $x^3y'' + xy' + y = 0$.

$$y'' + \frac{1}{x^2}y' + \frac{1}{x^3}y = 0.$$

This is of form

$$y'' + Py' + Qy = 0$$

$$P = \frac{1}{x^2}, \quad Q = \frac{1}{x^3}.$$

$$x[P(x)] = x\left(\frac{1}{x^{\cancel{2}}}\right) = \frac{1}{x}.$$

at $x = 0$, $x\,P[x] = \infty$.

$$x^2[Q(x)] = x^2\left[\frac{1}{x^3}\right] = \frac{1}{x}.$$

at $x = 0$, $x^2[Q(x)] = \infty$.

∴ It is irregular singular point

(1) $\rightarrow$ Expand the power series $y' - y = 0 \rightarrow (1)$

w.k.T $y'' + py' + Qy = 0 \rightarrow (2)$

comparing (1) & (2).

$P = 1, Q = -1$

it is ordinary point

Let us assume.

$$\left.\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ y' &= \sum_{n=1}^{\infty} a_n n x^{n-1} \end{aligned}\right\} \rightarrow (3)$$

from (1) & (3).

$$\sum_{n=\phi}^{\infty} a_n n x^{n-1} - \sum_{n=0}^{\infty} a_n x^n = 0$$

$\boxed{k = n-1}$ $\boxed{n = k}$.

$$\sum_{k+1=1}^{\infty} \cancel{a_n x^k}\, a_{k+1} (k+1) x^k - \sum_{k=0}^{\infty} a_k x^k = 0$$

$$\cancel{\sum_{k=0}^{\infty}} a_{k+1} (k+1) \cancel{x^k} = \cancel{\sum_{k=0}^{\infty}} a_k \cancel{x^k}$$

$$\boxed{a_{k+1} = \frac{a_k}{k+1}} \rightarrow (A) \quad \forall k = 0, 1, 2, \ldots$$

if $k=0$

$$(A) \Rightarrow a_1 = \frac{a_0}{1} = a_0$$

if $k=1$

$$(A) \Rightarrow a_2 = \frac{a_1}{2} = \frac{a_0}{2}.$$

if $k=2$.

$$(A) \Rightarrow a_3 = \frac{a_2}{3} = \frac{1}{3}\left(\frac{a_0}{2}\right) = \frac{a_0}{6}.$$

if $k=3$

$$(A) \Rightarrow a_4 = \frac{a_3}{4} = \frac{1}{4}\left(\frac{a_0}{6}\right) = \frac{a_0}{24}.$$

$\vdots$

from (3)

$$y = \sum_{n=0}^{\infty} a_n x^n.$$

$$= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots$$

$$= a_0 + \frac{a_0}{1} x + \frac{a_0}{2} x^2 + \frac{a_0}{6} x^3 + \frac{a_0}{24} x^4 + \cdots$$

$$= a_0\left[1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\right]$$

$$\boxed{y = a_0 e^x}$$

→ Expand the power series of:

i) $y''-y=0$.

$P=0,\ Q=-1$.

It is ordinary point.

let us assume,

$$y=\sum_{n=0}^{\infty} a_n x^n$$

$$y'=\sum_{n=1}^{\infty} a_n n x^{n-1}$$

$$y''=\sum_{n=2}^{\infty} a_n n(n-1) x^{n-2}$$

→ (2)

from (1) & (2)

$$\sum_{n=2}^{\infty} a_n n(n-1) x^{n-2} - \sum_{n=0}^{\infty} a_n x^n = 0$$

$\boxed{n-2=k}$ $\boxed{n=k}$

$$\Rightarrow \sum_{k+2=2}^{\infty} a_{k+2}(k+2)(k+1)x^k - \sum_{n=0}^{\infty} a_k x^k = 0$$

$$\Rightarrow \sum_{k=0}^{\infty} a_{k+2}(k+2)(k+1)x^k - \sum_{k=0}^{\infty} a_k x^k = \sum_{k=0}^{\infty} 0\, x^k.$$

$$\Rightarrow a_{k+2}(k+2)(k+1) = a_k.$$

$$\Rightarrow \boxed{a_{k+2} = \frac{a_k}{(k+2)(k+1)}} \rightarrow (A)$$

for

$k=0$

$$(A) \Rightarrow a_2 = \frac{a_0}{2}$$

$k=1$

$$(A) \Rightarrow a_3 = \frac{a_1}{6}.$$

$k=2$

$$(A) \Rightarrow a_4 = \frac{a_2}{12} = \frac{1}{12}\left(\frac{a_0}{2}\right) = \frac{a_0}{24}$$

$k=3$

$$(A) \Rightarrow a_5 = \frac{a_3}{20} = \frac{1}{20}\left(\frac{a_1}{6}\right) = \frac{a_1}{120},$$

$k=4$.

$$(A) \Rightarrow a_6 = \frac{a_4}{30} = \frac{1}{30}\left(\frac{a_0}{24}\right) = \frac{a_0}{720}.$$

$k=5$

$$(A) \Rightarrow a_7 = \frac{a_5}{42} = \frac{1}{42}\left(\frac{a_1}{120}\right) = \frac{a_1}{5040}$$

from,

$$y = \sum_{n=0}^{\infty} a_n x^n$$

$$= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5$$

$$= a_0 + a_1 x + \frac{a_0}{2} x^2 + \frac{a_1}{6} x^3 + \frac{a_0}{24} x^4 + \frac{a_1}{120} x^5$$

$$= a_0 \left[1 + \frac{1}{2} x^2 + \frac{1}{24} x^4 + \cdots \right] + a_1 \left[x + \frac{x^3}{6} + \frac{x^5}{120} \right]$$

$$y = a_0 \left[1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots \right] + a_1 \left[\frac{x}{1!} + \frac{x^3}{3!} + \frac{x^5}{5!} + \right]$$

16

ii) $y'' + y = 0$

$P = 0,\ Q = 1.$

It is ordinary point.

Let us assume,

$$y = \sum_{n=0}^{\infty} a_n x^n$$

$$y' = \sum_{n=1}^{\infty} a_n n \cdot x^{n-1}$$

$$y'' = \sum_{n=2}^{\infty} a_n n(n-1) x^{n-2}.$$

$y'' + y = 0$

$$\Rightarrow \sum_{n=2}^{\infty} a_n n(n-1) x^{n-2} + \sum_{n=0}^{\infty} a_n x^n = 0$$

let $n-2 = k$, $n = k$

$$\Rightarrow \sum_{k+2=2}^{\infty} a_{k+2}(k+2)(k+1)\, x^{k} + \sum_{k=0}^{\infty} a_k x^k = 0$$

$$\Rightarrow \sum_{k=0}^{\infty} a_{k+2}(k+2)(k+1)\, x^{k} + \sum_{k=0}^{\infty} a_k x^k = \sum_{k=0}^{\infty} 0\, x^k$$

$$\Rightarrow a_{k+2}(k+2)(k+1) = -a_k .$$

$$\Rightarrow \boxed{a_{k+2} = \frac{-a_k}{(k+1)(k+2)}}$$

for

$k=0$,

$$a_2 = \frac{-a_0}{2}$$

$k=1$,

$$a_3 = \frac{-a_1}{6}$$

$k=2$

$$a_4 = \frac{-a_2}{12} = \frac{-1}{12}\left(\frac{-a_0}{2}\right) = +\frac{a_0}{24}.$$

$k=3$

$$a_5 = \frac{-a_3}{20} = \frac{-1}{20}\left(\frac{-a_1}{6}\right) = \frac{a_1}{120}.$$

$k=4$.

$$a_6 = \frac{-a_4}{30} = \frac{-1}{30}\left(\frac{a_0}{24}\right) = \frac{-a_0}{720}.$$

from $y = \sum_{n=0}^{\infty} a_n x^n$.

$$= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + \cdots$$

$$= a_0 + a_1 x - \frac{a_0}{2} x^2 - \frac{a_1}{6} x^3 + \frac{a_0}{24} x^4 + \frac{a_1}{120} x^5 - \frac{a_0}{720} x^6 + \cdots$$

$$= a_0 \left[1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} \cdots \right] + a_1 \left[x - \frac{x^3}{3!} + \frac{x^5}{5!} \cdots \right]$$

$$y = a_0 \cos x + a_1 \underline{\sin x}.$$

③ $y'' + x^2 y = 0$

$P = 0,\ Q = x^2$

Ordinary point.

Let us assume.

$$y = \sum_{n=0}^{\infty} a_n x^n$$

$$y' = \sum_{n=1}^{\infty} a_n n x^{n-1}$$

$$y'' = \sum_{n=2}^{\infty} a_n n(n-1) x^{n-2}.$$

$$\Rightarrow \sum_{n=2}^{\infty} a_n n(n-1)x^{n-2} + x^2 \sum_{n=0}^{\infty} a_n x^n = 0.$$

$$\Rightarrow \sum_{n=2}^{\infty} a_n n(n-1)x^{n-2} + \sum_{n=0}^{\infty} a_n x^{n+2} = 0.$$

$n = k+2$ $\quad n-2 = k \qquad\qquad n+2 = k.$

$$\Rightarrow \sum_{k=0}^{\infty} a_{k+2}(k+2)(k+1) x^k + \sum_{k=2}^{\infty} a_{k-2} x^k = 0.$$

$$\Rightarrow a_2(2)x^0 + a_3(6)x^1 + \sum_{k=2}^{\infty} a_{k+2}(k+2)(k+1)x^k + \sum_{k=2}^{\infty} a_{k-2} x^k = 0.$$

$$\Rightarrow a_2(2)x^0 + a_3(6)x^1 + \sum_{k=2}^{\infty}\left[a_{k+2}(k+2)(k+1) + a_{k-2}\right]x^k = 0$$

$$\Rightarrow 2a_2 x^0 + 6a_3 x^1 + \sum_{k=2}^{\infty}\left[a_{k+2}(k+2)(k+1) + a_{k-2}\right]x^k = 0\cdot x^0 + 0\cdot x^1 + \sum_{n=2}^{\infty} 0\cdot x^k$$

$\Rightarrow \cancel{a_2 = 0, a_3 = 0}$

$\Rightarrow a_2 = 0, a_3 = 0$

$\cancel{2}a_2 + \cancel{6}^{3} a_3 x = 0.$

$a_2 = \frac{-3a_3 x}{\cancel{}}$

& $a_{k+2}(k+2)\frac{(k+1)}{a} + a_{k-2} = 0.$

$$a_{k+2} = \frac{-(a_{k-2})}{(k+2)(k+1)}$$

for.

$k = 2$.

$a_4 = \frac{-a_0}{12}$.

$k = 3$.

$a_5 = \frac{-a_1}{20}$.

$k = 4$

$a_6 = \frac{-a_2}{30} = \frac{+\cancel{3}a_3 x}{\cancel{4}} \quad \frac{a_3 x}{2} = 0.$

$k = 5$.

$a_7 = \frac{-a_3}{42} = 0.$

$y = \sum_{n=0}^{\infty} a_n x^n = a_0 x^0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + \ldots$

$y = a_0 + a_1 x + 0 + 0 + \frac{a_0}{12} x^4 - \frac{a_1}{20} x^5 + 0 + 0 +$

$\frac{a_0}{672} x^5 + \ldots$

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10 Sol) Given → Q) $4x\left(\frac{d^2y}{dx^2}\right)+2\left(\frac{dy}{dx}\right)+y=0$.

$4xy''+2y'+y=0$

divide $4x$ on b.s.

$y''+\frac{1}{2x}y'+\frac{1}{4x}y=0$

Here $P=\frac{1}{2x}=\infty$ (for $x=0$)

$Q=\frac{1}{4x}=\infty$ (for $x=0$)

not O.P.

$\therefore$ It is ~~regular~~ singular ~~point~~

i) $x(P(x))=x\left(\frac{1}{2x}\right)=\frac{1}{2}$

ii) $x^2(Q(x))=x^2\left(\frac{1}{4x}\right)=\frac{x}{4}$.

if $x=0$, $x(P(x))=\frac{1}{2}$ $(\neq\infty)$

$x=0$, $x^2(Q(x))=0$ $(\neq\infty)$

$\therefore$ It is regular singular point.

Let us assume,

$y=x^m\sum_{n=0}^{\infty}a_nx^n$ (or) $\sum_{n=0}^{\infty}a_nx^{m+n}$.

$y'=\sum_{n=0}^{\infty}a_nx^{m+n-1}(m+n)$

$y''=\sum_{n=0}^{\infty}a_n(m+n)(m+n-1)x^{m+n-2}$.

$$\Rightarrow 2\sum_{n=0}^{\infty} a_n(m+n)(m+n-1)x^{m+n-1} + \frac{1}{2x}$$

$$\Rightarrow 4x\sum_{n=0}^{\infty} a_n(m+n)(m+n-1)x^{m+n-2} + 2\sum_{n=0}^{\infty} a_n(m+n)x^{m+n-1}$$

$$+ \sum_{n=0}^{\infty} a_n x^{m+n} = 0.$$

$$\Rightarrow 4\sum_{n=0}^{\infty} a_n(m+n)(m+n-1)x^{m+n-1} + 2\sum_{n=0}^{\infty} a_n(m+n)x^{m+n-1}$$

$$+ \sum_{n=0}^{\infty} a_n x^{m+n} = 0.$$

$$\Rightarrow 4\sum_{n=0}^{\infty} a_n(m+n)(m+n-1)x^{n-1} + 2\sum_{n=0}^{\infty} a_n(m+n)x^{n-1}$$

$$+ \sum_{n=0}^{\infty} a_n x^{n} = 0$$

$\cancel{n-1=k}$ $\quad$ $\cancel{n-1=k}$ $\quad$ $\cancel{n=k}$

$$\Rightarrow 4\sum_{\cancel{n=0}}^{\infty}\left[\cancel{4}a_n(m+n)(m+n-1) + 2a_n(m+n)\right]x^{n-1} + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\Rightarrow \sum_{n=0}^{\infty} 2a_n(m+n)\left[2(m+n-1)+1\right]x^{n-1} + \sum_{n=0}^{\infty} a_n x^n = 0.$$

$\cancel{n-1}$ $n-1=k$ $\quad$ $n=k$

$$\Rightarrow \sum_{k+1=0}^{\infty} 2a_{k+1}(k+1+m)\left[2(k+\cancel{1}+n\cancel{-1})+1\right]x^k + \sum_{k=0}^{\infty} a_k x^k = 0$$

$$\Rightarrow \sum_{k=-1}^{\infty} 2a_{k+1}(k+m+1)\left[2(k+m)+1\right]x^{k} + \sum_{k=0}^{\infty} a_k x^k = 0$$

$$\Rightarrow \sum_{k=-1}^{\infty} a_{k+1}(2k+2m+2)\left[2k+2m+1\right]x^{k} + \sum_{k=0}^{\infty} a_k x^k = 0.$$

$$\Rightarrow a_0(2m)\left[2m-1\right]x^{-1} + \sum_{k=0}^{\infty} a_{k+1}(2k+2m+2)\left[2k+2m+1\right]x^{k} + \sum_{k=0}^{\infty} a_k x^k = 0$$

$$\Rightarrow a_0(2m)(2m-1)x^{-1} + \sum_{k=0}^{\infty} x^k\left[a_{k+1}(2k+2m+2)(2k+2m+1) + a_k\right] = 0.$$

$\Rightarrow a_0(2m)(2m-1) = 0 \quad (a_0 \neq 0)$

$\Rightarrow 2m(2m-1) = 0 \Rightarrow 4m^2 - 2m = 0 \Rightarrow \boxed{m = 0, 1/2}$

& $a_{k+1}(2k+2m+2)(2k+2m+1) + a_k = 0.$

$$\Rightarrow \boxed{a_{k+1} = \frac{-a_k}{(2k+2m+2)(2k+2m+1)}} \quad \forall\ k = 0, 1, 2, 3, \ldots$$

for $m = 0.$

$$a_{k+1} = \frac{-a_k}{(2k+2)(2k+1)}$$

for $k = 0$

$$a_1 = \frac{-a_0}{2}$$

$k = 1,$

$$a_2 = \frac{-a_1}{12} = \frac{a_0}{24}.$$

for $m = 1/2$

$$a_{k+1} = \frac{-a_k}{(2k+3)(2k+2)}$$

for $k = 0,$

$$a_1 = \frac{-a_0}{6}$$

$k = 1$

$$a_2 = \frac{-a_1}{20} = \frac{a_0}{120}$$

$k=1$

$$a_3 = \frac{-a_2}{30} = \frac{-a_0}{720}$$

$k=2$.

$$a_3 = \frac{-a_2}{42} = \frac{-a_0}{5040}$$

Case I (for $m=0$).

$$\Rightarrow y = \sum_{n=0}^{\infty} a_n x^{m+n}$$

$$= a_1 x^{m+1}$$

$$= a_0 x^m + a_1 x^{m+1} + a_2 x^{m+2} + \cdots$$

$$\boxed{y = a_0 + \left(-\frac{a_0}{2}\right)x + \frac{a_0}{24}x^2 - \frac{a_0}{720}x^3 + \cdots}$$

Case-II for $m = 1/2$

$$\Rightarrow y = \sum_{n=0}^{\infty} a_n x^{m+n}$$

$$= a_0 x^m + a_1 x^{m+1} + a_2 x^{m+2} + \cdots$$

$$= a_0 x^{1/2} + a_1 x^{3/2} + a_3 x^{5/2} + a_4 x^{7/2} \cdots$$

$$\boxed{y = a_0 x^{1/2} - \frac{a_0}{6} x^{3/2} + \frac{a_0}{120} x^{5/2} - \frac{a_0}{5040} x^{7/2} + \cdots}$$

$$\boxed{y = c_1 y_1 + c_2 y_2}$$

02/08/16

Unit - III

FUNCTION OF A COMPLEX VARIABLE

① We know that.

$z = x + iy = (x, y)$

$x \to$ RP, $y \to$ IP

$Re(z) = x$

$IP(z) = y$.

② $\bar{z} = x - iy$.

$z = x + iy$.

$\boxed{|z| = \sqrt{x^2 + y^2}}$

(or)

$|z| = \sqrt{r^2}$

$\boxed{|z| = r}$

Polar form:

We know that.

$x = r\cos\theta \longrightarrow ①$

$y = r\sin\theta \longrightarrow ②$

$①^2 + ②^2 \Rightarrow \boxed{x^2 + y^2 = r^2}$

$\frac{②}{①} \Rightarrow \frac{y}{x} = \frac{r\sin\theta}{r\cos\theta}$

→ Analytic function:

Let a $funct^n$ $f(z)$ be derivable at every point 'z' in an $\in$ neighbourhood of z_0

(or)

A $funct^n$ $f(z)$ is said to be analytic if it is differentiable in an $\in$ neighbourhood of z_0

→ Cauchy's-Riemann equations (C-R eq^{ns})

Statement: A necessary $condit^n$ for the derivative of the $funct^n$ $f(z) = \omega = u+iv = u(x,y)+iv(x,y)$ to exist $\forall z$ in domain $\{R\}$.

then,

i) $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ are continuous $funct^n$ in x & y

ii) $$\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ \& } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad \text{(or)} \quad u_x = v_y \quad u_y = -v_x}$$

(or)

Derive the neccessary $condit^n$ for $f(z)$ to be analytic in cartesian co-ordinates.

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P1) $f(z) = u + iv$.

$u + iv = (x+iy)^3$

$= x^3 + 3x^2 iy + 3x(iy)^2 + (iy)^3$

$= x^3 + 3x^2 iy - 3xy^2 - iy^3$

$u + iv = x^3 - 3xy^2 + i(3x^2y - y^3)$

$u = x^3 - 3xy^2$	$v = 3x^2y - y^3$
$u_x = 3x^2 - 3y^2$	$v_x = 6xy$
$u_y = -3x$.	$v_y = 3x^2 - 3y^2$

$\therefore u_x = v_y$ & $v_x = -u_y$

$\therefore$ CR eq^{ns} exists.

$\therefore f(z)$ is analytical

P2) $f(z) = z + 2\bar{z}$

$u + iv = x + iy + 2(x - iy)$

$= x + iy + 2x - 2iy$.

$u + iv = 3x - iy$.

$u = 3x$	$v = -y$
$u_x = 3$	$v_x = 0$
$u_y = 0$	$v_y = -1$

$\therefore u_x \neq v_y$ & $u_y \neq -v_x$

$\therefore$ CR eq^{ns} doesn't exist

$\therefore$ It is not analytical

Harmonic functions:

Any function $\phi(x,y)$ which passes continuous, partial derivatives of the I & II orders & satisfies laplace eqn: $\left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0\right)$ is called a Harmonic function.

P.T $\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|(Re(f(z))^2| = 2|f'(z)|^2$ where $f(z)$ is analytic.

Q4) Let $f(z)$ is analytic funct^n.

$\because f(z)$ is analytic.

$\Rightarrow$ It satisifies CR eq^{ns} & harmonic.

$f(z) = u + iv$.

$R.P(f(z)) = u$.

$IP(f(z)) = v$

$R.P(f(z)) = u$

$|RP(f(z))| = \sqrt{u^2}$

$|RP(f(z))|^2 = u^2$.

$IP(f(z)) = v$

$|RP(f(z))| = \sqrt{v^2}$.

$|RP(f(z))|^2 = v^2$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|RP(f(z))|^2 = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)u^2$$

$$= \frac{\partial^2 u^2}{\partial x^2} + \frac{\partial^2 u^2}{\partial y^2} \longrightarrow (1)$$

$$\frac{\partial^2}{\partial x^2}(u^2) = ?$$

$$\frac{\partial^2}{\partial x^2}(u^2) = \frac{\partial}{\partial x}\left(\frac{\partial u^2}{\partial x}\right)$$

$$= \frac{\partial}{\partial x}\left(2u \cdot \frac{\partial u}{\partial x}\right).$$

$$= 2\frac{\partial}{\partial x}\left(u \cdot \frac{\partial u}{\partial x}\right).$$

u v.

$$= 2\left[u\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x}\left(\frac{\partial u}{\partial x}\right)\right].$$

$$\frac{\partial^2}{\partial x^2}(u^2) = 2\left[u\frac{\partial^2 u}{\partial x^2} + \left(\frac{\partial u}{\partial x}\right)^2\right] \rightarrow (A)$$

Similarly $$\frac{\partial^2}{\partial y^2}(u^2) = 2\left[u\frac{\partial^2 u}{\partial y^2} + \left(\frac{\partial u}{\partial y}\right)^2\right] \rightarrow (B)$$

(A) + (B)

$$\left[\frac{\partial^2 (u^2)}{\partial x^2} + \frac{\partial^2}{\partial y^2}(u^2)\right] = 2\left[u\frac{\partial^2 u}{\partial x^2} + \left(\frac{\partial u}{\partial x}\right)^2\right] + 2\left[u\frac{\partial^2 u}{\partial x^2} + \left(\frac{\partial u}{\partial y}\right)^2\right]$$

$$= 2\left[\left(u\frac{\partial^2 u}{\partial x^2} + u\frac{\partial^2 u}{\partial y^2}\right) + \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right]$$

$$= 2\left[u\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right] + \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right]$$

$$= 2\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right] \quad [\because u \text{ is harmonic}]$$

$$= 2\,|f'(z)|^2$$

5) w.k.t

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)u^2 = 2|f'(z)|^2 \rightarrow (1)$$

Similarly $$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)v^2 = 2|f'(z)|^2 \rightarrow (2)$$

(1) + (2)

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$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)(u^2+v^2) = 4\,|f'(z)|^2$$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 4\,|f'(z)|^2.$$

Q. To prove that the functⁿ ~~f(z) = x^3(1+i)~~

$$f(z) = \begin{cases} \dfrac{x^3(1+i) - y^3(1-i)}{x^2+y^2}, & (z \neq 0) \\ 0 & ; z = 0. \end{cases}$$

is continuous & CR eqⁿˢ are satisfied at origin & $f'(0)$ doesnot exists at the origin.

