Unit - II

RL and RC Circuits:

Steady State and Transient Response:

A circuit having constant source is said to be in steady state if the currents and voltages do not change with time. Thus, circuits with currents and voltages having constant amplitude and constant frequency sinusoidal functions are also considered to be in a steady state. That means that the amplitude or frequency of a sinusoid never changes in a steady state circuit.

In a network containing energy storage elements, with change in excitation, the currents and voltages change from one state to another state. The behaviour of the voltage or current when it is changed from one state to another is called the transient state. The time taken for the circuit to change from one steady state to another steady state is called the transient time. The application of KVL and KCL to circuits containing energy storage elements results in differential, rather than algebraic equations.

Natural → Steady State

forced → Transient state

D.C → R (working) L (short) C → open → steady state (constant DC)

L (open) C → close : transient state

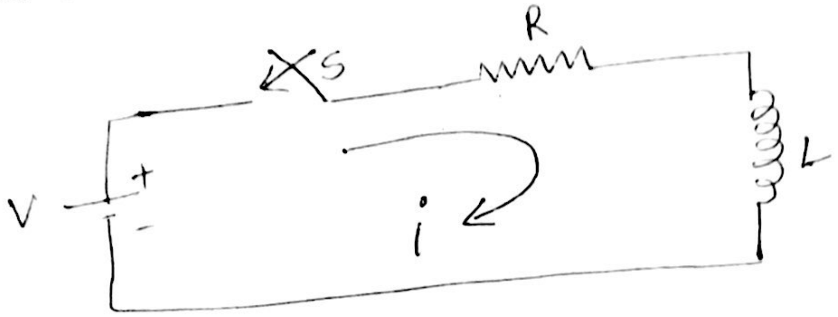
~~AC~~ → ~~working~~ ~~constant~~

When we consider a circuit containing storage elements which are independent of the sources, the response depends upon the nature of the circuit and is called the natural response. Storage elements deliver their energy to the resistances. Hence the response changes with time, gets saturated after some time, and is referred to as the transient response.

When we consider sources acting on a circuit, the response depends on the nature of the source or sources. This response is called forced response. In otherwords the complete response of a circuit consists of two parts: the forced response and the transient response. When we consider a differential equation, the complete solution consists of two parts: the complementary function and the particular solution. The complementary function dies out after short interval, and is referred to as the transient response or source free response. The particular solution is the steady state response, or the forced response. The first step in finding the complete solution of a circuit is to form a differential equation for the circuit. By obtaining the differential equation, several methods, can be used to find out the complete solution.

DC Response of an R-L Circuit :-

consider a circuit consisting of a resistance and inductance. The inductor in the circuit is initially uncharged and is in series with the resistor. When the switch 'S' is closed, we can find the complete solution for the current. Application of Kirchhoff's voltage law to the circuit result in the following differential equation



$$V = Ri + L\frac{di}{dt}$$

$$\rightarrow \cancel{\frac{di}{dt} + \frac{R}{L} i = \frac{V}{RL}} \quad (or) \quad \boxed{\frac{di}{dt} + \frac{R}{L} i = \frac{V}{L}}$$

$V \rightarrow$ applied constant voltage

$i \rightarrow$ the current

The voltage 'V' is applied to the circuit only when the switch 's' is closed. The above equation is a linear differential equation of 1st order. comparing it with a non-homogeneous differential equation

$$\frac{dx}{dt} + Px = K.$$

whose solution is,

$$x = e^{-Pt}\int K e^{+Pt} dt + C e^{-Pt}$$

where $C \rightarrow$ arbitrary constant.

In a similar way we can write the current equation as

$$i = C e^{-(R/L)t} + e^{-(R/L)t}\int \frac{V}{L} e^{(R/L)t} dt$$

$$i = C e^{-(R/L)t} + \frac{V}{R}$$

$\left(\because e^{-(R/L)t} \cdot \frac{V}{L} \cdot e^{(R/L)t} \cdot \frac{1}{(R/L)} \right.$

$\frac{V}{\cancel{L}} \times \frac{\cancel{L}}{R} = \frac{V}{R}$

To determine the value of 'C', we use the initial conditions the switch 'S' is closed at $t=0$. At $t=0^-$, i.e just before closing the switch 'S', the current in the Inductor is zero. Since the inductor does not allow sudden changes in currents, at $t=0^+$ just after the switch is closed, the current remains zero.

Thus at $t=0,\ i=0$

Substituting the above equation, we have

$$0 = C + \frac{V}{R}$$

$$C = -\frac{V}{R}.$$

$$\therefore \quad i = \frac{V}{R} - \frac{V}{R}\exp\left(-\frac{R}{L}t\right)$$

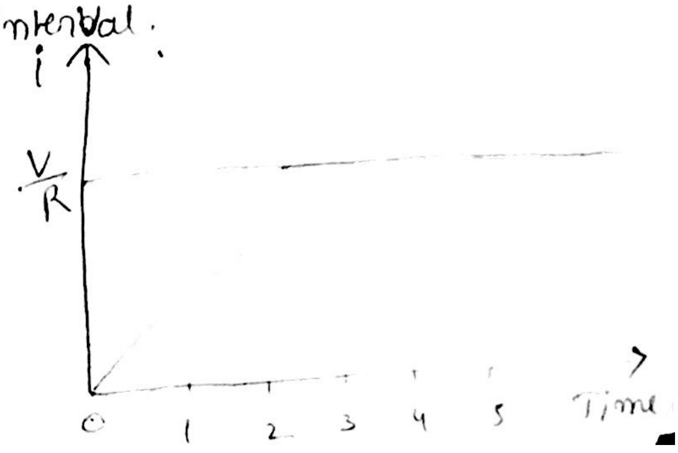
$$\boxed{i = \frac{V}{R}\left(1 - \exp\left(-\frac{R}{L}t\right)\right)}$$

The above equation consists of two parts,

The Steady State Part $= \frac{V}{R}$

The transient Part $= \frac{V}{R} e^{-\left(\frac{R}{L}t\right)}$.

When switch 'S' is closed, the response reaches a Steady state value after a time interval.



→ The transition Period is defined as the time taken for the current to reach its final or Steady State value from its initial value.

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In the transient part of the solution, the quantity $\frac{L}{R}$ is important in describing the curve since $\frac{L}{R}$ is the time required for the current to reach from its initial value of zero to the final value $\frac{V}{R}$. The time constant of a function $\frac{V}{R}e^{-(R/L)t}$ is the time at which the exponent of 'e' is unity, where 'e' is the base of the natural logarithm. The term $\frac{L}{R}$ is called the time constant and is denoted by 'τ'.

$$\boxed{\tau = \frac{L}{R}} \text{ sec.}$$

$\therefore$ The transient part of the solution is

$$i = -\frac{V}{R}\exp\left(-\frac{R}{L}t\right) = -\frac{V}{R}e^{-t/\tau}$$

At one time constant, the transient term reaches 36.8 percent of its initial value.

$$i(\tau) = -\frac{V}{R}e^{-t/\tau} \cdot -\frac{V}{R}e^{-\tau/\tau} = -\frac{V}{R}e^{-1} = -0.368\frac{V}{R}$$

Similarly,

$$i(2\tau) = -\frac{V}{R}e^{-2} = -0.135\frac{V}{R}$$

$$i(3\tau) = -\frac{V}{R}e^{-3} = -0.0498\frac{V}{R}$$

$$i(5\tau) = -\frac{V}{R}e^{-5} = -0.0067\frac{V}{R}$$

After 5 time constant, the transient part reaches more than 99 percent of its final value.

Voltage across the resistor is

$$V_R = Ri = R \times \frac{V}{R}\left[1 - \exp\left(-\frac{Rt}{L}\right)\right]$$

$$\therefore V_R = V\left[1 - \exp\left(-\frac{R}{L}t\right)\right]$$

Similarly,

The voltage across the inductance is,

$$V_L = L\frac{di}{dt}$$

$$= L\cdot\frac{d}{dt}\left\{\frac{V}{R}\left[1-\exp\left(-\frac{R}{L}t\right)\right]\right\}$$

$$= L\cdot\left[0 - \frac{V}{R}\exp\left(-\frac{R}{L}t\right)\cdot\left(-\frac{R}{L}\right)\right]$$

$$= \cancel{L}\cdot\frac{V}{\cancel{R}}\cdot\frac{\cancel{R}}{\cancel{L}}\exp\left(-\frac{R}{L}t\right)$$

$$= V\exp\left(-\frac{R}{L}t\right)$$

Power in the resistor is,

$$P_R = V_R\cdot i = V\left[1-\exp\left(-\frac{R}{L}t\right)\right]\cdot\left\{\frac{V}{R}\left[1-\exp\left(-\frac{R}{L}t\right)\right]\right\}$$

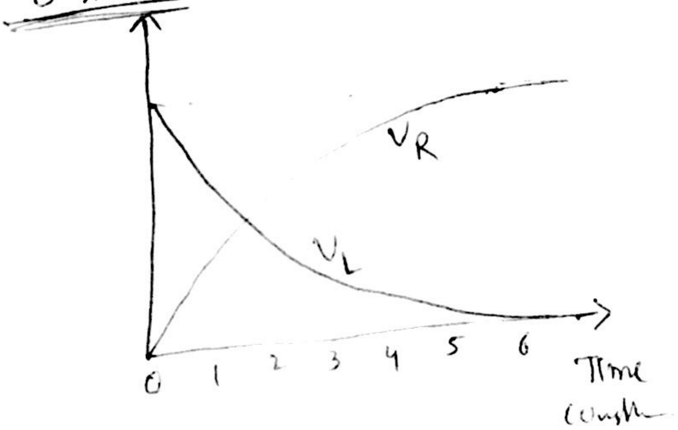
$$= \frac{V^2}{R}\left[1 - 2\exp\left(-\frac{R}{L}t\right) + \exp\left(-\frac{2R}{L}t\right)\right]$$

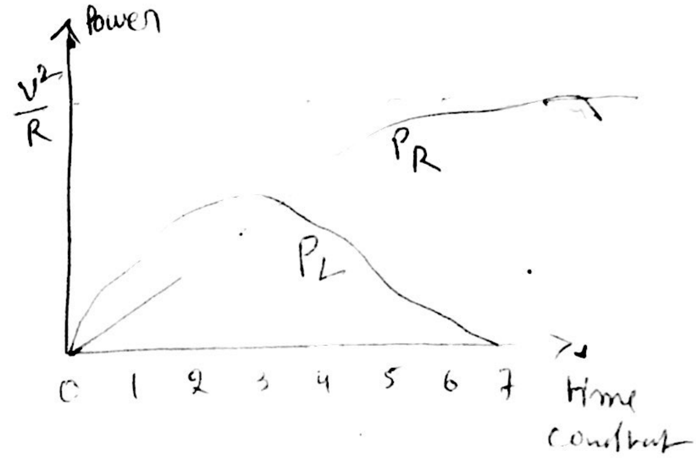
Power in the inductor is

$$P_L = V_L\cdot i = V\exp\left(-\frac{R}{L}t\right)\times\left[\frac{V}{R}\left(1-\exp\left(-\frac{R}{L}t\right)\right)\right]$$

$$= \frac{V^2}{R}\left[\exp\left(-\frac{R}{L}t\right) - \exp\left(-\frac{2R}{L}t\right)\right]$$

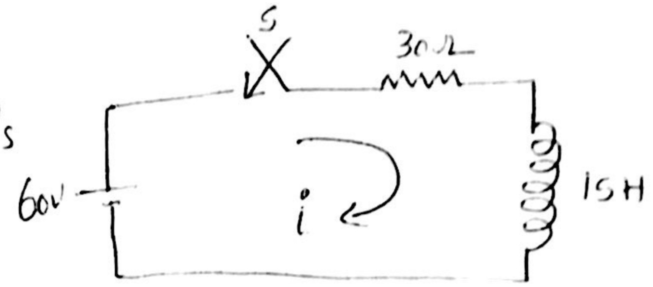
Responses:





EX: A series RL circuit with $R = 30\,\Omega$ and $L = 15H$ has, | II - (4)

a constant voltage $V = 60$ Volts applied at $t = 0$ as shown in fig. Determine the current i, the voltage across resistor and the voltage across the inductor.



Sol: By applying the Kirchhoff's voltage law, we get

$$15\frac{di}{dt} + 30i = 60$$

$$\Rightarrow \frac{di}{dt} + 2i = 4$$

The general solution for a linear differential equation is

$$i = ce^{-pt} + e^{-pt}\int Ke^{pt}\,dt$$

where $p = 2,\ K = 4.$

$$\therefore\ i = ce^{-2t} + e^{-2t}\int 4e^{2t}\,dt \qquad \left[\because 4\int e^{2t}\,dt = \cancel{4}^{2}\cdot\frac{e^{2t}}{\cancel{2}}\right]$$

$$\therefore\ i = ce^{-2t} + 2$$

At $t = 0$, the switch 'S' is closed.