1. $f(z) = \begin{cases} \dfrac{x^3(1+i) - y^3(1-i)}{x^2+y^2} & ; (z \neq 0) \\ 0 & ; z = 0. \end{cases}$

$$f(z) = \frac{x^3 + ix^3 - y^3 + iy^3}{x^2+y^2}$$

$$\cancel{f(z)=}\ u+iv = \frac{x^3 - y^3 + i(x^3+y^3)}{x^2+y^2}.$$

$$\therefore u = \frac{x^3 - y^3}{x^2+y^2}.$$

$$v = \frac{x^3+y^3}{x^2+y^2}$$

i) we have to show that $f(z)$ is continuous.

$$l = \underset{z \to z_0}{Lt} f(z).$$

$$l = \underset{z \to 0}{Lt} f(z).$$

$$\therefore l = \underset{(x+iy)\to(0,0)}{Lt} \frac{(x^3 - y^3) + i(x^3+y^3)}{x^2+y^2}$$

along path 1

$y \to 0, x \to 0.$

$$\text{(A)}\ l = \underset{\substack{y \to 0 \\ x \to 0}}{Lt} \frac{(x^3 - y^3) + i(x^3+y^3)}{x^2+y^2}$$

$$l = \underset{x \to 0}{Lt} \frac{x^3 + ix^3}{x^2}$$

$$l = \underset{x \to 0}{Lt} (x + ix).$$

$$\boxed{l = 0}$$

along path ②

$x \to 0, y \to 0.$

$$ⓐ \Rightarrow l = \underset{\substack{x\to 0 \\ y \to 0}}{Lt} \frac{(x^3-y^3)+i(x^3+y^3)}{x^2+y^2}$$

$$= \underset{y\to 0}{Lt} \frac{-y^3+iy^3}{y^2}$$

$$= \underset{y\to 0}{Lt} \; iy-y.$$

$$=0$$

$$\Rightarrow \boxed{l=0}.$$

along path ③

$y=mx, x\to 0.$

$$ⓐ \Rightarrow l = \underset{\substack{y\to mx \\ x \to 0}}{Lt} \frac{(x^3-y^3)+i(x^3+y^3)}{x^2+y^2}$$

$$= \underset{x\to 0}{Lt} \frac{(x^3-m^3x^3)+i(x^3+m^3x^3)}{x^2+m^2x^2}$$

$$= \underset{x\to 0}{Lt} \frac{x^{\cancel{3}}\left[(1-m^3)+i[1+m^3]\right]}{\cancel{x^2}[1+m^2]}$$

$$= \underset{x\to 0}{Lt} \frac{x\left[(1-m^3)+i(1+m^3)\right]}{1+m^2}$$

$$= 0.$$

$$\Rightarrow \boxed{l=0}.$$

along path 4

$x \to my^2, y \to 0.$

$$(A) \Rightarrow l = \underset{\substack{x \to my^2 \\ y \to 0}}{Lt} \frac{(x^3 - y^3) + i(x^3 + y^3)}{x^2 + y^2}$$

$$= \underset{y \to 0}{Lt} \frac{(m^3y^6 - y^3) + i(m^3y^6 + y^3)}{m^2y^4 + y^2}$$

$$= \underset{y \to 0}{Lt} \frac{y^{\cancel{3}}\left[[m^3y^3 - 1] + i[m^3y^3 + 1]\right]}{\cancel{y^2}[m^2y^2 + 1]}$$

$$= \underset{y \to 0}{Lt} \frac{y\left[[m^3y^3 - 1] + i[m^3y^3 + 1]\right]}{[m^2y^2 + 1]}$$

$$\Rightarrow \boxed{l = 0}.$$

along path 5

$y \to mx^2, x \to 0.$

$$(A) \Rightarrow l = \underset{\substack{y \to mx^2 \\ x \to 0}}{Lt} \frac{(x^3 - y^3) + i(x^3 + y^3)}{x^2 + y^2}$$

$$= \underset{x \to 0}{Lt} \frac{(x^3 - m^3x^6) + i(x^3 + m^3x^6)}{x^2 + m^2x^4}$$

$$= \underset{x \to 0}{Lt} \frac{x^{\cancel{3}}\left[(1 - m^3x^3) + i(1 + m^3x^3)\right]}{\cancel{x^2}[1 + m^2x^2]}$$

$$\Rightarrow \boxed{l = 0}$$

$\therefore f(z)$ is continuous.

ii) To prove CR eq^ns are satisified at origin.

$$u_x = \underset{x \to 0}{Lt} \frac{u(x,0) - u(0,0)}{x}$$

$$= \underset{x \to 0}{Lt} \frac{\frac{x^3}{x^2} - 0}{x}$$

$$\boxed{u_x = 1}$$

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$$u_y = \underset{y\to 0}{Lt} \frac{u(0,y) - u(0,0)}{y}$$

$$u_y = \underset{y\to 0}{Lt} \frac{\frac{-y^3}{y^2} - 0}{y}$$

$$\boxed{u_y = -1}$$

$$v_x = \underset{x\to 0}{Lt} \frac{v(x,0) - v(0,0)}{x}$$

$$v_x = \frac{\frac{x^3}{x^2} - 0}{x}$$

$$\boxed{v_x = 1}$$

$$v_y = \underset{y\to 0}{Lt} \frac{v(0,y) - v(0,0)}{y}$$

$$= \frac{\frac{y^3}{y^2}}{y}$$

$$\boxed{v_y = 1}$$

$\therefore u_x = v_y$ & $u_y = -v_x$

$\therefore$ CR eq^ns are satisified.

15/04/16

iii, To show that $f'(0)$ ~~doesnot~~ exist at the origin.

$$f'(0) = \underset{z\to 0}{Lt} \frac{f(z) - f(0)}{z - 0}$$

$$f'(0) = \underset{z\to 0}{Lt} \frac{f(z) - 0}{z}$$

$$= \underset{x+iy \to (0,0)}{Lt} \frac{\frac{x^3-y^3+i(x^3+y^3)}{x^2+y^2}}{x+iy}$$

$$f'(0) = \underset{x+iy \to (0,0)}{Lt} \frac{x^3-y^3+i(x^3+y^3)}{(x^2+y^2)x+iy} \to (A).$$

case i,

Along path (1)

$y \to 0, x \to 0.$

$$f'(0) = \underset{\substack{y \to 0 \\ x \to 0}}{Lt} \frac{x^3-y^3+i(x^3+y^3)}{(x^2+y^2)(x+iy)}$$

$$= \underset{x \to 0}{Lt} \frac{x^3+ix^3}{x^2(x)}$$

$$= \underset{x \to 0}{Lt} \frac{\cancel{x^3}(1+i)}{\cancel{x^3}}$$

$$\boxed{f'(0) = 1+i}$$

Along path (2)

$x \to 0, y \to 0.$

$$f'(0) = \underset{\substack{x \to 0 \\ y \to 0}}{Lt} \frac{x^3-y^3+i(x^3+y^3)}{(x^2+y^2)(x+iy)}$$

$$= \underset{y \to 0}{Lt} \frac{-y^3+iy^3}{y^2(iy)}$$

$$= \underset{y \to 0}{Lt} \frac{\cancel{y^3}(i-1)}{\cancel{y^3}\, i}$$

$$\boxed{f'(0) = \frac{i-1}{i} \times \cancel{x} = -i(i-1) = 1+i}$$

~~f'(0) doesnot exi~~

Along path ⑧

$y \to mx,\ x \to 0$

$$f'(0) = \underset{\substack{y \to mx \\ x \to 0}}{Lt} \frac{x^3 - y^3 + i(x^3 + y^3)}{(x^2+y^2)(x+iy)}$$

$$= \underset{x \to 0}{Lt} \frac{x^3 - m^3x^3 + i(x^3 + m^3x^3)}{(x^2+m^2x^2)(x+imx)}$$

$$= \underset{x \to 0}{Lt} \frac{\cancel{x^3}\left([1-m^3] + i[1+m^3]\right)}{\cancel{x^3}(1+m^2)(1+mi)}$$

$$f'(0) = \frac{(1-m^3) + i(1+m^3)}{(1+m^2)(1+im)} \neq 1+i$$

$\therefore f'(0)$ doesnot exists

$\Rightarrow f(z)$ is not analytic.

P1) Find the analytic funct? whose real part is

$u = x^2 - y^2$

$\frac{\partial u}{\partial x} = 2x.$

$\frac{\partial u}{\partial y} = -2y.$

put $x = z, y = 0$

$\frac{\partial u}{\partial x} = \phi(z,0) = 2z.$

$\frac{\partial u}{\partial y} = p(z,0) = 0.$

$\therefore f(z) = \int [2z - 0]\, dz.$

$\boxed{f(z) = z^2}$

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ST, $v = \sin x \cosh y + 2\cos x \sinh y + x^2 - y^2 + 4xy$ is harmonic & $f(z) = ?$

Given

$u = x^3 - 3xy^2$

$$\frac{\partial u}{\partial x} = 3x^2 - 3y^2$$

$$\frac{\partial^2 u}{\partial x^2} = 6x$$

$$\frac{\partial u}{\partial y} = -6xy.$$

$$\frac{\partial^2 u}{\partial y^2} = -6x.$$

$$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 6x - 6x = 0$$

Given $u = x^3 - 3xy^2$.

$\frac{\partial u}{\partial x} = 3x^2 - 3y^2 = \phi(x,y)$.

$\frac{\partial u}{\partial y} = 0 - 6xy = \psi(x,y)$.

$x = z,\ y = 0$.

$\phi(z,0) = 3z^2$

$\psi(z,0) = 0$.

$\therefore\ f(z) = \int \phi(z,0) - i\,\psi(z,0)\,dz$.

$= \int 3z^2\,dz$.

$\boxed{f(z) = z^3.}$

take $f(z) = z^3$.

$\cancel{f(z)}\ u + iv = (x+iy)^3$.

$= x^3 + 3x^2 iy + 3x(iy)^2 + (iy)^3$

$= x^3 + 3x^2 iy - 3xy - iy^3$

$u + iv = (x^3 - 3xy) + i(3x^2y - y^3)$

$u = x^3 - 3xy,\ \boxed{v = 3x^2y - y^3}$

11) ST $u = e^{2x}(x\cos 2y - y\sin 2y)$. is harmonic & find v.

$$u = \underset{u}{e^{2x}}\,\underset{v}{x\cos 2y} - \underset{u}{e^{2x}}\,\underset{v}{y\sin 2y}$$

$$\frac{\partial u}{\partial x} = \left(e^{2x}(\cos 2y) + x\cos 2y\,(2e^{2x})\right) - 2e^{2x} y\sin 2y$$

$$\frac{\partial u}{\partial x} = e^{2x}\cos 2y + 2e^{2x}x\cos 2y - 2e^{2x}y\sin 2y = \phi(x,y)$$

$$\frac{\partial^2 u}{\partial x^2} = 2e^{2x}\cos 2y + 2\left(e^{2x}(\cos 2y) + 2e^{2x}(x\cos 2y)\right) - 4e^{2x}y\sin 2y$$

$$= 2e^{2x}\cos 2y + 4xe^{2x}\cos 2y + 2e^{2x}\cos 2y - 4e^{2x}y\sin 2y$$

$$\frac{\partial^2 u}{\partial x^2} = 4e^{2x}\cos 2y + 4xe^{2x}\cos 2y - 4e^{2x}y\sin 2y$$

$$\frac{\partial u}{\partial y} = -e^{2x}x\,2\sin 2y + x\cos 2y\,(2e^{2x}) - \left(e^{2x}(\sin 2y) + ye^{2x}(2\cos 2y)\right)$$

$$= -e^{2x}x\,2\sin 2y + 2e^{2x}x\cos 2y - e^{2x}\sin 2y - 2ye^{2x}\cos 2y$$

$$= \psi(x,y)$$

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$$\frac{\partial^2 u}{\partial y^2} = -4xe^{2x}\cos 2y + 4ye^{2x}\sin 2y - \cancel{2e\cos 2y}\ 2e^{2x}\cos 2y - e^{2x} 2\cos 2y$$

$$\frac{\partial^2 u}{\partial y^2} = -4xe^{2x}\cos 2y + 4ye^{2x}\sin 2y - 4e^{2x}\cos 2y$$

$$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$\therefore u$ is harmonic.

Sol^n $f(z) = \int \phi(z,0) + i\psi(z,0)\, dz$

$= \int$

$\phi(x,y) = e^{2x}\cos 2y + 2e^{2x}x\cos 2y - 2e^{2x}y\sin 2y$

$\phi(z,0) = e^{2z}(1) + 2e^{2z}z(1) - 0$

$\phi(z,0) = e^{2z} + ze^{2z}$.

$\psi(x,y) = -e^{2x}x2\sin 2y + 2e^{2x}x\cos 2y - e^{2x}\sin 2y - 2ye^{2x}\cos$

$\psi(z,0) = 0 + \cancel{2e^{2z}z}\,0$

$\psi(z,0) = \cancel{2ze^{2z}}\,0$

$\psi(z,0) = 0$

$$f(z) = \int i0 + 1\left(e^{2z} + ze^{2z}\right)$$

$$= i\left[\frac{e^{2z}}{2} + 2z\left(\frac{e^{2z}}{2}\right) - z\left(\frac{e^{2z}}{2z}\right)\right]$$

$$= i\left[\cancel{\frac{e^{2z}}{2}} + ze^{2z} - \cancel{\frac{e^{2z}}{2}}\right]$$

$$f(z) = ze^{2z}$$

$$u+iv = (x+iy)e^{2(x+iy)}.$$

$$= (x+iy)\left(e^{2x}\cdot e^{2iy}\right)$$

$$= (x+iy)\, e^{2x}[\cos 2y + i\sin 2y].$$

$$= x+iy\left(e^{2x}\cos 2y + ie^{2x}\sin 2y\right)$$

$$= xe^{2x}\cos 2y + ixe^{2x}\sin 2y + ie^{2x}y\cos 2y - e^{2x}y\sin 2y.$$

$$= e^{2x}(x\cos 2y - y\sin 2y) + i\left(xe^{2x}\sin 2y + ye^{2x}\cos 2y\right)$$

$$\therefore v = e^{2x}(x\sin 2y + y\cos 2y).$$

07/08

$u = e^{x^2-y^2} \cos(2xy)$, $v = ?$.

$\frac{\partial u}{\partial x} = -e^{x^2-y^2} \sin(2xy)(2y) + \cos(2xy)\, e^{x^2-y^2}(2x) = \phi(x,y)$

$\phi(z,0) = -e^{z^2}(0) + \cos(0)\, e^{z^2}(2z)$.

$\phi(z,0) = 2z e^{z^2}$

$\frac{\partial u}{\partial y} = -e^{x^2-y^2} \sin(2xy)(2x) + \cos(2xy)\, e^{x^2-y^2}(-2y) = \psi(x,y)$

$\psi(z,0) = -e^{z^2}(0) + \cos(0)\, e^{z^2}(0)$.

$= 0$

$\psi(z,0) = 0$

$f(z) = \int \phi(z,0) + i\,\psi(z,0)\, dz$.

$= \int 2z e^{z^2} = 2\int z e^{z^2}$

e^{z^2} ... -1

$$= 2\int z e^{z^2} dz$$

Let $z^2 = t$

$2z\,dz = dt$

$$= \int dt\, e^t$$

$$= \cancel{t^2}$$

$$= e^t + c.$$

$$f(z) = e^{z^2} + c.$$

$$f(z) = e^{z^2}$$

$$u + iv = e^{(x+iy)^2}$$

$$= e^{x^2 - y^2 + 2xyi}$$

$$= e^{x^2-y^2} + e^{i2xy}.$$

$$= e^{x^2-y^2} + \cos 2xy + i \sin 2xy$$

$$u + iv = e^{x^2-y^2}\cos 2xy + i e^{x^2-y^2}\sin 2xy$$

$$\boxed{v = e^{x^2-y^2}\sin 2xy}$$

3

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ii) $u+v = \dfrac{\sin 2x}{\cosh 2y - \cos 2x}$, find $f(z)$

$$\frac{\partial Q}{\partial x} = \frac{(\cosh 2y - \cos 2x)(2\cos 2x) - (\sin 2x)(2\sin 2x)}{(\cosh 2y - \cos 2x)^2}$$

$$= \phi(x,y)$$

$$\phi(z,0) = \frac{(1-\cos 2z)(2\cos 2z) - (\sin 2z)(2\sin 2z)}{(1-\cos 2z)^2}$$

$$= \frac{2\cos 2z - 2\cos^2 2z - 2\sin^2 2z}{(1-\cos 2z)^2}$$

$$= \frac{2\cos 2z - 2}{(1-\cos 2z)^2}$$

$$= \frac{2(\cancel{\cos 2z - 1})}{(\cos 2z - 1)^{\cancel{2}}}$$

$$= \frac{2}{\cos 2z - 1}$$

$$= -2\left[\frac{1}{1-\cos 2z}\right]$$

$$= -2\left[\frac{1}{2\sin^2 z}\right]$$

$$= \cancel{2} - \operatorname{cosec}^2 z$$

$$\phi(z,0) = -\operatorname{cosec}^2 z$$

$$\frac{\partial Q}{\partial y} = \frac{(\cosh 2y - \cos 2x)(0) - \sin 2x\,(2\sinh 2y)}{(\cosh 2y - \cos 2x)}$$

$$= \frac{-2\sin 2x \sinh 2y}{\cosh 2y - \cos 2x} = \psi(x,y)$$

$$\psi(z,0) = 0$$

$$g(z) = \int \psi(z,0) + i\phi(z,0)\, dz.$$

$$= \int 0 + i(-\operatorname{cosec}^2 z).$$

$$= i\{[\cot z] + c\}.$$

$$f(z)(1+i) = i\cot z + c.$$

$$f(z) = \frac{i\cot z}{1+i} + \frac{c}{1+i}$$

$$f(z) = \frac{i(1-i)\cot z}{2} + c_1.$$

$$\boxed{f(z) = \frac{(i+1)\cot z}{2} + c_1}$$

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Find 'k' such that $f(x,y) = x^3 + 3kxy^2$ may be harmonic & find conjugate

v) $f(x,y) = x^3 + 3kxy^2$

$$\frac{\partial f}{\partial x} = 3x^2 + 3ky^2 = 3x^2 - 3y^2 = \phi(x,y).$$

$$\frac{\partial^2 f}{\partial x^2} = 6x$$

$$\frac{\partial f}{\partial y} = \cancel{3} \; 6kxy = -6xy = \psi(x,y)$$

$$\frac{\partial^2 f}{\partial y^2} = 6kx.$$

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 6x + 6kx = 0.$$

$$6x(1+k) = 0.$$

$$\boxed{k = -1}.$$

$\phi(z,0) = 3z^2$

$\psi(z,0) = 0.$

$f(z) = \int 3z^2 + 0.$

$f(z) = z^3$

$\cancel{f(z)} = (x+iy)^3$

$= x^3 + 3x^2 iy + 3x(iy)^2 + (iy)^3$

$= x^3 + 3x^2 iy - 3xy^2 - iy^3$

$f(z) = x^3 - 3xy^2 + i(3x^2y - y^3).$

$\overline{f(z)} = x^3 - 3xy^2 - i(3x^2y - y^3)$

25/07/16

UNIT-II

SPECIAL FUNCTIONS

→ **Bessel's function:**

It is denoted by $J_n(x)$ i.e.,

$$J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(n+k+1)} \left(\frac{x}{2}\right)^{n+2k}$$

is called 1st kind of Bessel funct^n.

and
$$J_{-n}(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(-n+k+1)} \left(\frac{x}{2}\right)^{-n+2k}$$

is called 2nd kind of Bessel funct^n.

Note: $\Gamma(k+1) = k!$ (or) $k\Gamma k$.

Ex: $\Gamma 5 = 4!$

$\Gamma 3 = 2!$

$\Gamma 1 = 0!$

2) $\Gamma \tfrac{1}{2} = \sqrt{\pi}$

$\Gamma -\tfrac{1}{2} = -2\sqrt{\pi}$

$\Gamma \tfrac{7}{2} = \tfrac{5}{2} \cdot \tfrac{3}{2} \cdot \tfrac{1}{2}\, \Gamma \tfrac{1}{2}$

$\Gamma \tfrac{11}{2} = \tfrac{9}{2} \cdot \tfrac{7}{2} \cdot \tfrac{5}{2} \cdot \tfrac{3}{2} \cdot \tfrac{1}{2}\, \Gamma \tfrac{1}{2}$

$\Gamma \tfrac{3}{2} = \tfrac{1}{2}\, \Gamma \tfrac{1}{2}$

(61)

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

→ properties:

- $J_0(0) = 1$
- $J_1(0) = 0.$
- $J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \cdot \sin x.$
- $J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \cos x.$

Proofs:

- $J_0(0) = 1.$

W.K.T.

$$J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(n+k+1)} \left(\frac{x}{2}\right)^{n+2k}.$$

$$J_0(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(k+1)} \left(\frac{x}{2}\right)^{2k}$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{(k!)^2} \left(\frac{x}{2}\right)^{2k}.$$

$$= \frac{(-1)^0}{(0!)^2}\left(\frac{x}{2}\right)^0 + \frac{(-1)^1}{(1!)^2}\left(\frac{x}{2}\right)^{2(2)} + \frac{(-1)^2}{(2!)^2}\left(\frac{x}{2}\right)^{2(3)} + \cdots$$

$$J_0(x) = 1 + \left(-\frac{x}{2}\right)^4 + \frac{1}{4}\left(\frac{x}{2}\right)^6 + \cdots$$

$$\boxed{J_0(0) = 1}$$

• $J_1(0) = 0$.

W.K.T.

$$J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma n+k+1} \left(\frac{x}{2}\right)^{n+2k}.$$

$$J_1(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma k+1+1} \left(\frac{x}{2}\right)^{1+2k}.$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma 2+k} \left(\frac{x}{2}\right)^{1+2k}$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^n}{k!\,(k+1)!} \left(\frac{x}{2}\right)^{1+2k}.$$

$$= \frac{(-1)^0}{0!\,(0+1)!} \left(\frac{x}{2}\right)^{1+0} + \frac{(-1)^1}{k!\,(k+1)!} \left(\frac{x}{2}\right)^{1+2(1)} + \cdots$$

$$J_1(x) = 1\left(\frac{x}{2}\right) + \left(\frac{-1}{k!\,(k+1)!}\left(\frac{x}{2}\right)^3\right) + \cdots$$

$$\boxed{J_1(0) = 0}$$

• $J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x.$

W.K.T.

$$J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma n+k+1} \left(\frac{x}{2}\right)^{n+2k}.$$

$$J_{1/2}(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma 1/2+k+1} \left(\frac{x}{2}\right)^{1/2+2k}$$

$$= \frac{(-1)^0}{0!\,\Gamma 1/2+1} \left(\frac{x}{2}\right)^{1/2+0} + \frac{(-1)^1}{1\,\Gamma 1/2+2} \left(\frac{x}{2}\right)^{1/2+2}$$

$$+ \frac{(-1)^2}{2\,\Gamma 1/2+2+1} \left(\frac{x}{2}\right)^{1/2+4}$$

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$$= \frac{1}{\Gamma 3/2}\left(\frac{x}{2}\right)^{1/2} - \frac{1}{\Gamma 3/2}\left(\frac{x}{2}\right)^{5/2} + \frac{1}{2\Gamma 7/2}\left(\frac{x}{2}\right)^{7/2} + \cdots$$

$$= \left(\frac{x}{2}\right)^{1/2}\left[\frac{1}{\Gamma 3/2} - \frac{1}{\Gamma 5/2}\left(\frac{x}{2}\right)^2 + \frac{1}{2\Gamma 7/2}\left(\frac{x}{2}\right)^4 + \cdots\right]$$

$$= \left(\frac{x}{2}\right)^{1/2}\left[\frac{1}{\frac{1}{2}\Gamma 1/2} - \frac{1}{\frac{3}{2}\frac{1}{2}\Gamma\frac{1}{2}}\frac{x^2}{2^2} + \frac{1}{2\frac{5}{2}\frac{3}{2}\frac{1}{2}\Gamma\frac{1}{2}}\frac{x^4}{(2)^4} + \cdots\right]$$

$$= \left(\frac{x}{2}\right)^{1/2}\frac{1}{\frac{1}{2}\Gamma 1/2}\left[1 - \frac{\cancel{2}}{3}\frac{x^2}{2^{\cancel{2}}} + \frac{\cancel{2}}{15}\cdot\frac{x^4}{2^{\cancel{3}}} + \cdots\right]$$

$$= \left(\frac{x}{2}\right)^{1/2}\frac{1}{\frac{1}{2}\Gamma 1/2}\left[1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \cdots\right].$$

$$= \left(\frac{x}{2}\right)^{1/2}\frac{2}{\sqrt{\pi}}\left[1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \cdots\right].$$

$$= \frac{\sqrt{x}\cdot\sqrt{2}}{x\sqrt{\pi}}\left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right].$$

$$\boxed{J_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\sin x.}$$

- $J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}}\cos x$

W.K.T.

$$J_n(x) = \sum_{k=0}^{\infty}\frac{(-1)^k}{k!\,\Gamma n+k+1}\left(\frac{x}{2}\right)^{n+2k}.$$

$$J_{-1/2}(x) = \sum_{k=0}^{\infty}\frac{(-1)^k}{k!\,\Gamma k+1/2}\left(\frac{x}{2}\right)^{-1/2+2k}.$$

$$= \cancel{(-1)^0}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!\,\Gamma k+1/2}\left(\frac{x}{2}\right)^{-1/2}\cdot\left(\frac{x}{2}\right)^{2k},$$

$$= \left(\frac{x}{2}\right)^{-1/2}\left[\frac{(-1)^0}{0!\,\Gamma 1/2}\left(\frac{x}{2}\right)^0 + \frac{(-1)^1}{1\,\Gamma 1+1/2}\left(\frac{x}{2}\right)^2 + \frac{(-1)^2}{2\Gamma 2+1/2}\left(\frac{x}{2}\right)^4 \cdots\right.$$

$$= \left(\frac{x}{2}\right)^{-1/2}\left[\frac{1}{\sqrt{\pi}} - \frac{1}{\Gamma 3/2}\frac{x^2}{2^2} + \frac{1}{2\Gamma 5/2}\frac{x^4}{2^4} + \dots\right]$$

$$= \left(\frac{x}{2}\right)^{-1/2}\left[\frac{1}{\sqrt{\pi}} - \frac{1}{\frac{1}{2}\sqrt{\pi}}\cdot\frac{x^2}{2^2} + \frac{1}{\cancel{2}\cdot\frac{3}{\cancel{2}}\cdot\frac{1}{2}\sqrt{\pi}}\frac{x^4}{2^4} + \dots\right]$$

$$= \left(\frac{x}{2}\right)^{-1/2}\frac{1}{\sqrt{\pi}}\left[1 - \frac{\cancel{2}}{\cancel{1}}\frac{x^2}{2^{\cancel{2}}} + \frac{\cancel{2}}{.3}\frac{x^4}{2^{\cancel{4}}3} + \dots\right]$$

$$= \frac{\sqrt{2}}{\sqrt{x}}\frac{1}{\sqrt{\pi}}\left[1 - \frac{x^2}{2!} + \frac{x^4}{4!} \dots\right]$$

$$\boxed{J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}}\ \cancel{\cos x}\ \cancel{\sin x}}$$

$\rightarrow$ S.T $\left(J_{1/2}(x)\right)^2 + \left(J_{1/2}(x)\right)^2 = \frac{2}{\pi/x}$

$$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\sin x$$

s.o.b.s

$$\left(J_{1/2}(x)\right)^2 = \frac{2}{\pi x}\sin^2 x \rightarrow (1)$$

$$J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}}\cos x$$

s.o.b.s

$$\left(J_{-1/2}(x)\right)^2 = \frac{2}{\pi x}\cos^2 x \rightarrow (2)$$

(1) + (2)

$$\left(J_{1/2}(x)\right)^2 + \left(J_{-1/2}(x)\right)^2 = \frac{2}{\pi x}$$

26/07/16

Recurrence relat^{ns} of Bessel funct^{ns}:

1. $$\frac{d}{dx}\left[x^n J_n(x)\right] = x^n J_{n-1}(x).$$

W.K.T.

$$J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(n+k+1)} \left(\frac{x}{2}\right)^{n+2k}.$$

$$x^n J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(n+k+1)} \frac{x^{2n+2k}}{2^{n+2k}}.$$

$$\frac{d}{dx}\left[x^n J_n(x)\right] = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(n+k+1)} (2n+2k)\frac{(x^{2n+2k-1})}{2^{n+2k}}$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\cancel{(n+k)}\,\Gamma(n+k)} 2\cancel{(n+k)} \cdot \frac{x^{2n+2k-1}}{2^{n+2k}}$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(n+k)} \frac{x^{2n+2k-1}}{2^{n+2k-1}}.$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(n+k)} x^n \left(\frac{x}{2}\right)^{n+2k-1}.$$

$$= x^n \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma((n-1)+k+1)} \left(\frac{x}{2}\right)^{(n-1)+2k}$$

$$= x^n J_{n-1}(x).$$

$$\boxed{\therefore \frac{d}{dx}\left[x^n J_n(x)\right] = x^n J_{n-1}(x).}$$

(2) $\frac{d}{dx}\left[x^{-n} J_n(x)\right] = -x^{-n} J_{n+1}(x)$.

WKT

$$J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma n+k+1} \left(\frac{x}{2}\right)^{n+2k}$$

$$x^{-n} J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma n+k+1} \left(\frac{x}{\cancel{2}}\right)^{n-n+2k} \cdot \frac{1}{2^{n+2k}}$$

$$x^{-n} J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma n+k+1} \left(\frac{x}{\cancel{2}}\right)^{2k} \cdot \frac{1}{2^{n+2k}}$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma n+k+1} \frac{x^{2k}}{2^{n+2k}}.$$

$$\sum_{k=0}^{\infty} = \frac{(-1)^0}{0!\,\Gamma n+k+1} \frac{x^0}{2^n} + \sum_{k=1}^{\infty} \frac{(-1)^k}{k!\,\Gamma n+k+1} \frac{x^{2k}}{2^{n+2k}}.$$

$$\frac{d}{dx}\left[x^{-n} J_n(x)\right] = 0 + \sum_{k=1}^{\infty} \frac{(-1)^k}{k!\,\Gamma n+k+1} 2k \frac{x^{2k-1}}{2^{n+2k}}.$$

$$= \sum_{k=1}^{\infty} \frac{(-1)^k}{k!\,\Gamma n+k+1} (-x^{-n})(-x^{n})\, 2k \frac{x^{2k-1}}{2^{n+2k}}$$

$$= -x^{-n} \sum_{k=1}^{\infty} \frac{(-1)^k}{\cancel{k}(k-1)!\,\Gamma n+k+1} (-x^{n}) \cancel{k} \frac{x^{2k-1}}{2^{n+2k-1}}.$$

$$= -x^{-n} \sum_{k=1}^{\infty} \frac{(-1)^k}{(k-1)!\,\Gamma n+k+1} (-1)(x^{n}) \frac{x^{2k-1}}{2^{n+2k-1}}.$$

Note: $(-1)^k(-1) = (-1)^{k-1}$

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$$= -x^{-n} \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{(k-1)!\,\Gamma(n+k+1)} \frac{x^n \cdot x^{2k-1}}{2^{n+2k-1}}$$

$$= -x^{-n} \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{(k-1)!\,\Gamma(n+k+1)} \frac{x^{2k+n-1}}{2^{n+2k-1}}$$

$$= -x^{-n} \sum_{k=1}^{\infty} \quad (4)$$

Let $k-1 = m$.

$k = m+1$.

$$= -x^{-n} \sum_{m=0}^{\infty} \frac{(-1)^{m}}{m!\,\Gamma(n+m+1+1)} \frac{x^{n+2(m+1)-1}}{2^{n+2(m+1)-1}}$$

$$= -x^{-n} \sum_{m=0}^{\infty} \frac{(-1)^{m}}{m!\,\Gamma(n+m+2)} \frac{x^{n+2m+1}}{2^{n+2m+1}}$$

$$= -x^{-n} \sum_{m=0}^{\infty} \frac{(-1)^{m}}{m!\,\Gamma((n+1)+m+1)} \left(\frac{x}{2}\right)^{(n+1)+2m}.$$

$$\boxed{\frac{d}{dx}\left[x^{-n} J_n(x)\right] = -x^{-n} J_{n+1}(x).}$$

3) $\frac{d}{dx}\left[x^n J_n(x)\right] = x^n J_{n-1}(x)$.

$x^n \cdot J_n'(x) + J_n(x) \cdot n \cdot x^{n-1} = x^n J_{n-1}(x)$.

$x^n J_n'(x) + J_n(x) \cdot n \cdot x^n \cdot x^{-1} = x^n J_{n-1}(x)$

$\cancel{x^n}\left[J_n'(x) + J_n(x) \cdot \frac{n}{x}\right] = \cancel{x^n} J_{n-1}(x)$

$$\boxed{J_n'(x) + \frac{n}{x} J_n(x) = J_{n-1}(x)} \rightarrow (3)$$

(4) $\frac{d}{dx}\left[x^{-n}J_n(x)\right] = -x^{-n}J_{n+1}(x)$.

$$x^{-n}\cdot J_n'(x) + J_n(x)(-n)x^{-n-1} = -x^{-n}J_{n+1}(x).$$

$$\cancel{x^{-n}}\left[J_n'(x) - J_n(x)\frac{n}{x}\right] = -\cancel{x^{-n}}J_{n+1}(x)$$

$$\boxed{J_n'(x) - \frac{n}{x}J_n(x) = -J_{n+1}(x)} \rightarrow (4)$$

(5). (3) + (4).

$$J_n'(x) + \cancel{\frac{n}{x}J_n(x)} - \cancel{\frac{n}{x}J_n(x)} + J_n'(x) = J_{n-1}(x) - J_{n+1}(x)$$

$$\boxed{2J_n'(x) = J_{n-1}(x) - J_{n+1}(x)} \rightarrow (5)$$

(6) (3) - (4).

$$\cancel{J_n'(x)} + \frac{n}{x}J_n(x) + \frac{n}{x}J_n(x) - \cancel{J_n'(x)} = J_{n-1}(x) + J_{n+1}(x)$$

$$\boxed{\frac{2n}{x}J_n(x) = J_{n-1}(x) + J_{n+1}(x)} \rightarrow (6).$$

P1) Evaluate $J_{-3/2}(x)$, $J_{-5/2}(x)$, $J_{3/2}(x)$, $J_{5/2}(x)$.

28/07/16

W.K.T

$$\frac{2n}{x}J_n(x) = J_{n-1}(x) + J_{n+1}(x).$$

i.e., $J_{n-1}(x) = \frac{2n}{x}J_n(x) - J_{n+1}(x) \rightarrow (1)$.

$\frac{5}{2}-1$

$-\frac{1}{2}-1$

$-\frac{3}{2}$

$n-1=\frac{3}{2}$

$n=\frac{3}{2}+1$

$n=\frac{5}{2}$

Sub, $n = -\frac{1}{2}$.

$$\Rightarrow J_{-\frac{1}{2}-1}(x) = \frac{2\left(-\frac{1}{2}\right)}{x}J_{-1/2}(x) - J_{-1/2+1}(x)$$

$$\Rightarrow J_{-3/2}(x) = -\frac{1}{x} J_{-1/2}(x) - J_{1/2}(x).$$

$$= -\frac{1}{x}\sqrt{\frac{2}{\pi x}}\cos x - \sqrt{\frac{2}{\pi x}}\sin x$$

$$\boxed{J_{-3/2}(x) = -\sqrt{\frac{2}{\pi x}}\left[\frac{\cos x}{x} + \sin x\right]}$$

sub $n = -3/2$ in (1)

$$J_{-3/2-1}(x) = \frac{2\left(-\frac{3}{2}\right)}{x} J_{-3/2}(x) - J_{-3/2+1}(x)$$

$$J_{-5/2}(x) = \frac{3}{x}\sqrt{\frac{2}{\pi x}}\left[\frac{\cos x}{x} + \sin x\right]$$

$$- J_{-1/2}(x).$$

$$J_{-5/2}(x) = \frac{3}{x}\sqrt{\frac{2}{\pi x}}\left[\frac{\cos x}{x} + \sin x\right] - \sqrt{\frac{2}{\pi x}}\cos x$$

$$\boxed{J_{-5/2}(x) = \sqrt{\frac{2}{\pi x}}\left[\frac{3}{x}\left(\frac{\cos x}{x} + \sin x\right) - \cos x\right]}$$

~~Sub. n=~~

Now, (1) $\Rightarrow$

$$\boxed{J_{n+1}(x) = \frac{2n}{x} J_n(x) - J_{n-1}(x).} \rightarrow (2)$$

$n = 1/2 \rightarrow J_{3/2}(x)$

$n = 3/2 \rightarrow J_{5/2}(x)$

$$J_{3/2}(x) = \frac{1}{x} J_{1/2}(x) - J_{-1/2}(x).$$

$$J_{3/2}(x) = \frac{1}{x}\sqrt{\frac{2}{\pi x}}\sin x - \sqrt{\frac{2}{\pi x}}\cos x$$

(1) Express $J_4(x)$ in terms of $J_0(x)$ & $J_1(x)$.

W.K.T.

$$\frac{2n}{x} J_n(x) = J_{n-1}(x) + J_{n+1}(x).$$

$$\boxed{J_{n+1}(x) = \frac{2n}{x} J_n(x) - J_{n-1}(x)} \rightarrow (1)$$

put $n=1$.

$$(1) \Rightarrow \boxed{J_2(x) = \frac{2}{x}J_1(x) - J_0(x).}$$

$(1) \Rightarrow n=2$.

$$J_3(x) = \frac{4}{x} J_2(x) - J_1(x).$$

$$\boxed{J_3(x) = \frac{4}{x}\left[\frac{2}{x}J_1(x) - J_0(x)\right] - J_1(x)}$$

$n=3$

$(1) \Rightarrow$

$$J_4(x) = \frac{6}{x} J_3(x) - J_2(x).$$

$$= \frac{6}{x}\left[\frac{4}{x}\left[\frac{2}{x}J_1(x) - J_0(x)\right] - J_1(x)\right] - \left[\frac{2}{x}J_1(x) - J_0(x)\right]$$

$$J_4(x) = \frac{48}{x^3}J_1(x) - \frac{24}{x^2}J_0(x) - \frac{6}{x}J_1(x) - \frac{2}{x}J_1(x) + J_0(x)$$

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$$\boxed{J_4(x) = J_1(x)\left[\frac{48}{x^3} - \frac{8}{x}\right] + J_0(x)\left[1 - \frac{24}{x^2}\right]}$$

P.T $\frac{d}{dx}\left(J_0(x)\right) = -J_1(x)$. (or) $J_0'(x) = -J_1(x)$.

2) $\frac{d}{dx}\left(x J_1(x)\right) = x \cdot J_0(x)$.

1) $\frac{d}{dx}\left(J_0(x)\right) = -J_1(x)$

w.k.t.

$$\frac{d}{dx}\left[x^{-n} J_n(x)\right] = -x^{-n} J_{n+1}(x).$$

put $n=0$.

$$\frac{d}{dx}\left[x^0 J_0(x)\right] = -x^0 J_1(x).$$

$$\Rightarrow \boxed{\frac{d}{dx}\left[J_0(x)\right] = -J_1(x).}$$

2) w.k.t.

$$\frac{d}{dx}\left[x^n J_n(x)\right] = x^n J_{n-1}(x).$$

put $n=1$.

$$\boxed{\frac{d}{dx}\left[x J_1(x)\right] = x \cdot J_0(x)}$$

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① P.T $\frac{d}{dx}\left[J_n^2 + J_{n+1}^2\right] = \frac{2}{x}\left[nJ_n^2 - (n+1)J_{n+1}^2\right]$

② $\frac{d}{dx}\left[xJ_n(x)J_{n+1}(x)\right] = x\left[J_n^2(x) - J_{n+1}^2(x)\right]$

③ P.T ~~$2\left[J_0^2 + 2J_1^2 + J_2^2 + J\right.$~~ $J_0^2 + 2\left[J_1^2 + J_2^2 + J_3^2 + \cdots\right] = 1$

④ P.T. $J_{-n}(x) = (-1)^n J_n(x)$ where n is a +ve integer.

① Sol:

$$\frac{d}{dx}\left[J_n^2 + J_{n+1}^2\right] = 2J_n' J_n + 2J_{n+1} J_{n+1}' \rightarrow ①$$

W.K.T.

$$\boxed{J_n' = \frac{n}{x}J_n - J_{n+1}} \rightarrow ②$$

W.K.T

$$J_n' = -\frac{n}{x}J_n + J_{n-1}.$$

put $n = n+1$.

$$\boxed{J_{n+1}' = \frac{-(n+1)}{x}J_{n+1} + J_n.} \rightarrow ③$$

sub, ②, ③ in ①

$$\frac{d}{dx}\left[J_n^2 + J_{n+1}^2\right] = 2J_n\left[\frac{n}{x}J_n - J_{n+1}\right] + 2J_{n+1}\left[\frac{-(n+1)}{x}J_{n+1} + J_n\right]$$

$$= \frac{2n}{x}J_n^2 - \cancel{2J_n J_{n+1}} \mp \frac{2J_{n+1}^2(n+1)}{x} + \cancel{2J_{n+1}J_n}$$

$$\boxed{\frac{d}{dx}\left[J_n^2 + J_{n+1}^2\right] = \frac{2}{x}\left[nJ_n^2 - (n+1)J_{n+1}^2\right]}$$

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1)

Sol:-

$\frac{d}{dx}\left[\underset{u}{x}\,\underbrace{J_n(x)\,J_{n+1}(x)}_{v}\right]$

$= x\left[J_n(x)\,J'_{n+1}(x) + J_{n+1}(x)\,J'_n(x)\right] + J_n(x)\,J_{n+1}(x)\,(1)$

$\to$ (1)

W.K.T,

$= xJ_n(x)J'_{n+1}(x) + xJ_{n+1}(x)J'_n(x) + J_n(x)J_{n+1}(x).$

$= x\left[J'_{n+1}(x)\right]J_n(x) + x\left[J'_n(x)\right]J_{n+1}(x) + J_n(x)J_{n+1}(x)$

$\to$ (1)

W.K.T.

$xJ'_n(x) = nJ_n(x) - xJ_{n+1}(x) \to$ (2).

W.K.T.

$xJ'_n(x) = -nJ_n(x) + xJ_{n-1}(x)$

sub $n = n+1$.

$xJ'_{n+1}(x) = -(n+1)\,J_{n+1}(x) + xJ_n(x) \to$ (3).

sub,

(2),(3) in (1)

$\Rightarrow \frac{d}{dx}\left[xJ_n(x)J_{n+1}(x)\right] = \left((-(n+1))J_{n+1}(x) + xJ_n(x)\right)J_n(x).$

$+\left(nJ_n(x) - xJ_{n+1}(x)\right)J_{n+1}(x) + J_n(x)J_{n+1}(x)$

$= -(n+1)J_{n+1}J_n(x) + xJ_n^2(x).$

$+ nJ_n(x)J_{n+1}(x) - xJ_{n+1}^2(x) + J_n(x)J_{n+1}(x)$

$= J_n(x)J_{n+1}(x)\left(\cancel{-n}\cancel{-1}\cancel{+n}\cancel{+1}\right) + x\left[J_n^2 - J_{n+1}^2\right]$

3) W.K.T,

$$\frac{d}{dx}\left[J_n^2 + J_{n+1}^2\right] = \frac{2}{x}\left[nJ_n^2 - (n+1)J_{n+1}^2\right] \rightarrow (1)$$

put $n=0$

$$(1) \Rightarrow \frac{d}{dx}\left[J_0^2 + J_1^2\right] = \frac{2}{x}\left[0 - J_1^2\right].$$

$$\Rightarrow \frac{d}{dx}\left[J_0^2 + J_1^2\right] = -\frac{2}{x}\left[J_1^2\right] \rightarrow (2)$$

put $n=1$.

$$(1) \Rightarrow \frac{d}{dx}\left[J_1^2 + J_2^2\right] = \frac{2}{x}\left[J_1^2 - (2)J_2^2\right] \rightarrow (3)$$

put $n=2$.

$$(1) \Rightarrow \frac{d}{dx}\left[J_2^2 + J_3^2\right] = \frac{2}{x}\left[2J_2^2 - 3J_3^2\right]$$

$\vdots$

Add (2), (3), ...

$$\Rightarrow \frac{d}{dx}\left[J_0^2 + J_1^2 + J_1^2 + J_2^2 + J_2^2 + J_3^2 + \cdots\right] = \frac{2}{x}\left[\cancel{-J_1^2} + \cancel{J_1^2} - \cancel{2J_2^2} + \cancel{2J_2^2} - 3J_3^2 \ldots\right]$$

$$\Rightarrow \frac{d}{dx}\left[J_0^2 + 2(J_1^2 + J_2^2 + J_3^2 + \cdots)\right] = 0$$

$$\Rightarrow \frac{d}{dx}\left[J_0^2 + 2(J_1^2 + J_2^2 + J_3^2 + \cdots)\right] = 0.$$

Here derivate = 0.

$\therefore$ the funct^n is constant.

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$J_0^2 + 2(J_1^2 + J_2^2 + \cdots) = c$ [where c is constant] → (A)

we have to s.T, $\boxed{c = 1}$.

Take eq^n (A)

$$J_0^2(x) + 2(J_1^2(x) + J_2^2(x) + J_3^2(x) + \cdots) = c.$$

put $x = 0$.

$$\Rightarrow J_0^2(0) + 2(J_1^2(0) + J_2^2(0) + J_3^2(0) + \cdots) = c.$$

$$\Rightarrow 1 + 2(0 + 0 + 0 \cdots) = c.$$

$\Gamma k+1 = k\Gamma k$
$k!$

$$\Rightarrow c = 1.$$

$$\therefore (A) \Rightarrow \boxed{J_0^2 + 2(J_1^2 + J_2^2 + \cdots) = 1}.$$

(Special case)

(4) PT. $J_{-n}(x) = (-1)^n J_n(x)$ where n is +ve integer.

W.K.T.

$$J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma n+k+1} \left(\frac{x}{2}\right)^{n+2k}.$$

$$J_{-n}(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma -n+k+1} \left(\frac{x}{2}\right)^{-n+2k}.$$

$$\therefore J_{-n}(x) = \sum_{k=n}^{\infty} \frac{(-1)^k}{k!\,\Gamma -n+k+1} \left(\frac{x}{2}\right)^{-n+2k}.$$

$(n+m)! = \Gamma n+m+1$.

$\Gamma m+1 = m!$

$\Gamma -n+k+1 \geq \phi$.

$-n+k+1 = \phi$.

$\boxed{k = n}$.

put $k = m+n$.

$$J_{-n}(x) = \sum_{m=0}^{\infty} \frac{(-1)^m \cdot (-1)^n}{(m+n)!\,\Gamma -n+(m+n)+1} \left(\frac{x}{2}\right)^{-n+2(m+n)}$$

$$J_{-n}(x) = (-1)^n \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma m+n+1} \left(\frac{x}{2}\right)^{n+2m}.$$

$$\therefore \boxed{J_{-n}(x) = (-1)^n J_n(x)}$$

Note:

$$J_{-n}(x) = (-1)^n J_n(x).$$

$n=1$.

$$J_{-1}(x) = (-1)^1 J_1(x)$$

$$\boxed{J_{-1}(x) = -J_1(x)}$$

$n=2$

$$\boxed{J_{-2}(x) = J_2(x)}$$

Similarly

$$\boxed{J_{-3}(x) = -J_3(x)}$$

⋮

30/07/06

→ Generating funct^n of Bessels:

$$e^{\frac{x}{2}\left[t-\frac{1}{t}\right]} = \sum_{n=-\infty}^{\infty} t^n J_n(x).$$

i) $\sin n\theta = \dfrac{e^{i\theta} - e^{-i\theta}}{2i}$

$$\boxed{2i\sin n\theta = e^{i\theta} - e^{-i\theta}}$$

$$\cos n\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

$$\boxed{2\cos n\theta = e^{i\theta} + e^{-i\theta}}$$

$$e^{in\theta} = \cos n\theta + i\sin n\theta$$

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W.K.T.

$$e^{x/2(t-1/t)} = \sum_{n=-\infty}^{\infty} t^n J_n(x)$$

$t = e^{i\theta}, \quad t^{-1} = e^{-i\theta}$.

$$e^{x/2(e^{i\theta} - 1/e^{i\theta})} = \left[t^0 J_0 + t J_1 + t^2 J_2 + \cdots\right] + \left[t^{-1} J_{-1} + t^{-2} J_{-2} + \cdots\right].$$

$$e^{x/2(e^{i\theta} - e^{-i\theta})} = \left[(J_0 + tJ_1 + t^2 J_2 + t^3 J_3 + \cdots) + (-t^{-1} J_1 + t^{-2} J_2 - t^{-3} J_3 + \cdots)\right]$$

$$= J_0 + (t - t^{-1}) J_1 + (t^2 - t^{-2}) J_2 + \cdots$$

$$e^{x\left(\frac{e^{i\theta} - e^{-i\theta}}{2}\right)} = J_0 + (2i\sin\theta) J_1 + (2\cos 2\theta) J_2 + \cdots$$

$$e^{i(x\sin\theta)} = J_0 + (2i\sin\theta) J_1 + 2(\cos 2\theta) J_2 + \cdots$$

$$\left[\cos(x\sin\theta) + i\sin(x\sin\theta)\right] = \left[J_0 + (2\cos 2\theta) J_2 + \cdots\right] + i\left[2\sin\theta\, J_1 + 2\sin 3\theta\, J_3 + \cdots\right]$$

comparing real & img parts.

$$\boxed{\cos(x\sin\theta) = J_0 + (2\cos 2\theta) J_2 + (2\cos 4\theta) J_4 + \cdots} \rightarrow (1)$$

$$\boxed{\sin(x\sin\theta) = (2\sin\theta) J_1 + (2\sin 3\theta) J_3 + \cdots} \rightarrow (2)$$

Replacing θ by $90-\theta$.