Since the inductor never allows sudden changes in currents At $t = 0^+$ the current in the circuit is zero.

$$\therefore \text{ at } t = 0^+,\ i = 0$$

$$\therefore\ 0 = c + 2$$

$$c = -2$$

Substituting the value of 'c' in the current equation, we have

$$i = -2e^{-2t} + 2$$

$$= 2(1 - e^{-2t}) \text{ Amp}$$

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Voltage across resistor $v_R = iR$

$$= 2(1-e^{-2t}) \times 30\,\Omega$$

$$= 60(1-e^{-2t}) \text{ volts.}$$

Voltage across inductor $v_L = L\frac{di}{dt}$

$$= 15 \times \frac{d}{dt}\{2(1-e^{-2t})\}$$

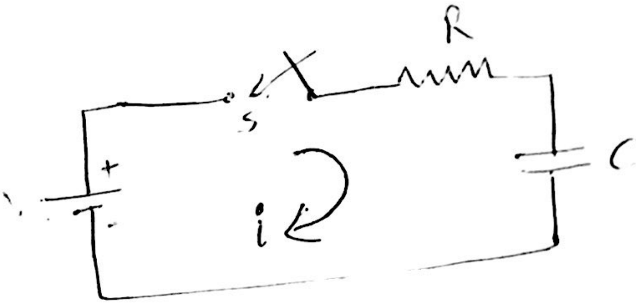
$$= 30\frac{d}{dt}(1-e^{-2t})$$

$$= 30[0 - e^{-2t}\cdot(-2)]$$

$$= 60e^{-2t} \text{ volts.}$$

DC RESPONSE OF AN R-C CIRCUIT:

Consider a circuit consisting of resistance and capacitance. The capacitor in the circuit is initially uncharged, and is in series with a resistor. when the switch 's' is closed at $t=0$. we can determine the complete solution for the current.



Applying Kirchhoff's Voltage Law,

$$Ri + \frac{1}{C}\int i\,dt = V$$

$$\Rightarrow V = Ri + \frac{1}{C}\int i\,dt$$

By differentiating the above equation, we get

$$0 = R\frac{di}{dt} + \frac{i}{C}$$

$\frac{di}{dt} + \frac{1}{RC} i = 0$, This is a linear differential equation with only the complementary function. The Perticular solution for the above equation is zero. The Solution of this type of differential equation is

$$i = Ce^{-t/RC}$$

Here, to find the value of C, we use the initial conditions

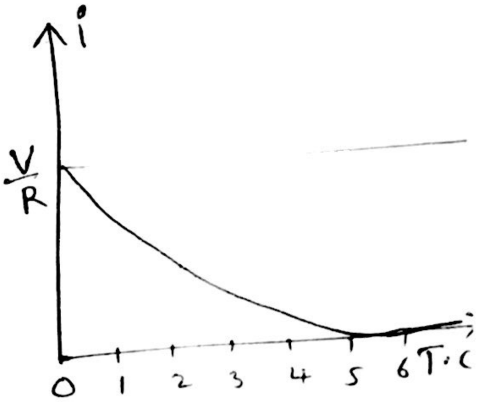
At t=0, the Switch 's' is closed. Since the capacitor never allows sudden changes in voltage, it will act as a short circuit at $t=0^+$. So the current in the circuit at $t=0^+$ is $\rightarrow \frac{V}{R}$.

At t=0, the current $i = \frac{V}{R}$

$$i = Ce^{-t/RC}$$

$$\Rightarrow \frac{V}{R} = Ce^{-0/RC}$$

$$\Rightarrow \frac{V}{R} = C$$



∴ The current equation becomes

$$\boxed{i = \frac{V}{R} e^{-t/RC}}$$

when switch 's' is closed, the response decays with time.

In the Solution, the quantity RC is the time constant, and is denoted by τ.

$\tau = RC$ ~~sections~~ sec

$$\therefore \boxed{\tau = RC \text{ seconds}}$$

After 5τ, the curve reaches 99% of its final value.

Voltage across the resistor is

$$v_R = Ri = \cancel{R} \times \frac{V}{\cancel{R}} e^{-(\frac{1}{RC})t}$$

$$\therefore V_R = Ve^{-t/R}$$

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$$\boxed{v_R = V e^{-t/RC}}$$

Similarly,

Voltage across the capacitor is

$$v_c = \frac{1}{C}\int i\,dt$$

$$= \frac{1}{C}\int \frac{V}{R} e^{-t/RC}\,dt$$

$$= \frac{V}{RC}\int e^{-t/RC}\,dt$$

$$= \frac{V}{RC} \times \frac{e^{-t/RC}}{\left(\frac{-1}{RC}\right)} + c = \frac{-V}{RC} \times RC\, e^{-t/RC}$$

$$= -V e^{-t/RC} + c$$

At $t=0$, voltage across capacitor is zero

$$c = V$$

$$\therefore \quad v_c = -V e^{-t/RC} + V$$

$$\boxed{v_c = V(1 - e^{-t/RC})}$$

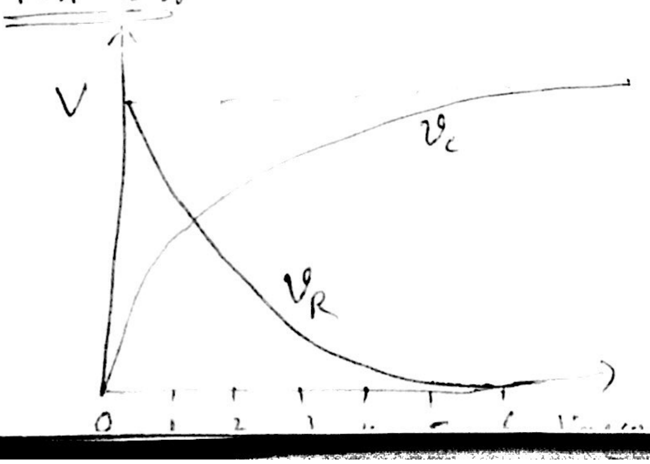
The Power in the resistor $P_R = v_R i = V e^{-t/RC} \times \frac{V}{R} e^{-t/RC}$

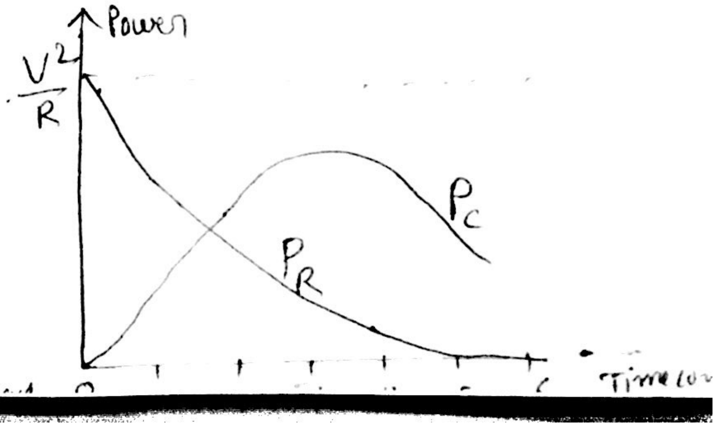
$$= \frac{V^2}{R} e^{-2t/RC}$$

The Power in the capacitor $P_c = v_c i = V(1 - e^{-t/RC}) \cdot \frac{V}{R} e^{-t/RC}$

$$= \frac{V^2}{R}\left(e^{-t/RC} - e^{-2t/RC}\right)$$

Responses:





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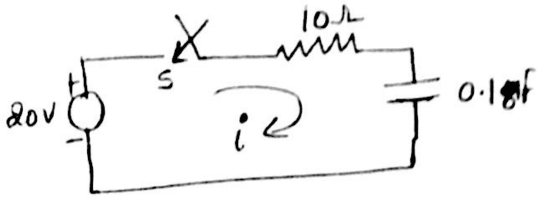
Ex: A series RC circuit consists of resistor of 10Ω and capacitor of $0.1F$ as shown in fig. A constant voltage of $20V$ is applied to the circuit at $t=0$. Obtain the current equation. Determine the voltages across the resistor and the capacitor. | II - (6)

Sol: By applying Kirchhoff's law, we get



$$10i + \frac{1}{0.1}\int i\,dt = 20$$

Differentiating with respect to 't' we get,

$$10\frac{di}{dt} + \frac{i}{0.1} = 0$$

$$\therefore \frac{di}{dt} + i = 0$$

The solution for the above equation is $i = ce^{-t}$

At $t=0$, $S \rightarrow$ closed. Since the capacitor does not allow sudden changes in voltage, the current in the circuit is

$$i = \frac{V}{R} = \frac{20}{10} = 2\text{ Amp.}$$

At $t=0$, $i = 2$ Amp.

$\therefore$ The current equation $i = 2e^{-t}$

Voltage across the resistor is $V_R = i \times R = 2e^{-t} \times 10$

$$= 20e^{-t}\text{ volts.}$$

Voltage across the capacitor is $v_c = V(1 - e^{-t/RC})$

$$= 20(1 - e^{-t})\text{ volts.}$$

$\therefore$ $V_R = 20e^{-t}$ volts

$V_c = 20(1 - e^{-t})$ volts.

DC RESPONSE OF AN R-L-C CIRCUIT :

Consider a circuit consisting of resistance, inductance and capacitance as shown in fig.

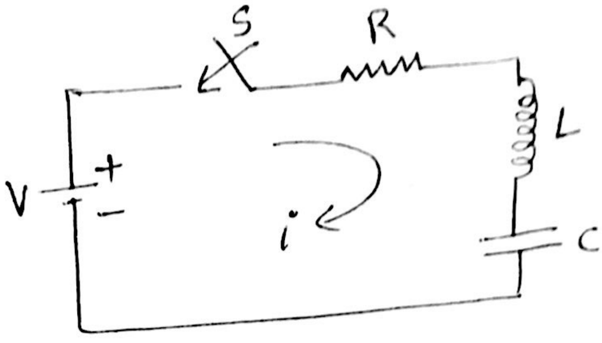


fig.

The capacitor and inductor are initially uncharged, and are in series with a resistor. when switch 'S' is closed at $t=0$, we can determine the complete solution for the current.

Applying the Kirchhoff's voltage Law, we get

$$Ri + L\frac{di}{dt} + \frac{1}{C}\int i\,dt = V$$

$$\Rightarrow \quad V = Ri + L\frac{di}{dt} + \frac{1}{C}\int i\,dt.$$

By differentiating the above equation, we have

$$0 = R\frac{di}{dt} + L\frac{d^2i}{dt^2} + \frac{1}{C}i$$

$$\Rightarrow \quad \frac{d^2i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{1}{LC}i = 0$$

It is a second order Linear differential equation, with only complementary function. The particular solution is zero.

The characteristic equation for the above differential equation is

$$\left(D^2 + \frac{R}{L}D + \frac{1}{LC}\right) = 0$$

The roots,

$$D_1, D_2 = -\frac{R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

By assuming $K_1 = -\frac{R}{2L}$ and $K_2 = \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$

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$D_1 = K_1 + K_2$ and $D_2 = K_1 - K_2$

Here K_2 may be +ve, −ve or zero.

Case (1):

K_2 is +ve, when $\left(\frac{R}{2L}\right)^2 > \frac{1}{LC}$

The roots are real and unequal, and give the over damped response as shown in fig.

$$[D-(K_1+K_2)][D-(K_1-K_2)]\,i = 0$$

The solution for the above equation is

$$i = C_1 e^{(K_1+K_2)t} + C_2 e^{(K_1-K_2)t}$$

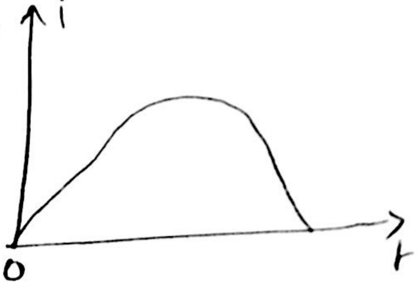


fig: overdamped.

Case (2):

K_2 is −ve, when $\left(\frac{R}{2L}\right)^2 < \frac{1}{LC}$

The roots are complex conjugate, and give the underdamped response.

$$[D-(K_1+jK_2)][D-(K_1-jK_2)]\,i = 0$$

The solution for the above equation is,

$$i = e^{K_1 t}[C_1 \cos K_2 t + C_2 \sin K_2 t]$$

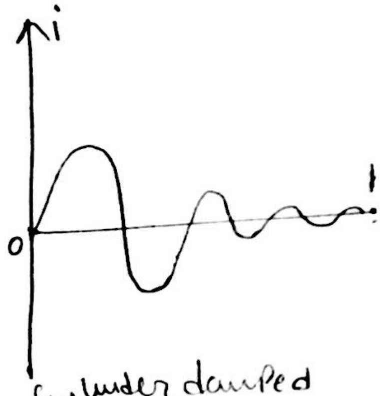


fig: under damped

Case (3):

K_2 is zero, when $\left(\frac{R}{2L}\right)^2 = \frac{1}{LC}$

The roots are equal, and give the critically damped response.

$$(D-K_1)(D-K_1)\,i = 0$$

The solution for the above equation is

$$i = e^{K_1 t}(C_1 + C_2 t)$$

The current curve for the critically damped case is shown in fig.