$(1) \Rightarrow \cos(x\sin(90-\theta)) = J_0 + 2\cos(2(90-\theta)) J_2 + 2\cos(4(90-\theta)) J_4 + \cdots$

$$\cos(x\cos\theta) = J_0 + 2\cos(180-2\theta) J_2 + 2\cos(360-4\theta) J_4 + \cdots$$

$$\boxed{\cos(x\cos\theta) = J_0 - 2\cos 2\theta\, J_2 + 2\cos 4\theta\, J_4 + \cdots} \rightarrow (3)$$

Replacing θ by $90-\theta$

$(2) \Rightarrow \sin(x\sin(90-\theta)) = 2\sin(90-\theta) J_1 + 2\sin 3(90-\theta) J_3 + 2\sin 5(90-\theta) J_5 + \cdots$

$$\sin(x\cos\theta) = 2\cos\theta J_1 + 2\sin(270-3\theta) J_3 + 2\sin(450-5\theta) J_5 + \dots$$

$$\sin(x\cos\theta) = 2\cos\theta J_1 - 2\cos 3\theta J_3 + 2\cos 5\theta J_5 + \dots$$

$$\boxed{\sin(x\cos\theta) = 2\left[(\cos\theta) J_1 - (\cos 3\theta) J_3 + (\cos 5\theta) J_5 + \dots\right]} \rightarrow (4)$$

Replacing θ by '0'.

eq^n (3) $\Rightarrow$

$$\cos(x\cos 0) = J_0 - 2\cos 0 J_2 + 2\cos 0 J_4 + \dots$$

$$\boxed{\therefore \cos x = J_0 - 2J_2 + 2J_4 + \dots} \rightarrow (5)$$

Replacing θ by '0'.

$$\sin(x\cos 0) = 2\left[(\cos 0) J_1 - (\cos 0) J_3 + (\cos 0) J_5 + \dots\right].$$

$$\boxed{\therefore \sin x = 2\left[J_1 - J_3 + J_5 + \dots\right]} \rightarrow (6)$$

→ Leibnitz formula:

(1) $\frac{d^n}{dx^n}(uv) = (uv)_n = {}^nC_0 u_n \cdot v + {}^nC_1 (u_{n-1}) v_1 + {}^nC_2 (u_{n-2}) v_2$
$+ \dots {}^nC_n (\cancel{u_{n-n}}) v_n$.

(or).

$$= {}^nC_0 \left(\frac{d^n}{dx^n} u\right) \cdot v + {}^nC_1 \left(\frac{d^{n-1} u}{dx^{n-1}}\right) v_1 + {}^nC_2 \left[\frac{d^{n-2} u}{dx^{n-2}}\right] v_2$$
$$+ \dots {}^nC_n \frac{\cancel{d^n}}{} u \left(\frac{d^n}{dx^n} v\right).$$

(2) $(y_1)_{n+1} = y_{n+2}$.

$(y_{+1})_{n-1} = y_n$.

→ Rodegue's formula:

$$\boxed{P_n(x) = \frac{1}{2^n \cdot n!} \cdot \frac{d^n}{dx^n}(x^2-1)^n}$$

Proof:- Let $y = (x^2-1)^n$.

$y' = y_1 = n(x^2-1)^{n-1}(2x)$

$\Rightarrow y_1 = n(x^2-1)^n \cdot (x^2-1)^{-1}(2x)$.

$\Rightarrow y_1 = \frac{ny}{x^2-1} 2x$.

$\Rightarrow y_1 = \frac{y(2nx)}{x^2-1}$

$$\underset{u}{y_1}\,\underset{v}{(x^2-1)} = \underset{u}{y}\,\underset{v}{(2nx)}.$$

Differentiate 'n+1' times w.r.t 'x'.

$$ {}^{n+1}C_0 (y_1)_{n+1}(x^2-1) + {}^{(n+1)}C_1 (y_1)_{n+1-1}(2x) + {}^{(n+1)}C_2 (y_1)_{(n+1)-2}(2).$$

$$= {}^{(n+1)}C_0 (y)_{n+1}(2nx) + {}^{(n+1)}C_1 (y)_{n+1-1}(2n)$$

$$+ \cancel{{}^{n+1}C}$$

$$\Rightarrow 1(y_{n+2})(x^2-1) + (n+1)y_{n+1}(2x) + \frac{(n+1)(n)}{\cancel{2}} y_n(\cancel{2})$$

$$= y_{n+1}(2nx) + (n+1)y_n(2n).$$

$$\Rightarrow (x^2-1)y_{n+2} + \cancel{(2nx)y_{n+1}} + (2x)y_{n+1} + n(n+1)y_n = \cancel{(2nx)y_{n+1}} + 2n(n+1)y_n.$$

$$\Rightarrow (x^2-1)y_{n+2} + 2xy_{n+1} + n(n+1)y_n - 2n(n+1)y_n = 0$$

$$\Rightarrow (x^2-1)y_{n+2} + 2xy_{n+1} - n(n+1)y_n = 0$$

$$\Rightarrow -[(1-x^2)y_{n+2} - 2xy_{n+1} + n(n+1)y_n] = 0 \rightarrow (1)$$

Let $y_n = z$

$y_{n+1} = \frac{dz}{dx}$

$y_{n+2} = \frac{d^2z}{dx^2}$ $\rightarrow (2)$

$$① \Rightarrow \left[(1-x^2) \frac{d^2 z}{dx^2} - 2x \frac{dz}{dx} + n(n+1) z \right] = 0.$$

This is legender's eq^n.

$\therefore$ The sol^n of the legender's eq^n is.

$$P_n(x) = k \cdot y^n \quad (\text{or})$$

$$P_n(x) = k \cdot y_n.$$

$$P_n(x) = k \left(\frac{d^n}{dx^n} y \right).$$

$$P_n(x) = k \frac{d^n}{dx^n} (x^2-1)^n. \rightarrow ③$$

$$P_n(x) = k \frac{d^n}{dx^n} ((x+1)(x-1))^n$$

$$= k \frac{d^n}{dx^n} \underbrace{(x+1)^n}_{u} \underbrace{(x-1)^n}_{v}.$$

$$\Rightarrow P_n(x) = k \left[{}^nC_0 \left(\frac{d^n}{dx^n} (x+1)^n \right) (x-1)^n + {}^nC_1 \left[\frac{d^{n-1}}{dx^{n-1}} (x+1)^n \right] n (x-1)^{n-1} \right.$$

$$+ {}^nC_2 \left[\frac{d^{n-2}}{dx^{n-2}} (x+1)^n \right] n(n-1)(x-1)^{n-2} + \cdots$$

$$\left. + {}^nC_n (x+1)^n \left[\frac{d^n}{dx^n} (x-1)^n \right] \right].$$

$$= k \left[1 \left(\frac{d^n}{dx^n} (x+1)^n \right) (x-1)^n + n \cdot \left(\frac{d^{n-1}}{dx^{n-1}} (x+1)^n \right) n \cdot (x-1)^{n-1} + \cdots \right.$$

$$\left. \underline{n(n+1)} + (x+1)^n \left(\frac{d^n}{dx^n} (x-1)^n \right) \right].$$

$$= k\left[n!\,(x-1)^n + n(n!)(x-1)^{n-1} + \cdots (x+1)^n n!\right] \quad \left[\because \frac{d^n}{dx^n}(x+1)^n = n!\right]$$

~~$P_n(x) =$~~

Put $x = 1$

$$P_n(1) = k\left[0 + 0 + \cdots + 2^n n!\right].$$

$$1 = k \cdot 2^n \cdot n!$$

$$k = \frac{1}{2^n n!} \rightarrow (4)$$

$$(3) \Rightarrow P_n(x) = \frac{\frac{d^n}{dx^n}(x^2-1)^n}{2^n n!}$$

$$\boxed{\therefore P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n}$$

(conti----) unit:3

5) PT the fn $v(x,y) = \sin x \cosh y + 2\cos x \sinh y + x^2 - y^2 + 4xy$ satisfies laplace equ & find corresponding analytic fn.

Sol $\quad v(x,y) = \sin x \cosh y + 2\cos x \sinh y + x^2 - y^2 + 4xy$

$v_x = \cos x \cosh y + 2(-\sin x)\sinh y + 2x + 4y$

$v_x = \cos x \cosh y - 2\sin x \sinh y + 2x + 4y$ —— (1)

$v_y = \sin x \sin y + 2\cos x \cosh y - 2y + 4x$ —— (2)

Let $u(x,y)$ be the real part of $f(z) = u + iv$

$\Rightarrow u$ & v satisfy CR equ's

i.e $u_x = v_y$; $u_y = -v_x$ —— (3)

WKT $f'(z) = u_x + iv_x$

$= v_y + iv_x$

$= (\sin x \sinh y + 2\cos x \cosh y - 2y + 4x) + i(\cos x \cosh y - 2\sin x \sinh y + 2x + 4y)$

By Milne Thomson method, $x = z$, $y = 0$

$f'(z) = (2\cos z + 4z) + i(\cos z + 2z)$

$\int f'(z)dz = \int (2\cos z + 4z) + i(\cos z + 2z)\, dz$

$f(z) = 2(\sin z) + \frac{2z^2}{2} + i\left(\sin z + \frac{2z^2}{2}\right) + c$

$\Rightarrow f(z) = 2\sin z + 2z^2 + i(\sin z + z^2) + c$

19.9.15

1) Find analytic fn $f(z) = u(r,\theta) + iv(r,\theta)$ when

$v(r,\theta) = r^2\cos 2\theta - r\cos\theta + 2$

Sol
$$v(r,\theta) = r^2\cos 2\theta - r\cos\theta + 2$$

Diff v wrt r & θ

$$\left.\begin{aligned} v_r &= 2r\cos 2\theta - \cos\theta \\ v_\theta &= -2r^2\sin 2\theta + r\sin\theta \end{aligned}\right\} \text{①}$$

Let $u(r,\theta)$ be the real part of $f(z) = u + iv$,

then u & v satisfy CR equ's

$$u_r = \frac{1}{r} v_\theta \quad ; \quad v_r = \frac{-1}{r} u_\theta \text{ — ②}$$

From ① & ②

$$u_r = \frac{1}{r}(-2r^2\sin 2\theta + r\sin\theta)$$

$$= -2r\sin 2\theta + \sin\theta$$

$$u_\theta = -2r^2\cos 2\theta + r\cos\theta \text{ — ③}$$

Integrate u_r wrt r

$$\int u_r\, dr = \int (-2r\sin 2\theta + \sin\theta)\, dr$$

$$u = -\sin 2\theta\,(r^2) + r\sin\theta + g(\theta)$$

Diff u wrt θ

$$u_\theta = -2\cos 2\theta\,(r^2) + r\cos\theta + g'(\theta) \text{ — ④}$$

Comparing ③ & ④

$g'(\theta) = 0 \Rightarrow g(\theta) = c$

$\therefore \; u = -r^2 \sin 2\theta + r\sin\theta + c$

Hence required analytic fn is

$f(z) = -r^2 \sin 2\theta + r\sin\theta + i(r^2 \cos 2\theta - r\cos\theta + 2) + c$

Note: To find interms of z

$= r^2(-\sin 2\theta + i\cos 2\theta) + r(\sin\theta - i\cos\theta) + 2i + c$

$= r^2 i(\cos 2\theta + i\sin 2\theta) - ir(\cos\theta + i\sin\theta)$

$= r^2 i e^{i2\theta} + (-ri)e^{i\theta} + 2i + c$

$= i(re^{i\theta})^2 - rie^{i\theta} + 2i + c \qquad (\because z = re^{i\theta})$

$= i(z^2 - z + 2) + c$

2) Find $f(z) = u + iv$, an analytic fn given that

$3u + 2v = y^2 - x^2 + 16x$ — ①

Sol Diff ① partially wrt x & y

$3u_x + 2v_x = -2x + 16$ — ②

$3u_y + 2v_y = 2y$

By CR equ's,

$\Rightarrow -3v_x + 2u_x = 2y$

i.e $2u_x - 3v_x = 2y$ — ③

Solving ② & ③

$$6U_x + 4V_x = -4x + 32$$

$$6U_x - 9V_x = -6y$$
$$\quad\;\;\, +$$

$$13V_x = -4x - 6y + 32$$

$$V_x = \frac{1}{13}(-4x - 6y + 32)$$

Similarly $U_x = \frac{1}{13}(-6x + 4y + 48)$

WKT $f'(z) = U_x + iV_x$

$$f'(z) = \frac{1}{13}(-6x + 4y + 48) + \frac{i}{13}(-4x - 6y + 32)$$

By Milne Thomson method, $x = z, y = 0$

$$f'(z) = \frac{1}{13}(-6z + 48) + \frac{i}{13}(-4z + 32)$$

IOBS

$$\int f'(z)\,dz = \frac{1}{13}\int(-6z + 48)\,dz + \frac{i}{13}\int(-4z + 32)\,dz$$

$$\therefore f(z) = \frac{1}{13}\left[-3z^2 + 48z + i(-2z^2 + 32z)\right] + C$$

3) Find $f(z) = u + iv$, an analytic fn given that

$3u + 2v = y^2 - x^2 + 16x$; $u - v = x - y(x^2 + 4xy + y^2) \hookrightarrow (1)$

Sol Diff (1) partially wrt x & y

$$u_x - V_x = \frac{d}{dx}\left[(x^3 + 4x^2y + xy^2) - (x^2y + 4xy^2 + y^3)\right]$$

$$= 3x^2 + 8xy + y^2 - 2xy + 4y^2$$

$$u_x - v_x = 3x^2 + 5y^2 + 6xy \quad \text{--- (2)}$$

$$u_x = 3x^2 + 5y^2 + 6xy + v_x$$

$$u_y - v_y = 3x^2 - 6xy - 3y^2$$

$$-v_x - u_x = 3x^2 - 6xy - 3y^2$$

i.e $$u_x + v_x = -3x^2 + 6xy + 3y^2 \quad \text{--- (3)}$$

(2) + (3)

$$\Rightarrow 2u_x = 12xy \Rightarrow u_x = 6xy$$

(2) − (3)

$$-2v_x = 6x^2 - 6y^2$$

$$\Rightarrow v_x = -3x^2 + 3y^2$$

WKT $$f'(z) = u_x + iv_x$$

$$= 6xy + i(3y^2 - 3x^2)$$

Replace $x = z$ & $y = 0$ By MT method,

$$f'(z) = i(-3z^2)$$

$$f(z) = -3i\frac{z^3}{3} + c$$

$$\therefore f(z) = -iz^3 + c$$

4) Find analytic fn whose realpart is

$r^2\cos 2\theta + r\sin\theta$

Sol

$$u(r,\theta) = r^2\cos 2\theta + r\sin\theta$$

$$u_r = 2r\cos 2\theta + \sin\theta$$

$$u_\theta = -2r^2\sin 2\theta + r\cos\theta \quad \text{--- (1)}$$

Let $v(r,\theta)$ be imaginary part of $f(z) = u+iv$

$$v_\theta = r(2r\cos 2\theta + \sin\theta)$$

$$v_r = -\frac{1}{r}(-2r^2\sin 2\theta + r\cos\theta)$$

$$v_\theta = 2r^2\cos 2\theta + r\sin\theta \quad \text{--- (2)}$$

$$v_r = 2r\sin 2\theta - \cos\theta$$

$$\int v_r\,dr = r^2\sin 2\theta - r\cos\theta + g(\theta)$$

$$v = r^2\sin 2\theta - r\cos\theta + g(\theta)$$

Diff wrt θ

$$v_\theta = 2r^2\cos 2\theta + r\sin\theta + g(\theta) \quad \text{--- (3)}$$

Comparing (2) & (3)

$$g(\theta) = c$$

$$\therefore v = r^2\sin 2\theta - r\cos\theta + c$$

$$f(z) = r^2\cos 2\theta + r\sin\theta + i(r^2\sin 2\theta - r\cos\theta) + c$$

5) PT $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|\text{Real} f(z)|^2 = 2|f'(z)|^2$ when $w = f(z)$ is analytic.

Sol

Let $f(z) = u+iv$

$\Rightarrow \text{Real} f(z) = u$

$\Rightarrow |\text{Real} f(z)|^2 = u^2$

Diff partially wrt x

$$\frac{\partial}{\partial x}(\text{Real} f(z))^2 = 2u\frac{\partial u}{\partial x}$$

$$\frac{\partial^2}{\partial x^2}[\text{Real} f(z)]^2 = 2u\frac{\partial^2 u}{\partial x^2} + 2\left(\frac{\partial u}{\partial x}\right)^2$$

Similarly, $\frac{\partial^2}{\partial y^2}|\text{Real} f(z)^2| = 2\left[u\frac{\partial^2 u}{\partial x^2} + \left(\frac{\partial u}{\partial x}\right)^2\right]$

Now $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}(\text{Real} f(z)^2) = 2\left[u\frac{\partial^2 u}{\partial x^2} + \left(\frac{\partial u}{\partial x}\right)^2 + u\frac{\partial^2 u}{\partial y^2} + \left(\frac{\partial u}{\partial y}\right)^2\right]$

$$= 2\left[u\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right]$$

From laplace equ, $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

$f'(x) =$
$f(x) \cdot e^{i0} = u_x + i v_x$
$f'(x) e^{i0} =$

$$= 2\left[0 + \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right]$$

$$= 2\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right] \quad \text{— (1)}$$

u_r also $f'(z) = u_x + iv_x$

$$|f'(z)| = \sqrt{u_x^2 + v_x^2}$$ $f'(x) = e^{-i0}\left[\frac{1}{r}\right.$

$u_r = \frac{1}{r} v_0$

$$|f'(z)|^2 = u_x^2 + v_x^2$$ $e^{-i0}\left[-\frac{1}{r}u_0\right)$

$v_r = -\frac{1}{r} u_0$

$$= \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2$$

$$2|f'(z)|^2 = 2\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2\right] \quad \text{— (2)}$$

From ① & ②

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right] (\text{Real } f(z))^2 = 2\,|f'(z)|^2$$

Hence Proved

7) Find analytic fn whose real part is $u=x^2-y^2-x$

Sol given: $u=x^2-y^2-x$

Diff u wrt x & y ①

$u_x = 2x-1$

$u_y = -2y$ —①

Let $v(x,y)$ be imaginary of $f(z)$, then u & v satisfy CR equ's

i.e $u_x = v_y$; $u_y = -v_x$ —②

From ① & ② we get

$v_y = 2x-1$; $v_x = 2y$ —③

Integrate v_y wrt y

$\int v_y \, dy = \int (2x-1)\, dy$

$v = 2xy - y + g(x)$

Diff v wrt x

$v_x = 2y + g'(x)$ —④

Compare ③ & ④

$\int g'(x)\, dx = \int 0 \, dx$

$g(x) = c$

$v = 2xy - y + c$ is the harmonic conjugate of u.

Now $f(z) = u + iv$

$$f(z) = x^2 - y^2 - x + i(2xy - y + c)$$

$$f'(z) = u_x + iv_x$$

$$= u_x - iv_y \quad (\text{from } 2)$$

$$f'(z) = (2x-1) + i2y \quad (\text{using } 1)$$

By MT, $x = z$ & $y = 0$

$$f'(z) = (2z - 1)$$

Integrate $f'(z)$ wrt z

$$\int f'(z)\,dz = \int (2z-1)\,dz$$

$$\therefore f(z) = z^2 - z + c$$

8) Find analytic fn whose real part is $u = \frac{x}{x^2+y^2}$ & also find its img part.

Sol given: $u = \frac{x}{x^2+y^2}$

Diff u partially wrt x & y

$$u_x = \frac{(x^2+y^2)1 - x(2x)}{(x^2+y^2)^2} = \frac{y^2-x^2}{x^2+y^2}$$

$$u_y = \frac{(x^2+y^2)0 - x(2y)}{(x^2+y^2)^2} = \frac{-2xy}{x^2+y^2} \quad \text{--- } (1)$$

Let $v(x,y)$ be img part of $f(z)$, then u & v satisfy CR eqns.

WKT $f'(z) = u_x + i v_x$

$$= u_x - i v_y \quad (\text{from } ②)$$

$$= \frac{y^2 - x^2}{(x^2+y^2)^2} - i\left[\frac{2xy}{(x^2+y^2)^2}\right]$$

But MT method, $x = z$, $y = 0$

$$f'(z) = \frac{-z^2}{(z^2)^2} - i\left(\frac{0}{z^2+0}\right)$$

$u + iv$

$= x+iy, \; ^2(x+iy) + c$

$x^2 - y^2 + 2ixy$

$-x - iy + c$

$(x^2 - y^2 - x) + i[2xy - y] + c$

$$f'(z) = \frac{-z^2}{z^4} = \frac{-1}{z^2}$$

Integrate $f'(z)$ wrt z

$$f(z) = \int \frac{-1}{z^2}\, dz$$

$$f(z) = \frac{1}{z} + c$$

$$z = x+iy \Rightarrow f(z) = \frac{1}{x+iy} + c$$

$$= \frac{x - iy}{x^2+y^2} + c$$

$$f(z) = \frac{x}{x^2+y^2} - i\,\frac{y}{x^2+y^2} + c$$

Hence given $f(z) = \dfrac{-y}{x^2+y^2} \rightarrow$ img part

9.) Find analytic fn whose imaginary part is

$v = \log(x^2+y^2) + x - 2y$

Sol given: $v = \log(x^2+y^2) + x - 2y$

. Diff v partially wrt x & y

$$V_x = \frac{2x}{x^2+y^2} + 1 \quad \text{—(1)}.$$

$$V_y = \frac{2y}{x^2+y^2} - 2$$

Let $u(x,y)$ be the real part of $f(z)$, then u & v satisfy CR equ's.

WKT $f'(z) = u_x + iV_x$.

$$= V_y + iV_x$$

$$f'(z) = \frac{2y}{x^2+y^2} - 2 + i\left[\frac{2x}{x^2+y^2} + 1\right] \text{ using (1)}$$

But MT, $x = z$, $y = 0$

$$f'(z) = -2 + i\left[\frac{2z}{z^2} + 1\right]$$

$$= -2 + i\left(\frac{2}{z} + 1\right).$$

Integrate $f'(z)$ wrt z

$$\int f'(z)\,dz = \int -2 + i(2/z + 1)\,dz$$

$$f(z) = -2z + i(2\log z + z) + C$$

If real part is to be obtained.

Put $z = x+iy$, then

$$f(z) = -2(x+iy) + i[2\log(x+iy) + (x+iy)] + c$$

$$= -2x - i2y + 2i\left[\frac{1}{2}\log(x^2+y^2) + i\tan^{-1}\left(\frac{y}{x}\right)\right] + i(x+iy) + c$$

$$= -2x - i2y + i\log(x^2+y^2) - 2\tan^{-1}\left(\frac{y}{x}\right) + i(x-iy) + c$$

$$= -2x - y - 2\tan^{-1}\left(\frac{y}{x}\right) + i[\log(x^2+y^2)] + x - 2y + c$$

$$\text{Real } f(z) = -2x - y - 2\tan^{-1}(y/x)$$

10) Find analytic fn $f(z) = u(r,\theta) + iv(r,\theta)$ where $v(r,\theta) = \left(r - \frac{1}{r}\right)\sin\theta \;\; ; r \neq 0$

Sol given: $v(r,\theta) = \left(r - \frac{1}{r}\right)\sin\theta$

Diff v wrt r & θ

$$V_r = \left(r - \frac{1}{r}\right)\sin\theta = r\sin\theta - \frac{\sin\theta}{r}$$

$$V_r = \sin\theta - \sin\theta\frac{1}{r^2} = \sin\theta\left(1 + \frac{1}{r^2}\right)$$

$$V_\theta = \left(r - \frac{1}{r}\right)\cos\theta$$

Let $u(r,\theta)$ be the real part of $f(z)$, then u & v satisfy CR equ's.

$$U_r = \frac{1}{r}V_\theta \quad ; \quad V_r = \frac{-1}{r}U_\theta \quad \text{--- (1)}$$

$$\therefore u_r = \frac{1}{r}\left(r - \frac{1}{r}\right)\cos\theta = \left(1 - \frac{1}{r^2}\right)\cos\theta \quad \text{—(2)}$$

$$u_\theta = -r V_r = -r\left(1 + \frac{1}{r^2}\right)\sin\theta$$

$$u_\theta = -\left(r + \frac{1}{r}\right)\sin\theta$$

Integrate u_θ wrt θ

$$\int u_\theta \, d\theta = -\int\left(r + \frac{1}{r}\right)\sin\theta \, d\theta$$

$$u = \left(r + \frac{1}{r}\right)\cos\theta + g(r)$$

Diff u wrt r

$$u_r = \left(1 - \frac{1}{r^2}\right)\cos\theta + g'(r) \quad \text{—(4)}$$

Comparing (3) & (4)

$$g'(r) = 0$$

$$\Rightarrow g(r) = c$$

$$u = \left(r + \frac{1}{r}\right)\cos\theta + c$$

$$f(z) = u + iv$$

$$f(z) = \left(r + \frac{1}{r}\right)\cos\theta + i\left(r - \frac{1}{r}\right)\sin\theta + c$$

is the required analytic fn.