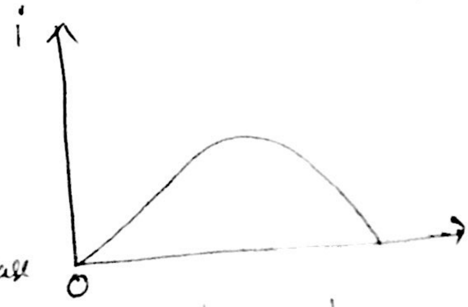
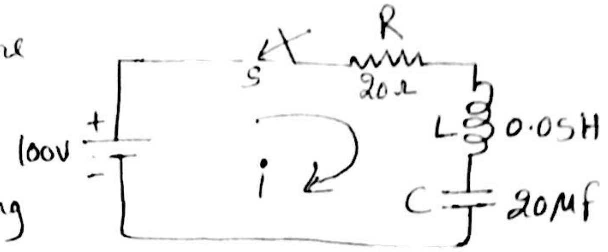


fig: critically damped

Ex: The circuit shown in fig. consists of resistance, inductance and capacitance in series with a 100V constant source when the switch is closed at $t=0$. Find the current transient.

Sol: At $t=0$, $S \to$ closed, when the 100V source is applied to the circuit and results in the following differential equation.



$$100 = 20i + 0.05\frac{di}{dt} + \frac{1}{20\times10^{-6}}\int i\,dt$$

differentiating above equation, we get

$$0.05\frac{d^2i}{dt^2} + 20\frac{di}{dt} + \frac{1}{20\times10^{-6}}i = 0$$

$$\frac{d^2i}{dt^2} + 400\frac{di}{dt} + 10^6 i = 0$$

$$\therefore (D^2 + 400D + 10^6)i = 0$$

$$D_1, D_2 = -\frac{400}{2} \pm \sqrt{\left(\frac{400}{2}\right)^2 - 10^6}$$

$$= -200 \pm \sqrt{(200)^2 - 10^6}$$

$$D_1 = -200 + j979.8$$

$$D_2 = -200 - j979.8$$

$$\therefore i = e^{+K_1 t}\left[C_1 \cos K_2 t + C_2 \sin K_2 t\right]$$

$$= e^{-200t}\left[C_1 \cos 979.8t + C_2 \sin 979.8t\right] \text{ Amp}$$

At $t=0$, the current flowing through the circuit is zero

$$i = 0 = e^{-200(0)}\left[C_1 \cos 979.8(0) + C_2 \sin 979.8(0)\right]$$

$$= 1\left[C_1 \cos 0 + C_2 \sin 0\right] = C_1 = 0$$

$\therefore \; i = e^{-200t} \, C_2 \sin 979.8t$ Amp.

$\therefore$ Differentiating, we have

$$\frac{di}{dt} = C_2 \left[e^{-200t} \, 979.8 \cos 979.8t + e^{-200t} (-200) \sin 979.8t \right]$$

At $t=0$, the voltage across inductor is 100V

$$L \frac{di}{dt} = 100$$

$$\frac{di}{dt} = 2000$$

$$\left(\frac{100}{0.05} = \frac{100 \times 10^2}{5 \times 10^{-2}} = \frac{10000}{5} = 2000 \right)$$

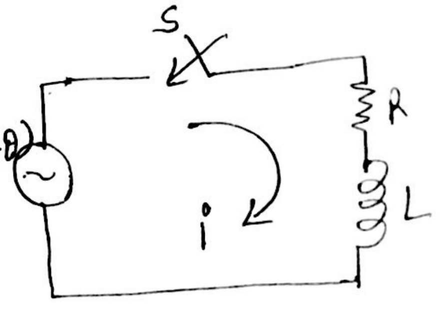
At $t=0$ $\quad \frac{di}{dt} = 2000 = C_2 \, 979.8 \cos 0$

$$C_2 = \frac{2000}{979.8} = 2.04$$

$\therefore$ The current equation is

$$i = e^{-200t} (2.04 \sin 979.8t) \text{ Amp.}$$

SINUSOIDAL RESPONSE OF R-L CIRCUIT:



Consider a circuit consisting of resistance and inductance.

The switch 'S' is closed at $t=0$. At $t=0$, a sinusoidal voltage $V\cos(\omega t+\theta)$ is applied to the series R-L circuit.

where $V \rightarrow$ the amplitude of the wave

$\theta \rightarrow$ the phase angle.

Applying the KVL to the circuit, we get

$$V \cos(\omega t + \theta) = Ri + L \frac{di}{dt}$$

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$$\therefore \frac{di}{dt} + \frac{R}{L} i = \frac{V}{L} \cos(\omega t + \theta)$$

The corresponding characteristic equation is

$$\left(D + \frac{R}{L}\right) i = \frac{V}{L} \cos(\omega t + \theta)$$

For the above equation, the solution consists of two parts, viz, complementary function and particular integral.

The complementary function of the solution i is

$$i_c = c e^{-t(R/L)}$$

The particular solution can be obtained by using undetermin[ed] coefficients

By assuming $i_p = A\cos(\omega t + \theta) + B\sin(\omega t + \theta)$

$$i_p' = -A\omega \sin(\omega t + \theta) + B\omega \cos(\omega t + \theta).$$

Substitute in the characteristic equation is,

$$\left\{-A\omega \sin(\omega t + \theta) + B\omega \cos(\omega t + \theta)\right\} + \frac{R}{L}\left\{A\cos(\omega t + \theta) + B\sin(\omega t + \theta)\right\}$$
$$= \frac{V}{L} \cos(\omega t + \theta)$$

$$\Rightarrow \left(-A\omega + \frac{BR}{L}\right) \sin(\omega t + \theta) + \left(B\omega + \frac{AR}{L}\right) \cos(\omega t + \theta) = \frac{V}{L} \cos(\omega t + \theta)$$

Comparing cosine terms and sine terms, we get

$$-A\omega + \frac{BR}{L} = 0$$

$$B\omega + \frac{AR}{L} = \frac{V}{L}$$

From the above equations, we have

To get max. value $\frac{dV_c}{d\omega} = 0$

If we solve for ω, we obtain the value of ω when V_c is m

$$\frac{dV_c}{d\omega} = \omega C \frac{1}{2}\left[R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2\right]^{-1/2}\left[2\left(\omega L - \frac{1}{\omega C}\right)\left(L + \frac{1}{\omega^2 C}\right)\right] + \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}\, C = 0$$