Complex potential function:-

1) In fluid fn an analytic fn is called as complex potential fn.

2) Real part is called 'vector potential fn'

3) Img part is called 'stream fn' or 'flux fn'

If $f(z) = w = \phi(x,y) + i\psi(x,y) \rightarrow$ C.P.F

$$\phi(x,y) \rightarrow \text{V.P.F}$$

$$\psi(x,y) \rightarrow \text{S.F}$$

CR equ's $\Rightarrow \phi_x = \psi_y$

$$\phi_y = -\psi_x$$

P1) If $w = \phi + i\psi$ represent the CP for an electric field, an $\psi = x^2 - y^2 + i\dfrac{x}{x^2+y^2}$. Determine the fn ϕ.

2) In a 2D flow of a fluid the velocity potential is $\phi = x^2 - y^2$. Find stream fn.

Sol: given: $\psi = x^2 - y^2 + \dfrac{x}{x^2+y^2}$

Diff ψ partially wrt x & y

$$\psi_x = 2x + \left[\frac{(x^2+y^2) - x(2x)}{(x^2+y^2)^2}\right]$$

$$= 2x + \frac{x^2+y^2-2x^2}{(x^2+y^2)^2} = 2x + \frac{y^2-x^2}{(x^2+y^2)^2}$$

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$$\psi_y = -2y + \left[\frac{(x^2+y^2)0 - x(2y)}{(x^2+y^2)^2} \right]$$

$$= -2y - \frac{2xy}{(x^2+y^2)^2}$$

$$= -\left[2y + \frac{2xy}{(x^2+y^2)^2} \right]$$

Let $\phi(x,y)$ be the real part,

CR equ's are $\phi_x = \psi_y$

& $\phi_y = -\psi_x$

$$\phi_x = -\left[2y + \frac{2xy}{(x^2+y^2)^2} \right]$$

$$\phi_y = -\left[2x + \frac{y^2-x^2}{(x^2+y^2)^2} \right]$$

$$f'(z) = \phi_x + i\psi_x$$

$$f'(z) = \phi_x - i\phi_y$$

$$f'(z) = -\left[2y + \frac{2xy}{(x^2+y^2)^2} \right] + i\left[2x + \frac{y^2-x^2}{(x^2+y^2)^2} \right]$$

By MT method, $x=z$ & $y=0$

$$f'(z) = -\left(\frac{0}{z+0} \right) + i\left(2z + \frac{0-z^2}{z^4} \right)$$

$$= i\left(2z - \frac{z^2}{z^4} \right)$$

$$f'(z) = i\left(2z - \frac{1}{z^2} \right)$$

Integrate $f'(z)$ wrt z

$$\int f'(z)\,dz = \int i\left(2z - \frac{1}{z^2}\right)dz$$

$$f(z) = i\left(\frac{2z^2}{2} + \frac{1}{z}\right) + c \quad (2)$$

$$f(z) = i\left(z^2 + \frac{1}{z}\right) + c \text{ is the req. CPF}$$

Put $z = x+iy$

$$f(x+iy) = i\left[(x+iy)^2 + \frac{1}{x+iy}\right] + c$$

$$= i\left[x^2 + i2xy - y^2 + \frac{1}{x+iy}\right] + c$$

$$= i\left[\frac{x^2(x+iy) + (i2xy - y^2)(x+iy) + 1}{x+iy}\right] + c$$

$$= i\left[\frac{x^3 + ix^2y + 2ix^2y - y^2x - y - iy^3 + 1}{x+iy}\right] + c$$

$$= i\left[\frac{x^3 - y + 1 - y^2x + i(x^2y + 2xy - y^3)}{x+iy}\right] + c$$

21-9-15

Complex Integration

1.) Evaluate $\int_c (x+y)dx + x^2y\,dy$ along $c: y=3x$ b/w $(0,0)$ and $(3,a)$

Sol

given $c: y=3x$

$dy = 3\,dx$

from $(0,0)$ to $(3,a)$

x varies from 0 to 3

$$\therefore \int (x+y)dx + x^2y\,dy = \int_{x=0}^{3} (x+3x)dx + x^2(3x)(3dx)$$

$$= \int_0^3 4x\,dx + 9x^3\,dx$$

$$= \int_0^3 (4x+9x^3)dx$$

$$= \left(2x^2 + 9\frac{x^4}{4}\right)_0^3$$

$$= 2(9) + \frac{9}{4}(81)$$

$$= \frac{801}{4}.$$

2.) Evaluate $\int (3x^2+4xy+ix^2)dz$ along $c: y=x^2$ b/w $(0,0)$ and $(1,1)$.

Sol

given: $c: y=x^2$

$dy = 2x\,dx$

from $(0,0)$ to $(1,1)$

x varies from 0 to 1

Let $z = x + iy$

$\Rightarrow dz = dx + i\,dy$

$$\therefore \int_C (3x^2 + 4xy + ix^2)\,dz$$

$$= \int_{(0,0)}^{(1,1)} (3x^2 + 4xy + ix^2)\,(dx + i\,dy)$$

$$= \int_0^1 (3x^2 + 4x^3 + ix^2)\,(dx + i2x\,dx)$$

$$= \int_0^1 (3x^2 + 4x^3 + ix^2 + 6ix^3 + 8ix^4 - 2x^3)\,dx$$

$$= \int_0^1 (3x^2 + 2x^3 + ix^2 + 6ix^3 + 8ix^4)\,dx$$

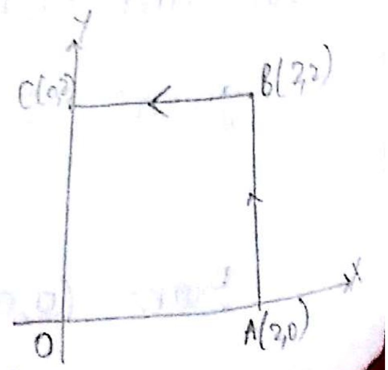
$$= \left[x^3 + \frac{x^4}{2} + \frac{ix^3}{3} + 3i\frac{x^4}{2} + 8i\frac{x^5}{5}\right]_0^1$$

$$= 1 + \frac{1}{2} + \frac{i}{3} + \frac{3i}{2} + \frac{8i}{2}$$

$$= \frac{3}{2} + i\left(\frac{103}{30}\right)$$

3) Evaluate $\int (z^2 + 3z)\,dz$ along st line c: st line from $(2,0)$ to $(2,2)$ and then $(2,2)$ to $(0,2)$

<u>Sol</u> Let $z = x + iy$



along AB:

here $x = 2$ & $y: 0$ to 2

$$z = 2+iy$$

$$\Rightarrow dz = i\,dy$$

$$\therefore \int_{AB} (z^2+3z)\,dz = \int_0^2 [(2+iy)^2 + 3(2+iy)]\, i\,dy$$

$$= i\int_0^2 (4-y^2+4iy+6+3iy)\,dy$$

$$= i\int_0^2 (10-y^2+7iy)\,dy$$

$$= i\left[10y - \frac{y^3}{3} + 7\frac{iy^2}{2}\right]_0^2$$

$$= i\left[20 - \frac{8}{3} + 14i\right]$$

$$= \frac{52i}{3} - 14$$

along BC: here $y = 2$ & $x: 2$ to 0

$$z = x+iy$$

$$z = x+2i$$

$$\Rightarrow dz = dx$$

$$\int_{BC} (z^2+3z)\,dz = \int_2^0 [(x+2i)^2 + 3(x+2i)]\,dx$$

$$= \int_2^0 (x^2+4i^2+4xi+3x+6i)\,dx$$

$$= \left[\frac{x^3}{3} - 4x + 2x^2 i + \frac{3x^2}{2} + 6ix\right]_2^0$$

$$= -\left(\frac{8}{3} - 8 + 8i + 6 + 12i\right)$$

$$= -\left(\frac{8}{3} - 2 + 20i\right)$$

$$= -\left(\frac{2}{3} + 20i\right)$$

$$\therefore \int_{z} (z^2 + 3z)\,dz = \int_{AB} (z^2+3z)\,dz + \int_{BC} (z^2+3z)\,dz$$

$$= \frac{52i}{3} - 14 - \frac{2}{3} - 20i$$

$$= \frac{-44}{3} - \frac{8}{3}i$$

4) Evaluate $\int_A^B (x^2 + ixy)\,dz$ where $A=(1,1)$ $B=(2,8)$

along (i) st line AB (ii) curve $x=t, y=t^3$

Sol (i) Equ of st line AB where $A=(1,1)$ $B(2,8)$

$$\Rightarrow y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

$$\Rightarrow y - 1 = \frac{8-1}{2-1}(x-1)$$

$$\Rightarrow y - 1 = \frac{7}{1}(x-1)$$

$$\Rightarrow y - 1 = 7x - 7$$

$$\Rightarrow y = 7x - 6$$

$$dy = 7dx$$

$$x : 1 \text{ to } 2$$

$$\therefore \int_A^B (x^2+ixy)\,dz = \int_A^B (x^2+ixy)(dx+idy)$$

$$= \int_A^B [x^2+ix(7x-6)]\,(dx+idy)$$

$$= \int_A^B (x^2+7ix^2-6ix)(dx+i7dx)$$

$$= \int_A^B (x^2+7ix^2-6ix)\,(1+7i)\,dx$$

$$= (1+7i)\int_A^B (x^2+7ix^2-6ix)\,dx$$

$$= (1+7i)\left[\frac{x^3}{3}+7i\frac{x^3}{3}-3ix^2\right]_1^2$$

$$= (1+7i)\left(\frac{8}{3}+\frac{56i}{3}-12i-\frac{1}{3}-\frac{7i}{3}+3i\right)$$

$$= (1+7i)\left(\frac{7}{3}+i\frac{22}{3}\right)$$

$$= \frac{7}{3}+\frac{22i}{3}+\frac{49i}{3}+\frac{154i^2}{3}$$

$$= \frac{7}{3}-\frac{83}{3}i$$

(ii) curve $C: x=t,\ y=t^3$ $\quad A(1,1)\quad B(2,8)$

$$dx=dt\ ;\ dy=3t^2\,dt$$

LL: $x=1 \Rightarrow t=1$	LL: $y=1 \Rightarrow t=1$
UL: $x=2 \Rightarrow t=2$	UL: $y=8 \Rightarrow t=2$

$\therefore t$ varies from 1 to 2

$$\int_A^B (x^2+ixy)\,dz = \int_1^2 [t^2+i(t)(t^3)]\,(dx+idy)$$

$$= \int_1^2 (t^2+it^4)\,(dt+i\,3t^2dt)$$

$$= \int_1^2 (t^2+it^4)\,(1+i3t^2)\,dt$$

$$= \int_1^2 (t^2+i3t^2+3it^4+3i^2t^6)\,dt$$

$$= \left(\frac{t^3}{3}+\frac{\cancel{3}it^3}{\cancel{3}}+\frac{3}{\cancel{4}}\frac{it^5}{5}-3\frac{t^7}{7}\right)_1^2$$

$$= \left[\frac{8}{3}+8i+\frac{3}{5}i(32)-\frac{3}{7}(128)\right]$$

$$-\left[\frac{1}{3}+i+\frac{3i}{5}-\frac{3}{7}\right.$$

$$= \left(-\frac{1096}{21}+i\frac{136}{5}\right)-\left(\frac{-2}{21}+i\frac{8}{5}\right)$$

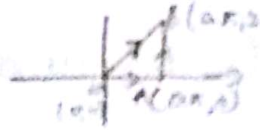
$$= \left(-\frac{1096}{21}+i\frac{136}{5}\right)+\frac{4}{21}-\frac{8i}{5}$$

$$= -52+i\,\frac{128}{5}$$

5) Evaluate $\int_{1-i}^{2+i} (2x+iy+1)\,dz$ along the st. line joining $(1,-1)$ and $(2,1)$

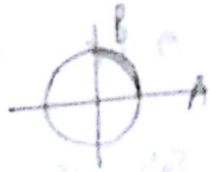
6) Evaluate $\int_C (y^2+2xy)dx+(x^2-2xy)dy$ where C is the boundary of the region by $y=x^2$ & $x=y^2$

7) Evaluate $\int_C (z^2+3z+2)dz$ where c is the arc of the cycloid $x = a(\theta + \sin\theta)$, $y = a(1-\cos\theta)$ b/w the points $(0,0)$ and $(a\pi, 2a)$



$z = x+iy$

8) Evaluate $\int_C (x-2y)dx + (y^2-x^2)dy$ where c is the boundary of first quadrant of the circle $x^2+y^2 =$



θ 0 to $\pi/2$

$x = 2\cos\theta$; $y = 2\sin\theta$

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Continuous arc:-

A set of points (x,y) where $x=x(t)$, $y=y(t)$, $a \le t \le b$ and $x(t)$, $y(t)$ are continuous fns of the real variable t is called a continuous arc.

Simple arc / Jordan arc:-

An arc $x=x(t)$, $y=y(t)$ is said to be simple or jordan arc if no 2 values of t correspond to same point (x,y).

Simple closed arc / Jordan curve:-

If $x(a)=x(b)$ & $y(a)=y(b)$ in a simple arc, then it is said to be a simple closed arc.

Smooth arc:-

An arc $x=x(t)$, $y=y(t)$ is said to be smooth is said to be smooth if $x(t)$, $y(t)$ possess continuous derivates and the curve has cont. turning tangents.

Contour:-

A chain of finite no. of smooth arcs is said to be a contour.

Closed contour:-

If a contour is closed, then it is said to be closed contour. (end points are same).

Cauchy's Theorem:-

If $f(z) = u(x,y) + iv(x,y)$ is analytic in a closed simple contour 'c' and $f'(z)$ is cont. there, then $\int_C f(z)dz = 0$.

Proof: $f(z) = u+iv$

$z = x+iy$

$dz = dx+idy$

$$\therefore f(z)dz = (u+iv)(dx+idy)$$

$$= udx - vdy + i(udy + vdx)$$

$$\int_C f(z)dz = \int_C udx - vdy + i(udy+vdx)$$

$$= \int_C udx - vdy + i\int_C vdx + udy$$

By green's theorem,

$$\int_C Mdx + Ndy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

$$= \iint_R \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy + i\iint_R \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dx\,dy$$

Since $f'(z)$ is cont., $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ are cont.

$$= \iint_R \left(\frac{\partial u}{\partial y} - \frac{\partial u}{\partial y}\right) dx\,dy + i \iint_R \left(\frac{\partial v}{\partial y} - \frac{\partial v}{\partial y}\right) dx\,dy$$

(by using CR equs)

$= 0$

Hence Proved

P1) Evaluate $\int_C e^z dz$ where $C: |z|=4$

$\because f(z) = e^z$ is analytic everywhere.

In particular, it is analytic on and within the circle $C: |z|=4$

$\therefore$ By Cauchy's theorem, $\int_C e^z dz = 0$

2) Evaluate $\int_C \frac{z^2}{z-1} dz$ ~~= 0~~ where $C: |z| = \frac{1}{2}$

Sol $f(z) = \frac{z^2}{z-1}$ is analytic everywhere except at $z=1$

but $z=1$ lies outside $C: |z| = \frac{1}{2}$.

$\therefore f(z)$ is analytic on and within C,

hence $\int_C \frac{z^2}{z-1} dz = 0$ by cauchy's theorem.

Note:

1) $|z| = r$, a circle with center origin & radius 'r'

$\because |x+iy| = r$

$\Rightarrow \sqrt{x^2+y^2} = r$

$\Rightarrow x^2+y^2 = r^2$

Eg:- $|z| = \frac{1}{2}$

$C(0,0)$, $r = \frac{1}{2}$

2) $|z-a| = r$, circle with center $(a,0)$ & radius r.

$\because |x+iy-a| = r$

$\Rightarrow |x-a+iy| = r$

$\Rightarrow (x-a)^2+y^2 = r^2$

Eg:- $|z-1| = 2$

Centre $(1,0)$, radius $= 2$

3) $z-a = re^{i\theta}$, a circle with centre $(0,0)$ and radius 'r'

$\because |z-a| = |re^{i\theta}|$

$\Rightarrow |z-a| = r \quad (\because |e^{i\theta}| = 1)$

4.) $|z-a| = r$, a circle

then, (i) $|b-a| > r$, $z=b$ lies outside C

(ii) $|b-a| < r$, $z=b$ lies inside C

(iii) $|b-a| = r$, $z=b$ lies on C

Eg:- (i) $|z-1| = 2$

$z=3$ lies ?

$|3-1| = 2$, lies on C

Eg:- $|z| = 2$

$z=3$ lies ?

$|3| = 3 > 2$, lies outside C

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Eg:- $|z-1| = 3$

$z = 1+2i$ lies?

$|1+2i-1| = |2i|$

$= 2 < 3$, lies inside C

Eg:- $|z| = \frac{1}{2}$

$z = 3$ lies ?

$|3| = \sqrt{9} = 3$, $z=3$ lies outside C.

P1.) Evaluate $\int_C \frac{z^2-z-1}{z-1} dz$ where $C: |z| = \frac{1}{2}$

Sol $f(z) = \frac{z^2-z-1}{z-1}$ is analytic everywhere

except at $z=1$ but $z=1$ lies outside the circle C

i.e $C = |z| = \frac{1}{2}$

$\left(|1| = 1 > \frac{1}{2} \right)$

$\therefore f(z)$ is analytic on and within C

Hence by cauchy's theorem $\int_C \frac{z^2-z-1}{z-1} dz = 0$

2) Evaluate $\int_C \frac{e^{2z}}{z-2} dz$ where $|z| = 1$

Sol $f(z) = \frac{e^{2z}}{z-2}$ is analytic ew except

at $z=2$

$z=2$ lies outside $C: |z|=1$

$(\because |2| = 2 > 1)$

$\therefore f(z)$ is analytic on and within C

Hence by Cauchy's theorem, $\int_C \frac{e^{2z}}{z-2}\, dz = 0$

Cauchy's integral formula:-

Let $f(z) = u + iv$ be an analytic fn on and within a closed simple contour C and 'a' be a point inside C, then $f(a) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z-a}\, dz$

i.e $\int_C \frac{f(z)}{z-a}\, dz = f(a) \cdot 2\pi i$

<u>Proof:-</u> given:

$f(z)$ is analytic fn on and within C and 'a' is a pt inside C.

then draw a circle C_0 with radius, so small r_0 such that the circle C_0 lies completely in C

Then $C_0 : z - a = r_0 e^{i\theta}$ and

$\frac{f(z)}{z-a}$ is analytic b/w C & C_0.

$$\therefore \int_C \frac{f(z)}{z-a}\, dz = \int_{C_0} \frac{f(z)}{z-a}\, dz \quad \text{, by known theorem.} \rightarrow (1)$$

On C_0, $z = a + r_0 e^{i\theta}$

$\Rightarrow dz = r_0 e^{i\theta} (i)\, d\theta$

& $\theta : 0$ to 2π

$$\int_{C_0} \frac{f(z)}{z-a}\, dz = \int_{\theta=0}^{2\pi} \frac{f(a + r_0 e^{i\theta})}{\cancel{r_0 e^{i\theta}}} \cancel{r_0 e^{i\theta}} \cdot i\, d\theta$$

$$= i\int_0^{2\pi} f(a+r_0e^{i\theta})\,d\theta$$

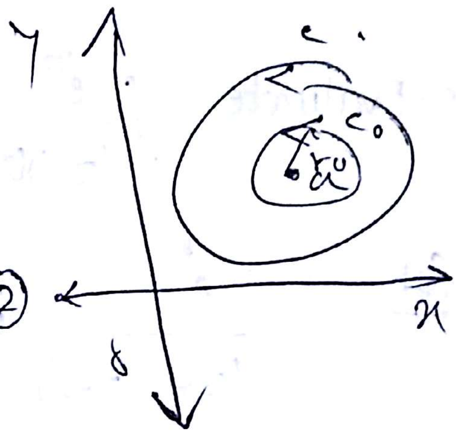
Letting $r_0 \to 0$

$$= i\int_0^{2\pi} f(a)\,d\theta$$

$$= i\,f(a)\int_0^{2\pi} d\theta$$

$$= i\,f(a)\,(\theta)_0^{2\pi}$$

$$= 2\pi i\,f(a) \quad \text{--- (2)}$$



From (1) & (2)

$$\int_C \frac{f(z)}{z-a}\,dz = 2\pi i\,f(a)$$

Hence $f(a) = \dfrac{1}{2\pi i}\displaystyle\int_C \frac{f(z)}{z-a}\,dz$ → Proved

P1) Evaluate $\displaystyle\int_C \frac{z^2-z-1}{z-1}\,dz$ where $c : |z| = 2$

Sol The fn $\dfrac{z^2-z-1}{z-1}$ is analytic ew except at $z=1$.

& $z=1$ lies inside $c : |z| = 2$

Take $f(z) = z^2-z-1$ and $a=1$

then by cauchys integral formula,

we have $\displaystyle\int_C \frac{f(z)}{z-a}\,dz = 2\pi i\,f(a)$

$$\Rightarrow \int_C \frac{z^2-z-1}{z-1}\,dz = 2\pi i f(1)$$

$$= 2\pi i\,(1^2-1-1)$$

$$= 2\pi i\,(-1)$$

$$= -2\pi i$$

2) Evaluate $\int_C \frac{e^{2z}}{(z-1)(z-2)}\,dz$ where $C: |z|=3$

Sol $\frac{e^{2z}}{(z-1)(z-2)}$ is not analytic at $z=1,2$

Pt. $z=1$ lies inside C

$\because |1|=1<3$

& Pt $z=2$ lies inside C.