From this,

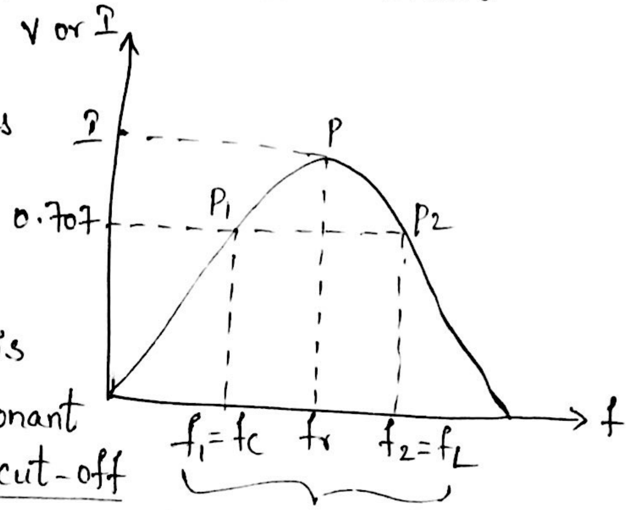
$$\omega_c^2 = \frac{1}{LC} - \frac{R^2}{2L}$$

$$\omega_c = \sqrt{\frac{1}{LC} - \frac{R^2}{2L}}$$

$$\therefore \boxed{f_c = \frac{1}{2\pi}\sqrt{\frac{1}{LC} - \frac{R^2}{2L}}}$$

Bandwidth of an RLC circuit :-

The Bandwidth of any system is the range of frequencies for which the current or output voltage is equal to 70.7% of its value at the resonant frequency, and it is denoted by BW. Fig. shows the response of a series RLC circuit.



The freq. f_1 is the freq. at which the current is 0.707 times the current at resonant value, and it is called the lower cut-off freq. The freq. f_2 is the freq. at which the current is 0.707 times the current at resonant value, and is called the upper cut-off freq. The bandwidth, or BW, is defined as the freq difference b/w f_2 and f_1.

$$\therefore BW = f_2 - f_1$$

The unit of BW is hertz (Hz).

If the current at P_1 is $0.707\, I_{max}$, the impedance of the circuit at this point is $\sqrt{2}R$, and hence

$$\frac{1}{\omega_1 C} - \omega_1 L = R \quad \text{— (1)}$$

Similarly, $$\omega_2 L - \frac{1}{\omega_2 C} = R \quad \text{— (2)}$$

If we equate both equations, we get

$$\frac{1}{\omega_1 C} - \omega_1 L = \omega_2 L - \frac{1}{\omega_2 C}.$$

$$L(\omega_1 + \omega_2) = \frac{1}{C}\left(\frac{\omega_1 + \omega_2}{\omega_1 \omega_2}\right) \quad \text{— (3)}$$

from eq. (3), we get

$$\omega_1 \omega_2 = \frac{1}{LC}$$

we have $\omega_r^2 = \frac{1}{LC}$

$$\therefore \omega_r^2 = \omega_1 \omega_2 \quad \text{— (4)}$$

If we add eq's (1) and (2), we get

$$\frac{1}{\omega_1 C} - \omega_1 L + \omega_2 L - \frac{1}{\omega_2 C} = 2R$$

$$(\omega_2 - \omega_1) L + \frac{1}{C}\left(\frac{\omega_2 - \omega_1}{\omega_1 \omega_2}\right) = 2R \quad \text{— (5)}$$

since $C = \frac{1}{\omega_r^2 L}$

and $\omega_1 \omega_2 = \omega_r^2$

$$(\omega_2 - \omega_1) L + \frac{\omega_r^2 L (\omega_2 - \omega_1)}{\omega_r^2} = 2R \quad \text{— (6)}$$

From eq. (6), we have

$$(\omega_2 - \omega_1) = \frac{R}{L} \quad \text{— (7)}$$

$$\therefore f_2 - f_1 = \frac{R}{2\pi L} \quad \text{— (8)}$$

or $BW = \frac{R}{2\pi L}$

from eq. (8), we have

$$f_2 - f_1 = \frac{R}{2\pi L}$$

$$\therefore f_r - f_1 = \frac{R}{4\pi L}$$

$$f_2 - f_r = \frac{R}{4\pi L}$$

The lower freq. limit $\boxed{f_1 = f_r - \frac{R}{4\pi L}}$ —— (9)

The upper freq. Limit $\boxed{f_2 = f_r + \frac{R}{4\pi L}}$ —— (10)

If we divide the eq. on both sides by f_r, we get

$$\frac{f_2 - f_1}{f_r} = \frac{R}{2\pi f_r L} \quad \text{—— (11)}$$

Here, an important property of a coil is defined. It is the ratio of the reactance of the coil to its resistance. The ratio is defined as the Q of the coil. 'Q' is known as a <u>figure of merit</u>, it is also called <u>quality factor</u> and is an indication of the quality of a coil.

$$\boxed{Q = \frac{X_L}{R} = \frac{2\pi f_r L}{R}} \quad \text{—— (12)}$$

If we substitute eq. (11) in eq. (12), we get

$$\frac{f_2 - f_1}{f_r} = \frac{1}{Q} \quad \text{—— (13)}$$

The upper and lower cut-off frequencies are sometimes called the half-power frequencies. At these frequencies, the power from the source is half of the power delivered at the resonant freq.

$$P_{max} = I^2_{max} R.$$

At freq. f_1, the power is $P_1 = \left(\frac{I_{max}}{\sqrt{2}}\right)^2 R = \frac{I^2_{max} R}{2}$

Similarly, at freq. f_2, the power is

$$P_2 = \left(\frac{I_{max}}{\sqrt{2}}\right)^2 R$$

$$P_2 = \frac{I^2_{max} R}{2}$$

The Quality factor (Q) and its Effect on Bandwidth :-

The quality factor, 'Q' is the ratio of the reactive power in the inductor or capacitor to the true power in the resistance in series with the coil or capacitor.

The Quality factor,

$$Q = 2\pi \times \frac{\text{max. Energy stored}}{\text{Energy dissipated per cycle}}.$$

In an inductor, the max. Energy stored is given by $\frac{LI^2}{2}$.

$$\text{Energy dissipated per cycle} = \left(\frac{I}{\sqrt{2}}\right)^2 R \times T = \frac{I^2RT}{2}.$$

$$\therefore \text{Quality factor of the coil } Q = 2\pi \times \frac{\frac{1}{2}LI^2}{\frac{I^2R}{2} \times \frac{1}{f}}$$

$$= \frac{2\pi fL}{R} = \frac{\omega L}{R}.$$

Similarly, in a capacitor, the max. Energy stored is given by $\frac{CV^2}{2}$.

The Energy dissipated per cycle $= \left(\frac{I}{\sqrt{2}}\right)^2 R \times T$.

The quality factor of the capacitance circuit

$$Q = \frac{2\pi \frac{1}{2} C \left(\frac{1}{\omega C}\right)^2}{\frac{I^2}{2} R \times \frac{1}{f}} = \frac{1}{\omega CR}.$$

In series circuits, the quality factor $Q = \frac{\omega L}{R} = \frac{1}{\omega CR}$.

The relation b/w bandwidth and quality factor, which is

$$Q = \frac{f_r}{BW}.$$

$$= \frac{\pi \times \cancel{C} \times \cancel{\frac{I^2}{2}} \times \frac{1}{W^2 C^2}}{\cancel{\frac{I^2}{2}} \cdot R \cdot 1/f} = \frac{\frac{\pi}{W^2 C}}{R \cdot 1/f}$$

Magnification in Resonance :-

If we assume that the voltage applied to the series RLC circuit is V, and the current at resonance is I, then the vol. across L is $V_L = IX_L = (V/R)\,\omega_r L$.

Similarly, the vol. across C

$$V_C = IX_C = \frac{V}{R\omega_r C}.$$

Since $$Q = \frac{1}{\omega_r CR} = \frac{\omega_r L}{R}.$$

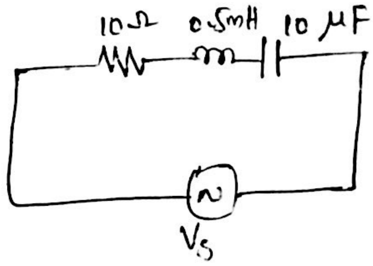
where ω_r is the freq. at resonance.

Therefore, $$V_L = VQ$$
$$V_C = VQ$$

The ratio of vol. across either L or C to the vol. applied at resonance can be defined as magnification.

$$\therefore \text{magnification} = Q = V_L/V \text{ or } V_C/V.$$

Problem :- Determine the resonant freq. for the ckt shown



Sol :- The resonant freq. is

$$f_r = \frac{1}{2\pi\sqrt{LC}}.$$

$$= \frac{1}{2\pi\sqrt{10\times10^{-6}\times0.5\times10^{-3}}}$$

$$= 2.25\,KHz.$$

Problem :- A series circuit with $R = 10\Omega$, $L = 0.1H$ and $C = 50\mu F$ has an applied voltage $V = 50\angle0°$ with a variable frequency. Find the resonant frequency, the value of freq. at which max. vol. occurs across the inductor and the value of freq. at which max vol. occurs across the capacitor.

Sol :- The freq. at which max. vol. occurs across the inductor is

$$f_L = \frac{1}{2\pi\sqrt{LC}}\sqrt{\frac{1}{\left(1-\frac{R^2C}{2L}\right)}}$$

$$= \frac{1}{2\pi\sqrt{0.1\times 50\times 10^{-6}}}\sqrt{\frac{1}{1-\left(\frac{(10)^2\times 50\times 10^{-6}}{2\times 0.1}\right)}}$$

$= 72.08$ Hz.

Similarly, $f_c = \frac{1}{2\pi}\sqrt{\frac{1}{LC} - \frac{R^2}{2L}}$

$$= \frac{1}{2\pi}\sqrt{\frac{1}{0.1\times 50\times 10^{-6}} - \frac{(10)^2}{2\times 0.1}}$$

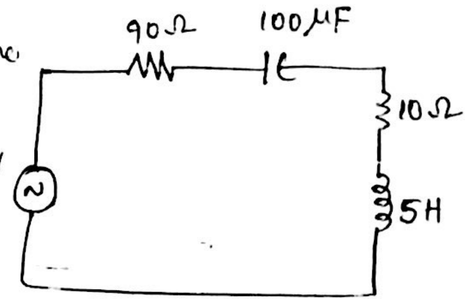
$$= \frac{1}{2\pi}\sqrt{200000 - 500}.$$

$= 71.08$ Hz.

Resonant freq. $f_r = \frac{1}{2\pi\sqrt{LC}} = \frac{1}{2\pi\sqrt{0.1\times 50\times 10^{-6}}} = 71.18$ Hz.

It is clear that the max. vol. across the capacitor occurs below the resonant freq. and the max. inductor vol. occurs above the resonant freq.

Problem :- For the ckt shown, determine the value of Q at resonance & bandwidth of the circuit.



Sol :- The resonant freq,

$$f_r = \frac{1}{2\pi\sqrt{LC}}$$

$$= \frac{1}{2\pi\sqrt{5\times 100\times 10^{-6}}}$$

$= 7.12$ Hz.

Quality factor $Q = \frac{X_L}{R} = \frac{2\pi f_r L}{R} = \frac{2\pi\times 7.12\times 5}{100} = 2.24$

Bandwidth of the ckt is $BW = \frac{f_r}{Q} = \frac{7.12}{2.24} = 3.178$ Hz.

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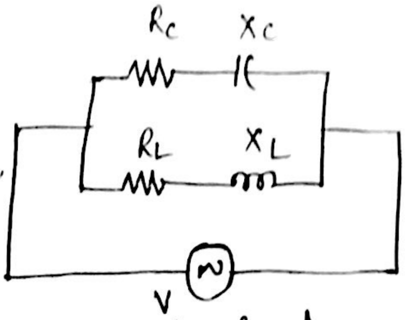
Parallel Resonance :-



Parallel Resonance occurs when $X_C = X_L$. The freq. at which resonance occurs is called the resonant freq. When $X_C = X_L$, the two branch currents are equal in magnitude and 180° out of phase with each other. Therefore, the two currents cancel each other out, and the total current is zero.

The condition for resonance occurs when $X_L = X_C$.

In circuit, The total admittance,

$$Y = \frac{1}{R_L + j\omega L} + \frac{1}{R_C - (j/\omega C)}$$

$$= \frac{R_L - j\omega L}{R_L^2 + \omega^2 L^2} + \frac{(R_C + j/\omega C)}{R_C^2 + \frac{1}{\omega^2 C^2}}$$

$$= \frac{R_L}{R_L^2 + \omega^2 L^2} + \frac{R_C}{R_C^2 + \frac{1}{\omega^2 C^2}} + j\left\{\left[\frac{(1/\omega C)}{R_C^2 + \frac{1}{\omega^2 C^2}}\right] - \left[\frac{\omega L}{R_L^2 + \omega^2 L^2}\right]\right\} \quad \text{--- (1)}$$