$\because |2|=2<3$

WKT $\frac{1}{(z-1)(z-2)} = \frac{1}{z-2} - \frac{1}{z-1}$ (partial fraction)

$$\Rightarrow \frac{e^{2z}}{(z-1)(z-2)} = \frac{e^{2z}}{z-2} - \frac{e^{2z}}{z-1}$$

By C.I.F we have

$$\int_C \frac{f(z)}{z-a}\,dz = 2\pi i f(a) \quad \text{--- (1)}$$

To evaluate $\int_C \frac{e^{2z}}{z-2}\,dz$:-

Take $f(z) = e^{2z}$ and $a = 2$

By using ①, $\int_C \frac{e^{2z}}{z-2}\,dz = 2\pi i f(2)$

$= 2\pi i e^4$

To evaluate $\int_C \frac{e^{2z}}{z-1}\,dz$:-

Here $f(z) = e^{2z}$ & $a = 1$

By using ① $\int_C \frac{e^{2z}}{z-1}\,dz = 2\pi i f(1)$

$= 2\pi i e^2$

Hence $\int_C \frac{e^{2z}}{(z-1)(z-2)}\,dz = \int_C \frac{e^{2z}}{z-2}\,dz - \int_C \frac{e^{2z}}{z-1}\,dz$

$= 2\pi i e^4 - 2\pi i e^2$

$= 2\pi i e^2(e^2 - 1)$

3) Evaluate $\int_C \frac{e^z}{(z-1)(z-4)}\,dz$ where $C: |z| = 2$

Sol $\frac{e^z}{(z-1)(z-4)}$ is not analytic at $z = 1, 4$

Pt. $z = 1$ lies inside C

$\because |1| = 1 < 2$

& pt $z = 4$ lies outside C

$\because |4| = 4 > 2$

Take $f(z) = \frac{e^z}{z-4}$

By C.I.F, we have

$$\int_C \frac{f(z)}{z-a}\, dz = 2\pi i f(a)$$

$$\Rightarrow \int_C \frac{e^z/z-4}{z-1}\, dz = 2\pi i f(1)$$

$$\Rightarrow \int_C \frac{e^z}{(z-1)(z-4)} = 2\pi i \frac{e}{-3}$$

$$= \frac{-2\pi i e}{3}$$

4) Evaluate $\int_C \frac{z+4}{z^2+2z+5}\, dz$ where $C: |z+1-i| = 2$

Sol $\frac{z+4}{z^2+2z+5}$ is not analytic when $z^2+2z+5=0$

$$z^2+2z+5=0$$

$$z = -1 \pm 2i$$

The pt. $z = -1+2i$, lies inside C

$$\because |\cancel{-1}+2i+\cancel{1}-i| = |i| = 1 < 2$$

& pt $z = -1-2i$, lies outside C.

$$|\cancel{-1}-2i+\cancel{1}-i| = |-3i| = \sqrt{9} = 3 > 2.$$

Take $f(z) = \frac{z+4}{z-(-1-2i)}$, $a = -1+2i$

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By C.I.F, we have

$$\int_C \frac{f(z)}{z-a}\,dz = 2\pi i f(a)$$

$$\Rightarrow \int_C \frac{\frac{z+4}{z-(-1-2i)}}{z-(-1+2i)}\,dz = 2\pi i f(-1+2i)$$

$$\Rightarrow \int_C \frac{z+4}{z^2+2z+5}\,dz = 2\pi i \left[\frac{-1+2i+4}{-1+2i+1+2i}\right]$$

$$= \cancel{+}2\pi\cancel{i}\left[\frac{3+2i}{\cancel{4i}}\right]$$

$$= \frac{\pi}{2}(3+2i)$$

5.) Evaluate $\int_C \frac{z+4}{z^2+2z+5}\,dz$ where $C: |z+1+i| = 2$

Sol $\frac{z+4}{z^2+2z+5}$ is not analytic when $z^2+2z+5=0$

$$z^2+2z+5=0$$

$$z=-1\pm 2i$$

Pt. $z=-1+2i$, lies outside C

$$\because |\cancel{-1}+2i+\cancel{1}+i| = |3i| = 3>2$$

& pt $z=-1-2i$, lies inside C

$$\because |\cancel{-1}-2i+\cancel{1}+i| = |-i| = 1<2$$

Take $f(z) = \frac{z+4}{z-(-1+2i)}$, $a=-1-2i$

By C.I.F,

$$\int_C \frac{f(z)}{z-a}\,dz = 2\pi i f(a)$$

$$\Rightarrow \int_C \frac{\frac{z+4}{z-(-1+2i)}}{z-(-1-2i)}\,dz = 2\pi i\, f(-1-2i)$$

$$\Rightarrow \int_C \frac{z+4}{z^2+2z+5}\,dz = 2\pi i\left[\frac{-1-2i+4}{-1-2i+1+2i}\right]$$

$$= \cancel{2}\pi \cancel{i}\left[\frac{3-2i}{-\cancel{4}\cancel{i}}\right]$$

$$= \frac{\pi}{2}(2i-3)$$

Generalised cauchy's integral formula:-

Let $f(z)$ be an analytic fn on and within a simple closed contour C and 'a' be a point inside C, then $f^n(a) = \frac{n!}{2\pi i}\int_C \frac{f(z)}{(z-a)^{n+1}}\,dz$

Proof:- By C.I.F,

$$\int_C \frac{f(z)}{z-a}\,dz = 2\pi i f(a) \quad \text{—①}$$

WKT $f'(a) = \lim_{\Delta a\to 0} \frac{f(a+\Delta a)-f(a)}{\Delta a}$

$$= \lim_{\Delta a\to 0}\left[\frac{1}{\Delta a}\left(\frac{1}{2\pi i}\int_C \frac{f(z)\,dz}{z-(a+\Delta a)} - \frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}dz\right)\right]$$

$$= \lim_{\Delta a\to 0}\frac{1}{\Delta a}\left[\frac{1}{2\pi i}\int_C\left(\frac{1}{z-a-\Delta a}-\frac{1}{z-a}\right)f(z)dz\right]$$

$$= \lim_{\Delta a\to 0}\frac{1}{\Delta a}\left[\frac{1}{2\pi i}\int_C \frac{(z-a)-(z-a-\Delta a)}{(z-a)(z-a-\Delta a)}f(z)dz\right]$$

$$= \lim_{\Delta a\to 0}\frac{1}{\Delta a}\cdot\frac{1}{2\pi i}\int_C \frac{\Delta a}{(z-a)(z-a-\Delta a)}f(z)\,dz$$

$$= \lim_{\Delta a\to 0}\frac{1}{2\pi i}\int_C \frac{1}{(z-a)(z-a-\Delta a)}f(z)\,dz$$

$$f'(a) = \frac{1!}{2\pi i}\int_C \frac{f(z)}{(z-a)^2}\,dz$$

$$f''(a) = \frac{2!}{2\pi i}\int \frac{f(z)}{(z-a)^3}\,dz$$

P1.) Evaluate $\int_C \frac{\cos z - \sin z}{(z+i)^3} dz$, $C: |z| = 2$

Sol . The fn $\frac{\cos z - \sin z}{(z+i)^3}$ has a singular pt, $z=$

and $z = -i$ lies inside C

$(\because |-i| = 1 < 2)$

Take $f(z) = \cos z - \sin z$

& $a = -i$, $n = 2$

By gen. CIF,

$$\int_C \frac{f(z)}{(z-a)^{n+1}} dz = \frac{f^n(a)\, 2\pi i}{n!}$$

$$\Rightarrow \int_C \frac{\cos z - \sin z}{(z-(-i))^{2+1}} dz = \frac{f''(-i) \cdot 2\pi i}{2!}$$

$$\Rightarrow \int_C \frac{\cos z - \sin z}{(z+i)^3} dz = \frac{[-\cos(-i) + \sin(-i)]\, 2\pi i}{\cancel{2}}$$

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$$= (-\cos i - \sin i)\pi i$$

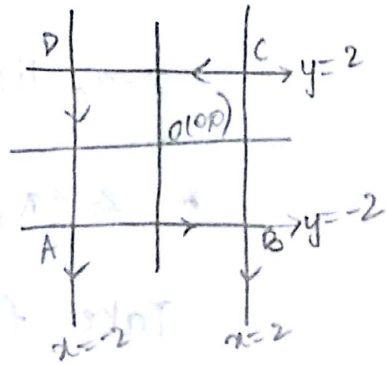
Since $f(z) = \cos z - \sin z$

$$f'(z) = -\sin z - \cos z$$

$$f''(z) = -\cos z + \sin z$$

2) Evaluate $\int_C \frac{\cosh z}{z^4}\, dz$ where c is a square,

$c : x = \pm 2, \; y = \pm 2$



Sol The fn $\frac{\cosh z}{z^4}$ has a singular pt $z = 0$ and $z = 0$ lies inside C

Take $f(z) = \cosh z$, $a = 0$, $n = 3$

By gen. C.I.F,

$$\int_C \frac{f(z)}{(z-a)^{n+1}}\, dz = \frac{f^n(a)\, 2\pi i}{n!}$$

$$\Rightarrow \int_C \frac{\cosh z}{(z-0)^{3+1}}\, dz = \frac{f'''(0)\, 2\pi i}{3!}$$

$$\Rightarrow \int \frac{\cosh z}{z^4}\, dz = \frac{\sinh(0)\, 2\pi i}{6}$$

$$= 0$$

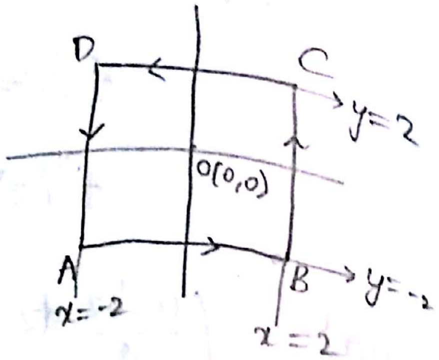
$[\because f(z) = \cosh z$

$f'(z) = \sinh z$

$f''(z) = \cosh z$

$f'''(z) = \sinh z\,]$

3) Evaluate $\int_C \frac{\tan z/2}{(z-x_0)^2}\, dz$, $|x_0|<2$ where $C: x=\pm 2$, $y=\pm 2$.



<u>Sol</u> The fn $\frac{\tan z/2}{(z-x_0)^2}$ has singular points $z=x_0$ and

$$\frac{z}{\cancel{2}} = (2n+1)\frac{\pi}{\cancel{2}}$$

$\Rightarrow z = x_0$, $z = \pm\pi, \pm 3\pi, \pm 5\pi \ldots$

$z=x_0$ lies inside C $(\because |x_0|<2)$.

& $z = \pm\pi, \pm 3\pi, \pm 5\pi \ldots$ lie outside C.

Take $f(z) = \tan\frac{z}{2}$, $a = x_0$, $n=1$

By Gen. C.I.F,

$$\int_C \frac{f(z)}{(z-a)^{n+1}}\, dz = \frac{f^n(a)\, 2\pi i}{n!}$$

$$\Rightarrow \int_C \frac{\overset{\tan}{\cancel{\cosh}}\, z/2}{(z-x_0)^{1+1}}\, dz = \frac{f'(x_0)\, 2\pi i}{1!}$$

$$\int_C \frac{\tan z/2}{(z-x_0)^2}\, dz = \frac{1}{2}\sec^2\frac{x_0}{2}\,(2\pi i)$$

$$\left[\because f'(z) = \sec^2\frac{z}{2}\left(\frac{1}{2}\right)\right]$$

$$= \pi i \sec^2\left(\frac{x_0}{2}\right)$$

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4) Evaluate $\int_C \frac{\sin^6 z}{(z-\frac{\pi}{6})^3} dz$, $C: |z|=1$

Sol The fn $\frac{\sin^6 z}{(z-\frac{\pi}{6})^3}$ has singular points $z=\frac{\pi}{6}$

$z=\frac{\pi}{6}$ lies inside C $(\because |\frac{\pi}{6}|<1)$

Take $f(z)=\sin^6 z$.

$a=\frac{\pi}{6}$, $n=2$

By gen. C.I.F,

$$\int_C \frac{f(z)}{(z-a)^{n+1}} dz = \frac{f^n(a)2\pi i}{n!}$$

$$\Rightarrow \int_C \frac{\sin^6 z}{(z-\frac{\pi}{6})^{2+1}} dz = \frac{f^2(\frac{\pi}{6})2\pi i}{2!}$$

$\because f(z)=\sin^6 z$

$f'(z)=6\sin^5 z\cos z$

$f''(z)=6(-\sin^6 z + 5\cos^2 z\sin^4 z)$

$$= \frac{-6\left(\frac{+1}{2}\right)^6+5\left(\frac{\sqrt{3}}{2}\right)^2\left(\frac{1}{2}\right)^6 \times 2\pi i}{2}$$

$$= \frac{21}{16}\pi i$$

5) Evaluate $\int_C \frac{\sin\pi z^2+\cos\pi z^2}{(z-1)(z-2)} dz$, $C: |z|=3$

Sol The fn $\frac{\sin\pi z^2+\cos\pi z^2}{(z-1)(z-2)}$ has singular points

$z=1$, $z=2$ & both $z=1,2$ lies inside C.

WKT $\frac{1}{(z-1)(z-2)} = \frac{1}{z-2} - \frac{1}{z-1}$.

$$\Rightarrow \frac{\sin\pi z^2 + \cos\pi z^2}{(z-1)(z-2)} = \frac{\sin\pi z^2+\cos\pi z^2}{z-2} - \frac{\sin\pi z^2+\cos\pi z^2}{z-1}$$

By C.I.F,

$$\int_C \frac{f(z)}{z-a}\,dz = 2\pi i f(a)$$

$$\Rightarrow \int_C \frac{\sin\pi z^2+\cos\pi z^2}{(z-1)(z-2)}\,dz = \int_C \frac{\sin\pi z^2+\cos\pi z^2}{z-2}\,dz - \int_C \frac{\sin\pi z^2+\cos\pi z^2}{z-1}$$

$$= f(2)\,2\pi i - f(1)\,2\pi i$$

WKT $f(z) = \sin\pi z^2 + \cos\pi z^2$

$f(2) = 0 + 1 = 1$

$f(1) = 0 - 1 = -1$

$\therefore \; = 2\pi i\,(f(2) - f(1))$

$= 2\pi i\,(1+1)$

$= 4\pi i$

*6) Evaluate $\int_C \left[\frac{e^z}{z^3} + \frac{z^4}{(z+i)^2}\right] dz, \quad C: |z|=2$

HW

7) Evaluate $\int_C \frac{e^z}{z(1-z)^3}\,dz$ inside pt $\Rightarrow a$

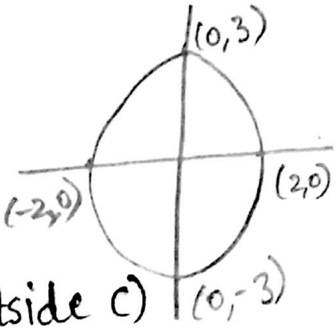
(i) if 0 lies inside c, 1 lies outside c $(a=0)$

(ii) if 0 lies outside c, 1 lies inside C (a=1)

(iii) both 0 & 1 lie inside c. (partial fractions)

8) Find $f(4), f'(-1), f''(-i)$ when $f(a) = \int_C \frac{4z^2+z+5}{z-a}\,dz$

and C is ellipse $\left(\frac{x}{2}\right)^2 + \left(\frac{y}{3}\right)^2 = 1$



Sol

$$f(4) = \int_C \frac{4z^2+z+5}{z-4}\,dz$$

$= 0$ ($\because z=4$ lies outside c)

By using cauchy's theorem

$$f'(a) = \int_C \frac{4z^2+z+5}{(z-a)^2}\,dz$$

$$f'(-1) = \int_C \frac{4z^2+z+5}{(z+1)^2}$$

$$f''(a) = 2\int \frac{4z^2+z+5}{(z-a)^3}\,dz$$

30-9-15

Unit:4 Complex Power Series

Taylor's series:-

The expansion of taylor's series of a comp fn about $z=a$ or in powers in $z-a$ or near $z=a$ is $f(z)=f(a)+(z-a)f'(a)+\frac{(z-a)^2}{2!}f''(a)\dots$

$$+\dots\frac{(z-a)^n}{n!}f^n(a)+$$

Maclurians Series:-

Expansion of $f(z)$ is

$$f(z)=f(0)+zf'(0)+\frac{z^2}{2!}f''(0)+\dots\frac{z^n}{n!}f^n(0)+\dots$$

Some important maclurians expansions:-

1) $e^z = 1+z+\frac{z^2}{2!}+\frac{z^3}{3!}+\dots\frac{z^n}{n!}+\dots, \quad |z|=\infty$

2) $\sin z = z-\frac{z^3}{3!}+\frac{z^5}{5!}-\frac{z^7}{7!}+\dots, \quad |z|<\infty$

3) $\cos z = 1-\frac{z^2}{2!}+\frac{z^4}{4!}+\dots, \quad |z|<\infty$

4) $\sinh z = z+\frac{z^3}{3!}+\frac{z^5}{5!}+\dots, \quad |z|<\infty$

5) $\cosh z = 1+\frac{z^2}{2!}+\frac{z^4}{4!}+\dots \quad |z|<\infty$

6) $\frac{1}{1-z}=(1-z)^{-1}=1+z+z^2+\dots, \quad |z|<1$

7) $\frac{1}{1+z}=(1+z)^{-1}=1-z+z^2-z^3+\dots, \quad |z|<1$

8) $\frac{1}{(1-z)^2} = (1-z)^{-2} = 1+2z+3z^2+4z^3+\cdots, |z|<1$

9) $\frac{1}{(1+z)^2} = (1+z)^{-2} = 1-2z+3z^2-4z^3+\cdots, |z|<1$

10) $\frac{1}{(1+z)^m} = (1+z)^{-m} = 1-mz+\frac{m(m+1)}{2!}z^2 - \frac{m(m+1)(m+2)}{3!}z^3$

$+ \cdots \quad |z|<1$

11) $\frac{1}{(1-z)^m} = (1-z)^{-m} = 1+mz+\frac{m(m+1)}{2!}z^2+\cdots, |z|<1$

Note:

To obtain a taylor series expansion of a fn $f(z)$ about z equal to a or in powers of $(z-a)$ will substitute $(z-a)=w$ i.e $z=a+w$, then $f(z)=f(a+w)$ and then will use Maclurians expansions.

A) Expand $f(z)=e^z$ in powers of $z-2$

Sol Put $z-2=w$

$\Rightarrow z=2+w$

$\therefore f(z) = e^z = e^{2+w} = e^2 e^w$

$= e^2\left(1+w+\frac{w^2}{2!}+\frac{w^3}{3!}+\cdots\right)$

when $|w|<\infty$

$= e^2\left[1+(z-2)+\frac{(z-2)^2}{2!}+\frac{(z-2)^3}{3!}+\cdots\right]$

when $|z-2|<\infty$

2) Expand $f(z)=\dfrac{1}{z^2}$ in powers of $z+1$.

Sol Put $z+1=w$

$$\Rightarrow z=w-1$$

$$f(z)=\frac{1}{(w-1)^2}$$

$$=\frac{1}{(1-w)^2}=(1-w)^{-2}$$

$$=1+2w+3w^2+4w^3+\cdots,\ |w|<1$$

$$=1+2(z+1)+3(z+1)^2+4(z+1)^3+\cdots,\ |z$$

3) ST $z^{-2}=1+\sum_{n=1}^{\infty}(n+1)(z+1)^n$ when $|z+1|<1$

Sol Let $f(z)=\dfrac{1}{z^2}$

Put $z+1=w$

$$\Rightarrow z=w-1$$

$$f(z)=\frac{1}{(w-1)^2}=\frac{1}{(1-w)^2}$$

$$=(1-w)^{-2}$$

$$=1+2w+3w^2+4w^3+\cdots,\ |w|<1$$

$$=1+2(z+1)+3(z+1)^2+\cdots,\ |z+1|<1$$

$$=1+\sum_{n=1}^{\infty}(n+1)(z+1)^n;\ |z+1|<1$$

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4) ST $f(z) = \dfrac{1}{(1+z)^2}$ with center at $-i$ (or) about $z=-i$ or in powers of $z+i$,

Sol $f(z) = \dfrac{1}{(1+z)^2} = \dfrac{1}{(1-i)^2}$

$= \dfrac{(1+i)^2}{(1-i)^2(1+i)^2}$

$= \dfrac{(1+i)^2}{(1-i^2)^2} = \dfrac{1-1+2i}{4}$

$f(z) = \dfrac{i}{2}$

$f'(z) = \dfrac{-2}{(z+1)^3} = \dfrac{-2}{(1-i)^3}$

$= \dfrac{-2}{(1-i)^3} \times \dfrac{(1+i)^3}{(1+i)^3}$

$= \dfrac{-2(1+i^3+3i+3i^2)}{2^3}$

$= \dfrac{-(1-i+3i-3)}{4} = -\left(\dfrac{-2+2i}{4}\right)$

$f'(z) = \dfrac{1-i}{2}$

$f''(z) = \dfrac{6}{(1+z)^4} \Rightarrow f''(-i) = \dfrac{6}{(1-i)^4}$

$= 6 \cdot \dfrac{i}{2} \cdot \dfrac{i}{2}$

$= \dfrac{-3}{2}$

$$f(z) = f(a) + (z-a) f''(a) + \frac{(z-a)^2}{2!} f''(a) + \cdots$$

$$\Rightarrow \frac{1}{(1+z)^2} = f(-i) + (z+i) f'(-i) + \frac{(z+i)^2}{2!} f''(-i) + \cdots$$

$$= \frac{i}{2} + (z+i)\left(\frac{1-i}{2}\right) + \left[\frac{(z+i)^2}{2!}\right]\left(\frac{-3}{2}\right) + \cdots$$

5) Expand $f(z) = \frac{1}{z^2-z-6}$ about $z = -1$

Sol given: $f(z) = \frac{1}{z^2-z-6}$

$$= \frac{1}{(z-3)(z+2)} = \left[\frac{1}{z-3} - \frac{1}{z+2}\right]\frac{1}{5}$$

Put $z+1 = w \Rightarrow z = w-1$

$$f(z) = \frac{1}{5}\left[\frac{1}{w-1-3} - \frac{1}{w-1+2}\right]$$

$$= \frac{1}{5}\left[\frac{1}{w-4} - \frac{1}{w+1}\right]$$

$$= \frac{1}{5}\left[\frac{1}{-4\left(1-\frac{w}{4}\right)} - \frac{1}{1+w}\right]$$

$$= \frac{1}{5}\left[\frac{-1}{4}\left(1-\frac{w}{4}\right)^{-1} - (1+w)^{-1}\right]$$

$$= \frac{1}{5}\left[\frac{-1}{4}\left(1+\frac{w}{4}+\left(\frac{w}{4}\right)^2+\cdots\right) - \left(1-w+w^2-w^3+\cdots\right)\right]$$

The first series is valid when $\left|\frac{w}{4}\right| < 1$
The second series is valid when $|w| < 1$ $\Big\}$ $|w| < 1$

$$\therefore f(z)=\frac{1}{5}\left[\frac{-1}{4}\left(1+\frac{z+1}{4}+\left(\frac{z+1}{4}\right)^2+\cdots\right]-\right.$$

$$\left[1-(z+1)+(z+1)^2-(z+1)^3+\cdots\right]$$

when $|z+1|<1$

6) Applying Taylors series expansion of e^{1+z} in powers of $z-1$.

Sol

Put $z-1=w$

$z=w+1$

$$f(z)=e^{1+z}=e^{1+w+1}=e^{2+w}$$

$$=e^2\left[1+w+\frac{w^2}{2!}+\cdots\right] \quad |w|<\infty$$

$$=e^2\left[1+(z-1)+\frac{(z-1)^2}{2!}+\cdots\right], \quad |z-1|<\infty$$

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3-10-15

Laurent's series expansion:- ①

1.) Find laurent's series expansion of $\frac{1}{(1-z)(z-2)}$

(i) Maclurians exp of $f(z)$

(ii) Laurents series exp in the region $1<|z|<2$

(iii) Laurents series exp in $|z|>2$

Sol given:- $f(z) = \frac{1}{(1-z)(z-2)}$

$$= -\frac{1}{(z-1)(z-2)}$$

$$= -\left[\frac{1}{z-2} - \frac{1}{z-1}\right]$$

(i) Maclurians exp:

$$f(z) = \frac{1}{z-1} - \frac{1}{z-2}$$

$$= \frac{-1}{1-z} - \frac{1}{-2\left(1-\frac{z}{2}\right)}$$

$$= -(1-z)^{-1} + \frac{1}{2}\left(1-\frac{z}{2}\right)^{-1}$$

$$= -(1+z+z^2+\ldots) + \frac{1}{2}\left[1+\frac{z}{2}+\left(\frac{z}{2}\right)^2+\ldots\right]$$

1st exp is valid when $|z|<1$

2nd exp is valid when $\left|\frac{z}{2}\right|<1$

(ii) $1 < |z| < 2$

i.e $1 < |z|$ & $|z| < 2$

$\Rightarrow \frac{1}{|z|} < 1$; $\frac{|z|}{2} < 1$

$$f(z) = \frac{1}{z-1} - \frac{1}{z-2}$$

$$= \frac{1}{z(1-\frac{1}{z})} + \frac{1}{2(1-\frac{z}{2})}$$

$$= \frac{1}{z}\left(1-\frac{1}{z}\right)^{-1} + \frac{1}{2}\left(1-\frac{z}{2}\right)^{-1}$$

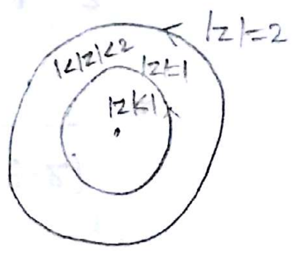
$$= \frac{1}{z}\left(1+\frac{1}{z}+\frac{1}{z^2}+\cdots\right) + \frac{1}{2}\left[1+\frac{z}{2}+\left(\frac{z}{2}\right)^2+\cdots\right]$$

1st exp is valid since $\frac{1}{|z|} < 1$

2nd exp is valid since $\frac{|z|}{2} < 1$

(iii) $|z| > 2$

~~i.e $2 < |z|$~~



$|z| > 2 > 1$

$\Rightarrow |z| > 2$ & $|z| > 1$

$\Rightarrow 2 < |z|$ & $1 < |z|$

$\Rightarrow \frac{2}{|z|} < 1$ & $\frac{1}{|z|} < 1$

$$f(z) = \frac{1}{z-1} - \frac{1}{z-2}$$

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$$= \frac{1}{z\left(1-\frac{1}{z}\right)} - \frac{1}{z\left(1-\frac{2}{z}\right)}$$ (3)

$$= \frac{1}{z}\left(1-\frac{1}{z}\right)^{-1} - \frac{1}{z}\left(1-\frac{2}{z}\right)^{-1}$$

$$= \frac{1}{z}\left(1+\frac{1}{z}+\frac{1}{z^2}+\cdots\right) + \frac{1}{z}\left(1+\frac{2}{z}+\left(\frac{2}{z}\right)^2+\cdots\right.$$

1^{st} exp is valid since $\frac{1}{|z|} < 1$

2^{nd} exp is valid since $\frac{2}{|z|} < 1$

2) Find laurents expansion of $\frac{1}{z^2-4z+3}$ for

a) $1 < |z| < 3$

b) $|z| < 1$

c) $|z| > 3$

Sol
$$f(z) = \frac{1}{z^2-4z+3}$$

$$= \frac{1}{(z-3)(z-1)}$$

$$= \left(\frac{1}{z-3} - \frac{1}{z-1}\right)\frac{1}{2}$$

i) $1 < |z| < 3$

i.e $1 < |z|$ & $|z| < 3$

$\frac{1}{|z|} < 1$ & $\frac{|z|}{3} < 1$

(4)

$$f(z) = \frac{1}{2}\left[\frac{1}{z-3} - \frac{1}{z-1}\right]$$

$$= \frac{1}{2}\left[\frac{1}{-3\left(1-\frac{z}{3}\right)} - \frac{1}{z\left(1-\frac{1}{z}\right)}\right]$$

$$= \frac{1}{2}\left[-\frac{1}{3}\left(1-\frac{z}{3}\right)^{-1} - \frac{1}{z}\left(1-\frac{1}{z}\right)^{-1}\right]$$

$$= \frac{1}{2}\left[\frac{-1}{3}\left(1+\frac{z}{3}+\left(\frac{z}{3}\right)^2+ \cdots\right) - \frac{1}{z}\left(1+\frac{1}{z}+\frac{1}{z^2} \cdots\right)\right]$$

1st exp is valid since $\frac{|z|}{3} < 1$

2nd exp is valid since $\frac{1}{|z|} < 1$

b) $|z| < 1 < 3$

$\Rightarrow \frac{|z|}{1} < 1$ and $|z| < 3 \Rightarrow \frac{|z|}{3} < 1$

$$f(z) = \left(\frac{1}{z-3} - \frac{1}{z-1}\right)\frac{1}{2}$$

$$= \frac{1}{2}\left[\frac{1}{-3\left(1-\frac{z}{3}\right)} + \frac{1}{1-z}\right]$$

$$= \frac{1}{2}\left[-\frac{1}{3}\left(1-\frac{z}{3}\right)^{-1} + (1-z)^{-1}\right]$$

$$= \frac{1}{2}\left[\frac{-1}{3}\left(1+\frac{z}{3}+\left(\frac{z}{3}\right)^2+ \cdots\right) + \left(1+z+z^2+ \cdots\right)\right]$$

1st exp is valid since $\frac{|z|}{3} < 1$

2nd exp is valid since $|z| < 1$

(5)

c) $|z| > 3 > 1$ & $|z| > 1$

i.e. $|z| > 3$, $|z| > 1 \Rightarrow 1 < |z|$

$\frac{|z|}{3} > 1$; $\frac{1}{|z|} < 1$

$\Rightarrow \frac{3}{|z|} < 1$

$$f(z) = \frac{1}{2}\left[\frac{1}{z-3} - \frac{1}{z-1}\right]$$

$$= \frac{1}{2}\left[\frac{1}{-3\left(1-\frac{z}{3}\right)} - \frac{1}{\quad}\right.$$

$$= \frac{1}{2}\left[\frac{1}{z\left(1-\frac{3}{z}\right)} - \frac{1}{z\left(1-\frac{1}{z}\right)}\right]$$

$$= \frac{1}{2}\left[\frac{1}{z}\left(1-\frac{3}{z}\right)^{-1} - \frac{1}{z}\left(1-\frac{1}{z}\right)^{-1}\right]$$

$$= \frac{1}{2}\left[\frac{1}{z}\left(1+\frac{3}{z}+\left(\frac{3}{z}\right)^2+\cdots\right) - \frac{1}{z}\left(1+\frac{1}{z}+\frac{1}{z^2}+\cdots\right.\right.$$

1st exp is valid since $\frac{3}{|z|} < 1$

2nd exp is valid since $\frac{1}{|z|} < 1$

3.) Obtain Laurents exp for $f(z) = \frac{1}{(z+2)(1+z)^2}$ in

a) $|z| < 2$

b) $|1+z| > 1$

c) $|z| < 1$

d) $1 < |z| < 2$

(6)

Sol
$$f(z)=\frac{1}{(z+2)(1+z)^2}$$

$$=\frac{A}{z+2}+\frac{B}{1+z}+\frac{C}{(1+z)^2}$$

$$1=A(1+z)^2+B(1+z)(z+2)+C(z+2)$$

$$\Rightarrow 1=A(1+z^2+2z)+B(z^2+2z+2+z)+C(z+2)$$

z^2 coeff $\Rightarrow$ $0=A+B$

$A=-B$

z coeff $\Rightarrow$ $0=2A+3B+C$

$0=-2B+3B+C$

$B=-C$

const $\Rightarrow$ $1=A+2B+2C$

$1=-B+2B-2B$

$B=-1$

$\therefore C=1$ & $A=1$

$$\therefore f(z)=\frac{1}{z+2}-\frac{1}{1+z}+\frac{1}{(1+z)^2}$$

a) $|z|<2 \Rightarrow \frac{|z|}{2}<1$

$$f(z)=\frac{1}{z+2}-\frac{1}{1+z}+\frac{1}{(1+z)^2}$$

$$=\frac{1}{2\left(1+\frac{z}{2}\right)}-(1+z)^{-1}+(1+z)^{-2}$$

$= \frac{1}{2}\left(1+\frac{z}{2}\right)^{-1} - (1+z)^{-1} + (1+z)^{-2}$ (7)

$= \frac{1}{2}\left(1 - \frac{z}{2} + \left(\frac{z}{2}\right)^2 - \cdots\right) - (1 - z + z^2 - \cdots) + (1 - 2z + 3z^2 \cdots$

1^{st} exp is valid since $\frac{|z|}{2} < 1$

2^{nd} exp & 3^{rd} exp are valid when $|z| < 1$

b) $|1+z| > 1$

Put $1+z = w$; $z = w-1$

$\Rightarrow |w| > 1$

$$f(z) = \frac{1}{z+2} - \frac{1}{1+z} + \frac{1}{(1+z)^2}$$

$$= \frac{1}{w+1} - \frac{1}{w} + \frac{1}{w^2}$$

$$= \frac{1}{w\left(1+\frac{1}{w}\right)} - \frac{1}{w} + \frac{1}{w^2}$$

$$= \frac{1}{w}\left(1+\frac{1}{w}\right)^{-1} - \frac{1}{w} + \frac{1}{w^2}$$

$$= \frac{1}{w}\left(1 - \frac{1}{w} + \left(\frac{1}{w}\right)^2 - \cdots\right) - \frac{1}{w} + \frac{1}{w^2}, \text{ since } \frac{1}{|w|}$$

$$= \frac{1}{z+1}\left[1 - \frac{1}{1+z} + \left(\frac{1}{1+z}\right)^2 \cdots\right] - \frac{1}{1+z} + \frac{1}{(1+z)^2}$$

is valid when $\frac{1}{|1+z|} < 1$

i.e $|1+z| > 1$

c) $|z| < 1 < 2$

$|z| < 1$ & $|z| < 2$

$$\frac{|z|}{2} < 1$$

$$f(z) = \frac{1}{z+2} - \frac{1}{1+z} + \frac{1}{(1+z)^2}$$

$$= \frac{1}{2\left(1+\frac{z}{2}\right)} - \frac{1}{1+z} + \frac{1}{(1+z)^2}$$

$$= \frac{1}{2}\left(1+\frac{z}{2}\right)^{-1} - (1+z)^{-1} + (1+z)^{-2}$$

$$= \frac{1}{2}\left(1 - \frac{z}{2} + \left(\frac{z}{2}\right)^2 - \cdots\right) - \left(1 - z + z^2 - \cdots\right)$$

$$+ \left(1 - 2z + 3z^2 - \cdots\right)$$

1st exp is valid when $\frac{|z|}{2} < 1$

2nd & 3rd exp are valid when $|z| < 1$

d) $1 < |z| < 2$

$1 < |z|$; $|z| < 2$

$\frac{1}{|z|} < 1$; $\frac{|z|}{2} < 1$

$$f(z) = \frac{1}{z+2} - \frac{1}{1+z} + \frac{1}{(1+z)^2}$$

$$= \frac{1}{2\left(1+\frac{z}{2}\right)} - \frac{1}{z\left(1+\frac{1}{z}\right)} + \frac{1}{z\left(1+\frac{1}{z}\right)^2}$$

$$= \frac{1}{2}\left(1+\frac{z}{2}\right)^{-1} - \frac{1}{z}\left(1+\frac{1}{z}\right)^{-1} + \frac{1}{z}\left(1+\frac{1}{z}\right)^{-2}$$

$$= \frac{1}{2}\left(1-\frac{z}{2}+\left(\frac{z}{2}\right)^2 - \cdots\right) - \frac{1}{z}\left(1-\frac{1}{z}-\cdots\right) + \frac{1}{z}\left(1-\frac{2}{z}\cdots\right)$$

1st exp is valid when $\frac{|z|}{2} < 1$

2nd & 3rd exp are valid when $\frac{1}{|z|} < 1$

4) $f(z) = \dfrac{z+3}{z(z^2-z-2)}$

a) $|z| < 1$

b) $1 < |z| < 2$

c) $|z| > 2$

5) $f(z) = \dfrac{z^2-6z-1}{(z-1)(z-3)(z+2)}$ in the region $3 < |z+2| < 5$

Sol $f(z) = \dfrac{A}{z-1} + \dfrac{B}{z-3} + \dfrac{C}{z+2}$

$z^2-6z-1 = A(z-3)(z+2) + B(z-1)(z+2) + C(z-1)(z-3)$

z^2 coeff $\Rightarrow 1 = A+B+C$

z coeff $\Rightarrow -6 = -A+B-4C$

const $\Rightarrow -1 = -6A-2B+3C$

On solving,

$A=1$, $B=-1$, $C=1$

$$f(z) = \frac{1}{z-1} - \frac{1}{z-3} + \frac{1}{z+2}$$

$3 < |z+2| < 5$

i.e $3 < |z+2|$; $|z+2| < 5$

Put $\cancel{z=}\, z+2 = w$

$z = w-2$

$3 < |w|$; $|w| < 5$

$\frac{3}{|w|} < 1$; $\frac{|w|}{5} < 1$

$$f(z) = \frac{1}{w-3} - \frac{1}{w-5} + \frac{1}{w}$$

$$= \frac{1}{w\left(1-\frac{3}{w}\right)} - \frac{1}{-5\left(1-\frac{w}{5}\right)} + \frac{1}{w}$$

$$= \frac{1}{w}\left(1-\frac{3}{w}\right)^{-1} + \frac{1}{5}\left(1-\frac{w}{5}\right)^{-1} + \frac{1}{w}.$$

$$= \frac{1}{w}\left(1+\frac{3}{w}+\left(\frac{3}{w}\right)^2+\cdots\right) + \frac{1}{5}\left(1+\frac{w}{5}+\left(\frac{w}{5}\right)^2\cdots\right) + \frac{1}{w}$$

1st exp is valid when $\frac{3}{|w|} < 1$

2nd exp is valid when $\frac{|w|}{5} < 1$

6.) Expand $\dfrac{7z-2}{(z+1)z(z-2)}$ about the points $z=-1$,

$1<|z+1|<3$ as Laurents series.

7.) Expand $f(z)=\dfrac{1}{z^2-3z+2}$ in the region

(i) $0<|z-1|<1$

(ii) $1<|z|<2$

8.) $f(z)=\dfrac{(z-2)(z+2)}{(z+1)(z+4)}$, $1<|z|<4$, $|z|<1$

9.) $f(z)=\dfrac{2z^3+1}{z^2+z}$, find Taylors exp in neighbourhood of i, i.e $z=i$. A laurents series valid within the annular of which centre is origin.

10.) $f(z)=\dfrac{4z+4}{z(z-3)(z+2)}$, $|z|\le 1$, $1\le|z|\le 2$, $|z|>2$

11.) $f(z)=\dfrac{1}{z(z+2)^3(z+1)^2}$, $\frac{5}{4}\le|z|\le\frac{7}{4}$

12.) $f(z)=\dfrac{1}{z^2(z-3)^2}$, $|z|<1$, $|z|>3$

13.) $f(z)=\dfrac{1}{(z^2+1)(z^2+2)}$, $|z|<1$, $|z|>\sqrt{2}$

14.) $f(z)=\dfrac{z}{(z+1)(z+2)}$ about $z=2$

15) $f(z) = \dfrac{z^2-4}{z^2+5z+4}$, $1<|z|<4$

16) $f(z) = \dfrac{4z+3}{(z-3)(z+2)}$ in the circular region b/w

$|z|=2$ & $|z|=3$

3-10-15

Ch:7. Contour Integration

Zero of an analytic fn:-

A point $z=a$ is said to be a zero of a analytic fn if $f(a)=0$.

Note: If $f(z) = (z-a)^n \phi(z)$ where $\phi(z)$ is an analytic fn & $\phi(a) \neq 0$, then $z=a$ is a zero of $f(z)$ of order n.

Eg:- $f(z) = \dfrac{z^3 (z+2)}{(z-3)(z-2)^3}$

Here $z=0$ is zero of $f(z)$ of order 3

and $z=-2$ is zero of $f(z)$ of order 1

Singular point:-

A pt $z=a$ is said to be a singular point of an analytic fn $f(z)$ if $f(z)$ fails to be analytic at $z=a$.

Eg:- $f(z) = \dfrac{z^3 (z+2)}{(z-3)(z-2)^3}$

Here the fn fails to be analytic at $z=3,2$

$z=3$ & 2 are singular points.

Isolated singular point:-

A pt $z=a$ is said to be an isolated singular pt of an analytic fn $f(z)$ if

(i) $f(z)$ is not analytic at $z=a$

(ii) $f(z)$ is analytic in the deleted neighbourhood of $z=a$.

Pole of an analytic fn:-

If $z=a$ is an isolated singular pt of an analytic fn $f(z)$, then $f(z)$ can be expanded in Laurents series as given below.

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-a)^n}$$

where $$a_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-a)^{n+1}} dz$$

$$b_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-a)^{-n+1}} dz$$

Here the series $\sum_{n=1}^{\infty} \frac{b_n}{(z-a)^n}$ is known as the principle part of $f(z)$.

Pole of order n:-

If $b_n = 0 \ \forall \ n \geq m$ i.e

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \frac{b_1}{(z-a)} + \frac{b_2}{(z-a)^2} + \cdots \frac{b_m}{(z-a)^m}$$

then $z=a$ is called pole of order m

Eg:- $f(z) = \dfrac{z^3(z+2)}{(z-3)(z-2)^3}$

$z=2$ is a pole of order 3

A pole of order 1 is called a simple pole.

$\therefore z=3$ is simple pole.

Essential singularity:-

If $b_n \neq 0 \ \forall \ n$

i.e $f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \dfrac{b_1}{z-a} + \dfrac{b_2}{(z-a)^2} + \cdots \dfrac{b_m}{(z-a)^m} + \cdots$

then $z=a$ is called a essential singular pt of $f(z)$ ~~pole of order m~~.

Eg:- $f(z) = e^{1/z}$

$$= 1 + \frac{1}{z} + \frac{1}{z^2}\frac{1}{2!} + \frac{1}{z^3}\frac{1}{3!} + \cdots$$

$\therefore z=0$ is an essential singular pt.

Removable singularity:-

If $b_n = 0 \ \forall \ n$

i.e $f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n$, then $z=a$ is called removable ~~regular~~ singular pt of $f(z)$.

Eg:- (i) $f(z) = \frac{\sin z}{z}$

$$= \frac{1}{z}\left[\cancel{1}z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots\right]$$

$$= 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots$$

$\therefore z = 0$ is a ~~regular~~ removable singular pt.

(ii) $f(z) = \frac{1-\cos z}{z}$

Residue of $f(z)$ at $z = a$:-

Residue of an analytic fn $f(z)$ at a singular point $z = a$ is the coeff of $\frac{1}{z-a}$ in the Laurents expansion, $f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-a)^n}$

i.e coeff of $\frac{1}{z-a} = b_1 = \frac{1}{2\pi i}\int_C f(z)\,dz$

$$\because b_n = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-a)^{-n+1}}\,dz$$

$$b_1 = \frac{1}{2\pi i}\int_C f(z)\,dz$$

$\therefore$ Residue of $f(z)$ at $z = a$ is

$$\operatorname*{Res}_{z=a} f(z) = \frac{1}{2\pi i}\int_C f(z)\,dz$$ when $z = a$ is a singular point of $f(z)$ lying inside C.

Cauchy's residue theorem:

Let $f(z)$ be analytic on and within a closed curve C except at a finite singular points $z_1, z_2 \cdots z_n$. Let $R_1, R_2 \cdots R_n$ be the residues of then singular points respectively, then

$$\int_C f(z)\,dz = 2\pi i [R_1 + R_2 + \cdots R_n]$$

Proof: given: $f(z)$ is analytic on & within C except at $z_1, z_2, \cdots z_n$

Draw circles $C_1, C_2 \cdots C_n$ with centres at $z_1, z_2 \cdots z_n$ and radii $r_1, r_2, \cdots r_n$ respectively so small such that the circles entirely lie inside C and do not intersect each other.

By cauchy's multiconnected theorem,

$$\int_C f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz + \cdots + \int_{C_n} f(z)\,dz \quad \text{--- (1)}$$

By def of residue,

$$\operatorname*{Res}_{z=z_1} f(z) = R_1 = \frac{1}{2\pi i}\int_{C_1} f(z)\,dz \Rightarrow \int_{C_1} f(z)\,dz = 2\pi i R_1$$

$$\operatorname*{Res}_{z=z_2} f(z) = R_2 = \frac{1}{2\pi i}\int_{C_2} f(z)\,dz \Rightarrow \int_{C_2} f(z)\,dz = 2\pi i R_2$$

$$\vdots$$

$$\operatorname*{Res}_{z=z_n} f(z) = R_n = \frac{1}{2\pi i}\int_{C_n} f(z)\,dz \Rightarrow \int_{C_n} f(z)\,dz = 2\pi i R_n \quad \text{--- (2)}$$

From (1) & (2),

$$\int_C f(z)\,dz = 2\pi i R_1 + 2\pi i R_2 + \dots + 2\pi i R_n$$

$$\int_C f(z)\,dz = 2\pi i\,(R_1 + R_2 + \dots R_n) \rightarrow \text{Proved}$$

<u>Note</u>: By cauchys residue theorem, $\int_C f(z)\,dz = 2\pi i \times$ sum of residues at poles which lie inside C.

* Residue of $f(z)$ at $z=a$ of order m is

$$\operatorname*{Res}_{z=a} f(z) = \lim_{z\to a} \frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}\left[(z-a)^m f(z)\right]$$

P1) Find residues of $\frac{ze^z}{(z-1)^3}$ at its pole

<u>Sol</u> given: $f(z) = \frac{ze^z}{(z-1)^3}$

$z=1$ is a pole of $f(z)$ of order $3 \Rightarrow a=1$ & $m=3$

$$\therefore \operatorname*{Res}_{z=1} f(z) = \lim_{z\to 1}\frac{1}{2!}\frac{d^2}{dz^2}\left[\cancel{(z-1)^3}\cdot\frac{ze^z}{\cancel{(z-1)^3}}\right]$$

$$= \lim_{z\to 1}\frac{1}{2}\frac{d}{dz}(ze^z + e^z)$$

$$= \lim_{z\to 1}\frac{1}{2}\cancel{\frac{d}{dz}}(ze^z + e^z + e^z)$$

$$= \frac{1}{2}(3e) = \frac{3e}{2}$$

2) Find residues of $\frac{z+1}{z^2(z-2)}$ at its poles

<u>Sol</u> $z=0$ is a pole of $f(z)$ order 2

$z=2$ is a pole of $f(z)$ order 1

$$\text{WKT } \underset{z=0}{\text{Res}}\, f(z) = \lim_{z\to 0} \frac{1}{1!} \frac{d}{dz}\left[\cancel{z^2} \cdot \frac{z+1}{\cancel{z^2}(z-2)}\right]$$

$$= \lim_{z\to 0} \frac{\cancel{z}-2-\cancel{z}-1}{(z-2)^2}$$

$$= \frac{-2-1}{4} = \frac{-3}{4}$$

$$\underset{z=2}{\text{Res}}\, f(z) = \lim_{z\to 2} \frac{1}{1!} \cancel{(z-2)} \frac{z+1}{z^2\cancel{(z-2)}}$$

$$= \frac{3}{2^2} = \frac{3}{4}$$

3) Find residues of $f(z) = \frac{z^3}{(z-1)^4(z-2)(z-3)}$ at $z=1$

<u>Sol</u> $z=1$ is a pole of order 4

$$\underset{z=1}{\text{Res}}\, f(z) = \lim_{z\to 1} \frac{1}{3!} \frac{d^3}{dz^3}\left[\frac{\cancel{(z-1)^4} \cdot z^3}{\cancel{(z-1)^4}(z-2)(z-3)}\right]$$

$z^2-3z-2z+6$

$$\cancel{= \frac{1}{6} \lim_{z\to 1} \frac{d^2}{dz^2}\left[\frac{3z^2(z-2)(z-3)-(2z-5)z^3}{(z^2-3z-2z+6)^2}\right]}$$

$$= \lim_{z\to 1} \frac{1}{6} \frac{d^3}{dz^3}\left[\frac{z^3}{z^2-5z+6}\right]$$

$$= \lim_{z\to 1} \frac{1}{6}\frac{d^3}{dz^3}\left[z+5+\frac{19z-30}{z^2-5z+6}\right]$$

$z^2-5z+6)z^3(z+5$

z^3-5z^2+6z

$5z^2-6z$

$5z^2-25z+30$

$19z-30$

$$= \lim_{z\to 1} \frac{1}{6}\frac{d^3}{dz^3}\left[z+5+\frac{19z-30}{(z-2)(z-3)}\right]$$

$$= \lim_{z\to 1} \frac{1}{6}\frac{d^3}{dz^3}\left[z+5-\frac{8}{z-2}+\frac{27}{z-3}\right]$$

$\frac{19z-30}{(z-2)(z-3)} = \frac{A}{z-2}+\frac{B}{z-3}$

$$= \lim_{z\to 1} \frac{1}{6}\frac{d^2}{dz^2}\left[1+\frac{8}{(z-2)^2}-\frac{27}{(z-3)^2}\right]$$

$19z-30 = A(z-3)+B(z-2)$

$A=-8 \; ; B=27$

$$= \lim_{z\to 1} \frac{1}{6}\frac{d}{dz}\left[-\frac{8\cdot 2}{(z-2)^3}+\frac{27\cdot 2}{(z-3)^3}\right]$$

$$= \lim_{z\to 1} \frac{1}{6}\left[\frac{8\cdot \cancel{2}\cdot 3}{(z-2)^4}-\frac{27\cdot \cancel{2}\cdot 3}{(z-3)^4}\right]$$

$$= \frac{8}{1}-\frac{27}{16}$$

$$= \frac{101}{16}$$

4) Find residue of $f(z) = \frac{1+e^z}{\sin z + z\cos z}$ at $z=0$

Sol $z=0$ is a pole of $f(z)$ of order 1

WKT $\operatorname{Res}_{z=0} f(z) = \lim_{z\to 0} z\frac{(1+e^z)}{\sin z + z\cos z}$

$$= \lim_{z\to 0} \frac{\cancel{z}(1+e^z)}{\cancel{z}\left(\frac{\sin z}{z}+\cos z\right)}$$

$$= \frac{1+1}{\dots} = 1$$

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5) Find residue of $f(z) = \frac{z^2}{z^4-1}$ at each pole

Sol given: $f(z) = \frac{z^2}{z^4-1}$

Pole of $f(z)$ are obtained from $z^4-1=0$

i.e $(z^2)^2 - 1^2 = 0$

$\Rightarrow (z^2-1)(z^2+1)=0$

$\Rightarrow (z+1)(z-1)(z+i)(z-i)=0$

$z = \pm 1, \pm i$ are

poles of $f(z)$ of order 1

$$\operatorname*{Res}_{z=1} f(z) = \lim_{z \to 1} \cancel{(z-1)} \cdot \frac{z^2}{(z+1)\cancel{(z-1)}(z^2+1)}$$

$$= \frac{1}{2(2)} = \frac{1}{4}$$

$$\operatorname*{Res}_{z=-1} f(z) = \lim_{z \to -1} \cancel{(z+1)} \cdot \frac{z^2}{\cancel{(z+1)}(z-1)(z^2+1)}$$

$$= \frac{1}{-2(2)} = \frac{-1}{4}$$

$$\operatorname*{Res}_{z=i} f(z) = \lim_{z \to i} \cancel{(z-i)} \cdot \frac{z^2}{(z^2-1)(z+i)\cancel{(z-i)}}$$

$$= \frac{-1}{-2(2i)} = \frac{+1}{4i}$$

$$\underset{z=-i}{\text{Res}} = \lim_{z\to -i} \cancel{(z+i)} \frac{z^2}{(z^2-1)\cancel{(z+i)}(z-i)}$$

$$= \frac{-1}{-2(-2i)} = \frac{-1}{4i}$$

6.) Find residue of $f(z) = \dfrac{\cos(z-i)}{(z+2i)^3}$ at its pole

7) $f(z) = \dfrac{1}{z(e^z-1)}$ at $z=0$

Sol Poles of $f(z)$ are

$z=0$ & $e^z - 1 = 0$

$\Rightarrow e^z = 1$

$\Rightarrow e^z = e^{2n\pi i}$

$z = 2n\pi i$; $n = 0, \pm1, \pm2 \dots$

$$\underset{z=0}{\text{Res}\, f(z)} = \lim_{z\to 0} \cancel{z} \cdot \frac{1}{\cancel{z}(e^z-1)}$$

$$= \lim_{z\to 0} \frac{1}{e^z - 1}$$

$$= \lim_{z\to 0} \frac{1}{\left(\cancel{1}+z+\frac{z^2}{2!}+\dots\right) - \cancel{1}}$$

$$= \lim_{z\to 0} \frac{1}{z\left[1+\left(\frac{z}{2!}+\frac{z^2}{3!}+\dots\right)\right]}$$

$$= \lim_{z\to 0} \frac{1}{z}\left[1+\left(\frac{z}{2!}+\frac{z^2}{3!}+\dots\right)\right]^{-1}$$

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$$= \lim_{z\to 0} \frac{1}{z}\left[1-\left(\frac{z}{2!}+\frac{z^2}{3!}+\cdots\right)+\left(\frac{z}{2!}+\frac{z^2}{3!}+\cdots\right)^2\right]$$

$$= \lim_{z\to 0}\left[\frac{-1}{2!}+\frac{2z}{3!}+\cdots+2\left(\frac{z}{2!}+\frac{z^2}{3!}+\cdots\right)\left(\frac{1}{2!}+\frac{2z}{3!}\right.\right.$$

↑
using L-Hosp rule

$$= \frac{-1}{2!} = \frac{-1}{2}$$

HW

8) $f(z) = \dfrac{z^2-2z}{(z+1)^2(z^2+1)}$ at each pole.

9) Evaluate $\displaystyle\int_C \frac{4-3z}{z(z-1)(z-2)}\,dz$; $C: |z| = \frac{3}{2}$ by using cauchys residue theorem.

Sol $\quad f(z) = \dfrac{4-3z}{z(z-1)(z-2)}$

$z=0, z=1, z=2$ are poles of order 1

$z=0,1$ lie inside C, $z=2$ lies outside C

$$\operatorname*{Res}_{z=0} f(z) = \lim_{z\to 0} \cancel{z}\cdot\frac{4-3z}{\cancel{z}(z-1)(z-2)}$$

$$= \frac{4}{2} = 2$$

(or)

formula: $\operatorname*{Res}_{z=a} f(z) = \dfrac{\phi(a)}{\psi'(z)}$

$$\operatorname*{Res}_{z=1} f(z) = \lim_{z\to 1} \frac{\cancel{(z-1)}(4-3z)}{z\cancel{(z-1)}(z-2)}$$

$$= \frac{1}{-1} = -1$$

By cauchys residue theorem,

$$\int_C f(z)\,dz = 2\pi i(\text{sum of residues})$$

$$\Rightarrow \int_C \frac{4-3z}{z(z-1)(z-2)}\,dz = 2\pi i\,[2+(-1)]$$

$$= 2\pi i$$

10) Evaluate $\int_C \frac{z^2}{(z-1)^2(z+2)}\,dz$, $C: |z|=3$

11) Evaluate $\int_C \frac{z-3}{z^2+2z+5}\,dz$; $C: |z+1+i|=2$ (ans: $\pi(2+i)$)

12) Evaluate $\int_C \frac{e^z}{(z^2+\pi^2)^2}\,dz$, $C: |z|=4$ (ans: $\frac{i}{\pi}$)

13) Evaluate $\int_C \frac{\sin z}{z\cos z}\,dz$, $C: |z|=\pi$ (ans: 0)

14) Evaluate $\int_C \frac{z-1}{(z+1)^2(z-2)}$, $C: |z-i|=2$ (ans: $\frac{-2\pi i}{9}$)

15) Evaluate $\int_C \frac{ze^z}{z^2+9}\,dz$, $C: |z|=5$

Integrals of type $\int_0^{2\pi} F(\cos\theta, \sin\theta)\, d\theta$:-

Put $e^{i\theta} = z$

$$\Rightarrow e^{i\theta}\, i\, d\theta = dz$$

$$\Rightarrow d\theta = \frac{dz}{iz}$$

and $|e^{i\theta}| = |z|$

$$\Rightarrow 1 = |z| : C$$

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} = \left(z + \frac{1}{z}\right)\frac{1}{2}$$

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} = \left(z - \frac{1}{z}\right)\frac{1}{2i}$$

$$\cos 2\theta = \frac{e^{i2\theta} + e^{-i2\theta}}{2} = \left(z^2 + \frac{1}{z^2}\right)\frac{1}{2}$$

$$\sin 2\theta = \frac{e^{i2\theta} - e^{-i2\theta}}{2i} = \left(z^2 - \frac{1}{z^2}\right)\frac{1}{2i}$$

$$\cos 4\theta = \left(z^4 + \frac{1}{z^4}\right)\frac{1}{2}$$

$$\sin 4\theta = \left(z^4 - \frac{1}{z^4}\right)\frac{1}{2i}$$

(1) Evaluate $\int_0^{2\pi} \frac{\cos 2\theta}{5+4\cos\theta}\,d\theta$

Sol

Put $e^{i\theta} = z$

$\Rightarrow e^{i\theta}\, i\, d\theta = dz$

$\Rightarrow d\theta = \frac{dz}{iz}$

& $|e^{i\theta}| = |z|$

$\Rightarrow 1 = |z| : c$

$$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} = \left(z+\frac{1}{z}\right)\frac{1}{2}$$

$$\cos 2\theta = \frac{e^{i2\theta}+e^{-i2\theta}}{2} = \left(z^2+\frac{1}{z^2}\right)\frac{1}{2}$$

$$\therefore I = \int_c \frac{\left(z^2+\frac{1}{z^2}\right)\frac{1}{2}}{5+4\left(z+\frac{1}{z}\right)\frac{1}{2}}\,\frac{dz}{iz}, \quad c:|z|=1$$

$$= \frac{1}{2i}\int_c \frac{z^4+1}{z^2}\cdot\frac{1}{5+\left(z+\frac{1}{z}\right)2}\,\frac{dz}{z}$$

$$= \frac{1}{2i}\int_c \frac{z^4+1}{z^2}\cdot\frac{1}{5z+2z^2+2}\cdot\frac{\cancel{z}}{\cancel{z}}\,dz$$

$$= \frac{1}{2i}\int_c \frac{z^4+1}{z^2(2z^2+5z+2)}\,dz$$

Let $f(z) = \frac{z^4+1}{z^2(2z^2+5z+2)}$

Poles are obtained from $z^2(2z^2+5z+2) = 0$

$$\Rightarrow z^2(2z+1)(z+2)=0$$

$z=0$ is a pole of order 2

$z=\frac{-1}{2}, -2$ are poles of order 1

But $z=0$ and $z=\frac{-1}{2}$ only lie inside $C: |z|=1$

$\therefore$ We find residues at $z=0, \frac{-1}{2}$

$$\underset{z=0}{\text{Res}}\, f(z) = \lim_{z\to 0} \frac{d}{dz}\left[z^2 \cdot \frac{z^4+1}{z^2(2z^2+5z+2)}\right]$$

$$= \lim_{z\to 0} \frac{(2z^2+5z+2)(4z^3)-(z^4+1)(4z+5)}{(2z^2+5z+2)^2}$$

$$= -\frac{5}{4}$$

$$\underset{z=\frac{-1}{2}}{\text{Res}}\, f(z) = \lim_{z\to\frac{-1}{2}} \left(z+\frac{1}{2}\right) \cdot \frac{z^4+1}{z^2(2)(z+2)\left(z+\frac{1}{2}\right)}$$

$$= \frac{\frac{1}{16}+1}{\frac{2}{4}\left(\frac{-1}{2}+2\right)}$$

$$= \frac{17}{12}$$

By Cauchy's residue theorem,

$$\int \frac{z^4+1}{z^2(2z^2+5z+2)}\,dz = 2\pi i\,(\text{Sum of residues})$$

$$= 2\pi i\left(-\frac{5}{4}+\frac{17}{12}\right)$$

$$= \frac{\pi i}{3}$$

hence $I = \int_0^{2\pi} \frac{\cos 2\theta}{5+4\cos\theta}\, d\theta$

$$= \frac{1}{2i}\int \frac{z^4+1}{z^2(2z^2+5z+2)}\, dz$$

$$= \frac{1}{2i}\cdot\frac{\pi i}{3}$$

$$= \frac{\pi}{6}$$

$1-2a\left[\frac{z^2+1}{z}\right]$

2) ST $\quad I = \int_0^{\pi} \frac{\cos 2\theta}{1-2a\cos\theta+a^2}\, d\theta = \frac{\pi a^2}{1-a^2} \quad (a^2<1)$

Sol $\quad I = \frac{1}{2}\int_0^{2\pi} \frac{\cos 2\theta}{1-2a\cos\theta+a^2}\, d\theta$

$$= \frac{1}{2}\ \text{RP of} \int_0^{2\pi} \frac{e^{i2\theta}}{1-2a\cos\theta+a^2}\, d\theta$$

$2z^2+4z+z+2$

$2z[z+2]+1[z+2]$

$[2z+1][z+2]$

put $e^{i\theta}=z$, $|z|=1$: C

$$\cos\theta = \left(z+\frac{1}{z}\right)\frac{1}{2}$$

$$e^{i2\theta} = z^2$$

$$\therefore I = \text{RP}\ \frac{1}{2}\int \frac{z^2}{1-2a\left(z+\frac{1}{z}\right)\frac{1}{2}+a^2}\,\frac{dz}{iz}$$

2

$$= RP\ \frac{1}{2}\int_C \frac{z^2}{z-az^2-a+a^2z}\cdot\frac{z}{iz}\,dz$$

$$= RP\cdot\frac{1}{2i}\int_C \frac{z^2}{az^2-(a^2+1)z+a}\,dz$$

$$f(z) = \frac{z^2}{az^2-(a^2+1)z+a}$$

$$z = \frac{a^2+1 \pm \sqrt{(a^2+1)^2-4a^2}}{2a}$$

$$z = \frac{a^2+1 \pm \sqrt{(a^2-1)^2}}{2a}$$

$$z = \frac{a^2+1 \pm (a^2-1)}{2a}$$

$z = a, \frac{1}{a}$ are poles of order 1

& $z=a$ lies inside C

$z=\frac{1}{a}$ lies outside C

$$\operatorname*{Res}_{z=a} f(z) = \lim_{z\to a} \cancel{(z-a)}\cdot\frac{z^2}{a\cancel{(z-a)}\left(z-\frac{1}{a}\right)}$$

$$= \frac{a^2}{a\left(a-\frac{1}{a}\right)} = \frac{a^2}{a^2-1}$$

$$I = RP\ \frac{1}{2i}\int_C \frac{z^2}{az^2-(a^2+1)z+a}\,dz$$

$$= RP \frac{(-1)}{2i} 2\pi i \frac{a^2}{a^2-1}$$

$$= \frac{\pi a^2}{1-a^2} \longrightarrow \text{Proved}$$

13-10-15

Unit: 5 Conformal mapping

Mapping / Transformation / Function:-

The relation defined by the equ $f(z) = \omega$ b/w the points in z-plane & ω-plane is called a mapping or transformation from z-plane to ω-plane.

Conformal mapping:-

Let $f(z) = \omega$ be a transformation from z-plane to ω-plane. Let 'P' be a pt in z-plane & P' be its image in ω-plane. Suppose C_1 & C_2 are any 2 curves intersecting at P, Suppose the mapping takes C_1 & C_2 into the curves C_1' & C_2' which intersect at P'.

If the transformation is such that the angle b/w C_1 & C_2 at P is equal both in magnitude & sense to the angle b/w C_1' & C_2' at P', then it is said to be a conformal transformation at P.

Isogonal:-

If the trans. preserves the magnitude but not necessarily sense, then it is called isogonal.

Note: Sufficient condition for $f(z)=w$ to represent a conformal mapping is $f'(z)\neq 0 \ \forall z$ in its domain.

Standard transformations:-

1) Translation:-

A transformation ~~f(z~~ $w=f(z)=z+c$ where c is a complex constant is called a complex translation.

Here a figure in z-plane transforms to a figure of same size & shape in w-plane but the figure will be shifted through a distance given by c.

2) Expansion, Contraction, Rotation:-

A transformation $w=f(z)=cz$ where c is a complex constant is said to be expansion when $|c|>1$, contraction when $0<|c|<1$, rotation when $|c|=1$.

Here any figure in z-plane transforms into a singular figure in ω-plane.

3) Linear transformation:-

A transformation $\omega=f(z)=az+c$ where a and c are complex constants is called linear transformations.

Result:- Circles are invariant under linear transformation.

Reflection:-

A transformation $\omega=f(z)=\frac{1}{z}$ is called reflection.

1) Find the points at which $w = \cosh z$ is not conformal.

Sol given:- $w = \cosh z = f(z)$

The points at which transformation is not conformal are obtained from $f'(z) = 0$

i.e $\sinh z = 0$

$$\Rightarrow \frac{e^{z} - e^{-z}}{2} = 0$$

$$\Rightarrow e^{z} - \frac{1}{e^{z}} = 0$$

$$\Rightarrow e^{2z} - 1 = 0$$

$$\Rightarrow e^{2z} = 1$$

$$\Rightarrow 2z = 2n\pi i$$

$$\Rightarrow z = n\pi i, \quad n = 0, \pm 1, \pm 2$$

2) Find the Image of $|z| = 2$ under the transf. $w = 3z$.

Sol given : $w = f(z) = 3z$

Let $z = x+iy$, $w = u+iv$

$$\Rightarrow u+iv = 3(x+iy)$$

$u = 3x$; $v = 3y$

$x = \frac{u}{3}$; $y = \frac{v}{3}$ —— (1)

given: a circle $|z|=2$; $|x+iy|=2$

$$x^2+y^2=4$$

in z-plane. —②

From ① & ②

$$\left(\frac{u}{3}\right)^2+\left(\frac{v}{3}\right)^2=4$$

$$\Rightarrow u^2+v^2=36$$

which is a circle with center at origin and radius 6 in w-plane.

3) For the mapping $w=\frac{1}{z}$, find the image of family of circles $x^2+y^2=ax$ where a is real.

Sol

given: $w=f(z)=\frac{1}{z} \Rightarrow z=\frac{1}{w}$

Let $z=x+iy$, $w=u+iv$

$$\Rightarrow x+iy=\frac{1}{u+iv}=\frac{u-iv}{u^2+v^2}$$

$$x=\frac{u}{u^2+v^2} \quad ; \quad y=\frac{-v}{u^2+v^2} \quad \text{—①}$$

given a family of circles $x^2+y^2=ax$ —②

in z-plane

From ① & ②

$$\left(\frac{u}{u^2+v^2}\right)^2+\left(\frac{-v}{u^2+v^2}\right)^2=\frac{a\cdot u}{u^2+v^2}$$

$$\Rightarrow \frac{u^2}{(u^2+v^2)^2}+\frac{v^2}{(u^2+v^2)^2}=\frac{au}{\cancel{u^2+v^2}}$$

$$\Rightarrow \quad 1=au$$

$\Rightarrow \quad u=\frac{1}{a}$, family of St. lines in w-plane

is the image of $x^2+y^2=ax$.

4) Under the trans. $w=\frac{1}{z}$, find the image of the circle $|z-2i|=2$,

5) Find image of circle $|z|=c$ under trans. $w=z-2+4i$.

6) Find & plot the image of the triangular region with vertices $(0,0)$ $(1,0)$ $(0,1)$ under transf. $w=(1-i)z+3$

7) Find the image of a Δ^{le} with vertices at $i, 1+i, 1-i$ in the z-plane under the transformation (i) $w=3z+4-2i$ and (ii) $w=e^{\frac{5\pi i}{3}}z-2+4i$

(iii) $w=e^{\frac{5\pi i}{3}}(z-2+4i)$

14-10-15

P1) Find image of infinite strip $0 < y < \frac{1}{2}$ under the transformation $w = \frac{1}{z}$

Sol

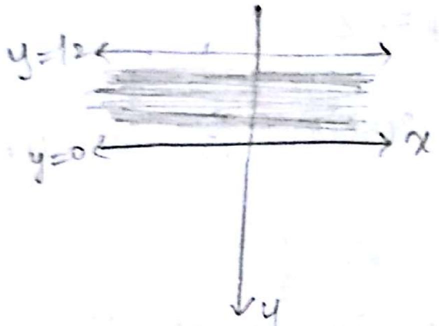
given : $w = \frac{1}{z}$

$$\Rightarrow z = \frac{1}{w}$$

$$\Rightarrow x + iy = \frac{1}{u + iv}$$

$$x = \frac{u}{u^2+v^2} \; ; \; y = \frac{-v}{u^2+v^2} \quad \text{--- (1)}$$

given: an infinite strip $0 < y < \frac{1}{2}$ in z-plane.



To find the image of $y=0$

using (1) $\frac{-v}{u^2+v^2} = 0$

$\Rightarrow v = 0$, a st. line in w-plane

To find the image of $y = \frac{1}{2}$

using (1), $\frac{-v}{u^2+v^2} = \frac{1}{2}$

$$\Rightarrow u^2 + v^2 = -2v$$

$\Rightarrow u^2+v^2+2v=0$

$\Rightarrow u^2+v^2+2v+1=1$

$\Rightarrow u^2+(v+1)^2=1$

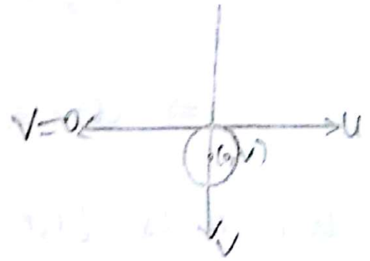
a circle with centre $(0,-1)$ & radius 1

is w-plane.

Hence the image of $0<y<\frac{1}{2}$ is the region

lying b/w the st. line $v=0$ & the circle

$u^2+(v+1)^2=1$



2) ST the image of hyperbola $x^2-y^2=1$ under

the trans. $w=\frac{1}{z}$ is the lamniscate $\rho^2=\cos 2\phi$

Sol given: $\rho^2=\cos 2\phi$

$w=\frac{1}{z}$

$\Rightarrow z=\frac{1}{w}$

Let $z=re^{i\theta}$, $w=\rho e^{i\phi}$

$\Rightarrow x+iy=r\cos\theta+ir\sin\theta$

$\Rightarrow x=r\cos\theta,\ y=r\sin\theta$ —— (1)

Now $z=\frac{1}{w}$ $\Rightarrow re^{i\theta}=\frac{1}{\rho e^{i\phi}}$

$$re^{i\theta} = \frac{e^{-i\phi}}{p} \Rightarrow r = \frac{1}{p} \quad \& \quad \theta = -\phi$$
$\rightarrow$ (2)

To find the image of $x^2 - y^2 = 1$

$$r^2\cos^2\theta - r^2\sin^2\theta = 1, \text{ using (1)}$$

$$\Rightarrow r^2(\cos^2\theta - \sin^2\theta) = 1$$

$$\Rightarrow \frac{1}{p^2}\left[\cos^2(-\phi) - \sin^2(-\phi)\right] = 1, \text{ using (2)}$$

$$\Rightarrow \cos^2\phi - \sin^2\phi = p^2$$

$$\Rightarrow \cos 2\phi = p^2$$

which is the image of $x^2 - y^2 = 1$.

Hence Proved.

3) ST the transformation $\omega = z + \frac{1}{z}$ converts the st. line $\arg z = a$ where $|a| < \frac{\pi}{2}$ into a branch of hyperbola of eccentricity $\sec a$.

Sol given: $\omega = z + \frac{1}{z}$

Let $\omega = u + iv$ & $z = re^{i\theta}$

$$\Rightarrow u + iv = re^{i\theta} + \frac{1}{re^{i\theta}}$$

$$u + iv = r(\cos\theta + i\sin\theta) + \frac{\cos\theta - i\sin\theta}{r}$$

$$u+iv = r\cos\theta + \frac{\cos\theta}{r} + i\left(r\sin\theta - \frac{\sin\theta}{r}\right)$$

comparing $\Rightarrow$ $u = \cos\theta\left(r+\frac{1}{r}\right)$

$\quad$ & $v = \sin\theta\left(r-\frac{1}{r}\right)$ —①

WKT $\left(r+\frac{1}{r}\right)^2 - \left(r-\frac{1}{r}\right)^2 = 4$

$\Rightarrow \left(\frac{u}{\cos\theta}\right)^2 - \left(\frac{v}{\sin\theta}\right)^2 = 4$, from ①

since $\arg z = a$ i.e $\theta = a$

we get $\frac{u^2}{\cos^2 a} - \frac{v^2}{\sin^2 a} = 4$

$\Rightarrow \frac{u^2}{4\cos^2 a} - \frac{v^2}{4\sin^2 a} = 1$, a hyperbola

is the image of $\arg z = a$

WKT $(\text{eccentricity})^2 = e^2 = 1 + \frac{b^2}{a^2}$

$$= 1 + \frac{\cancel{4}\sin^2 a}{\cancel{4}\cos^2 a}$$

$$= \frac{1}{\cos^2 a}$$

$$e^2 = \sec^2 a$$

$\Rightarrow e = \sec a \rightarrow$ proved.

Scanned by CamScanner

Bilinear transformations:-

A transformation $f(z) = w = \frac{az+b}{cz+d}$ where $ad-bc \neq 0$ is called a bilinear transformation.

Note: Bilinear transformation is a conformal transformation.

Proof: Let $f(z) = \frac{az+b}{cz+d}$ be a B.trans, then $ad-bc \neq 0$

$$\Rightarrow f'(z) = \frac{(cz+d)a - (az+b)c}{(cz+d)^2}$$

$$= \frac{a\cancel{cz} + ad - a\cancel{cz} - bc}{(cz+d)^2}$$

$$= \frac{ad-bc}{(cz+d)^2} \neq 0$$

~~Invalid points~~ Hence every B.T is C.T

Note2: A B.T preserves a cross ratio property of 4 points.

1)* CR (crossratio) property of 4 points t_1, t_2, t_3, t_4 is $\frac{(t_1-t_2)(t_3-t_4)}{(t_1-t_4)(t_3-t_2)}$

2) If w_1, w_2, w_3, w_4 are the images of z_1, z_2, z_3, z_4

then $$\frac{(w_1-w_2)(w_3-w_4)}{(w_1-w_4)(w_3-w_2)} = \frac{(z_1-z_2)(z_3-z_4)}{(z_1-z_4)(z_3-z_2)}$$

3) To find a B·T when 3 points z_1, z_2, z_3 & their images w_1, w_2, w_3 are given,

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

Fixed points or invariant points:-

The points z such that $f(z)=z$ are called fixed / invariant points.

Fixed points for B·T:-

$$f(z)=z$$

$$\Rightarrow \frac{az+b}{cz+d} = z$$

$$\Rightarrow az+b = z(cz+d)$$

$$\Rightarrow cz^2+(d-a)z-b=0$$

The points satisfying above equ are fixed pts of $f(z)$.

B.T when α & β are fixed points:-

A Q.E in z with α & β are roots is

$z^2 - (\alpha+\beta)z + \alpha\beta = 0$ for any complex λ

$$z^2 - (\alpha+\beta)z + \alpha\beta + \lambda z - \lambda z = 0$$

$$\Rightarrow z^2 - (\alpha+\beta)z + \lambda z = \lambda z - \alpha\beta$$

$$\Rightarrow z = \frac{\lambda z - \alpha\beta}{z - (\alpha+\beta) + \lambda}$$