At resonance, the susceptance part becomes zero

$$\therefore \frac{\omega_r L}{R_L^2 + \omega_r^2 L^2} = \frac{(1/\omega_r C)}{R_C^2 + (1/\omega_r^2 C^2)} \quad \text{--- (2)}$$

$$\omega_r L\left[R_C^2 + \frac{1}{\omega_r^2 C^2}\right] = \frac{1}{\omega_r C}\left(R_L^2 + \omega_r^2 L^2\right)$$

$$\omega_r^2\left(R_C^2 + \frac{1}{\omega_r^2 C^2}\right) = \frac{1}{LC}\left(R_L^2 + \omega_r^2 L^2\right)$$

$$\omega_r^2 R_C^2 - \frac{\omega_r^2 L}{C} = \frac{1}{LC} R_L^2 - \frac{1}{C^2}$$

$$\omega_r^2\left(R_C^2 - \frac{L}{C}\right) = \frac{1}{LC}\left(R_L^2 - \frac{L}{C}\right).$$

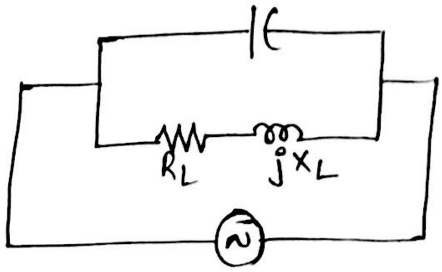
$$\omega_r = \frac{1}{\sqrt{LC}}\sqrt{\frac{R_L^2 - (L/C)}{R_C^2 - (L/C)}} \quad \text{--- (3)}$$

The condition for resonant freq. is given by eq. (3).
As a special case, if $R_L = R_C$, then eq. (3) becomes

$$\omega_r = \frac{1}{\sqrt{LC}}$$

Therefore, $f_r = \frac{1}{2\pi\sqrt{LC}}$.

Resonant freq. for a Tank circuit :-



The parallel resonant circuit is generally called a tank circuit because of the fact that the circuit stores energy in the magnetic field of the coil and in the electric field of the capacitor.
The circuit is said to be in resonant condition when the susceptance part of admittance is zero.
The total admittance is

$$Y = \frac{1}{R_L + jX_L} + \frac{1}{-jX_C} \quad \text{——} (1)$$

Simplifying eq. (1) we have

$$Y = \frac{R_L - jX_L}{R_L^2 + X_L^2} + \frac{j}{X_C}$$

$$= \frac{R_L}{R_L^2 + X_L^2} + j\left[\frac{1}{X_C} - \frac{X_L}{R_L^2 + X_L^2}\right]$$

To satisfy the condition for resonance, the susceptance part is zero.

$$\therefore \frac{1}{X_C} = \frac{X_L}{R_L^2 + X_L^2} \quad \text{——} (2)$$

$$\omega C = \frac{\omega L}{R_L^2 + \omega^2 L^2} \quad \text{——} (3)$$

From eq. (3) we get

$$R_L^2 + \omega^2 L^2 = \frac{L}{C}$$

$$\omega^2 L^2 = \frac{L}{C} - R_L^2$$

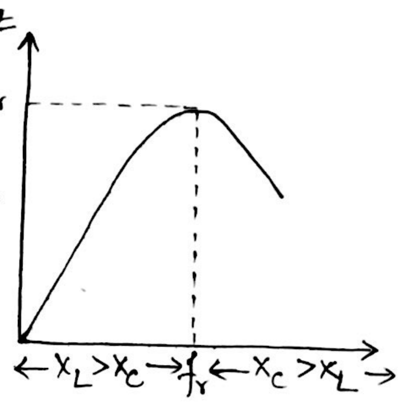
$$\omega^2 = \frac{1}{LC} - \frac{R_L^2}{L^2}.$$

$$\therefore \omega = \sqrt{\frac{1}{LC} - \frac{R_L^2}{L^2}} \quad \text{——— (3)}$$

The resonant freq. for the tank circuit is

$$\boxed{f_r = \frac{1}{2\pi}\sqrt{\frac{1}{LC} - \frac{R_L^2}{L^2}}} \quad \text{——— (4)}$$

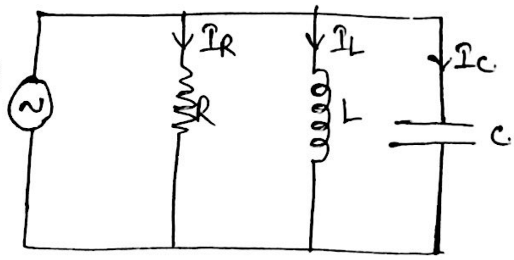
Variation of Impedance with frequency :-



The impedance of a parallel resonant circuit is maximum at the resonant freq. and decreases as shown.
At very low frequencies, X_L is very small and X_C is very large, so the total impedance is essentially inductive. As the freq. increases, the impedance also increases, and the inductive reactance dominates until the resonant freq. is reached. At this point $X_L = X_C$, and the impedance is at its maximum. As the freq. goes above resonance capacitive reactance dominates and the impedance decreases.

Q-factor of parallel resonance :-

Consider the parallel RLC circuit as shown.



In the circuit shown, the condition for resonance occurs when the susceptance part is zero.

Admittance $Y = G + jB.$ ——— (1)

$$= \frac{1}{R} + j\omega C + \frac{1}{j\omega L}$$

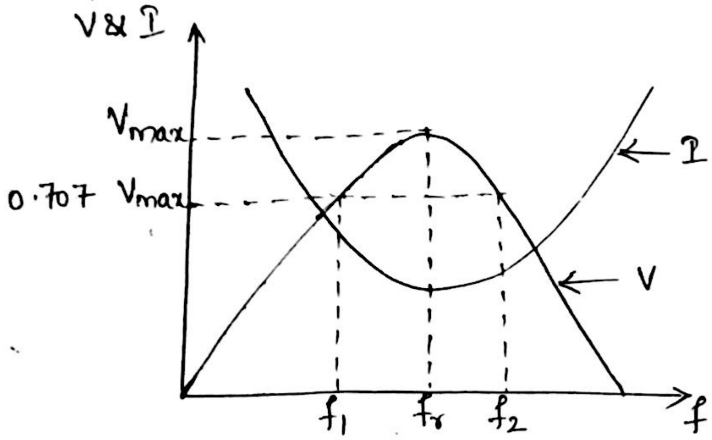
$$= \frac{1}{R} + j\left(\omega C - \frac{1}{\omega L}\right) \quad \text{——— (2)}$$

The freq. at which resonance occurs is

$$\omega_r C - \frac{1}{\omega_r L} = 0 \quad \text{--- (3)}$$

$$\omega_r = \frac{1}{\sqrt{LC}} \quad \text{--- (4)}$$

The voltage and current variation with freq. is as shown. At resonant freq., the current is minimum.



The bandwidth, $BW = f_2 - f_1$.

For parallel circuit, to obtain the lower half-power freq,

$$\omega_1 C - \frac{1}{\omega_1 L} = -\frac{1}{R} \quad \text{--- (5)}$$

From Eq (5), we have

$$\omega_1^2 + \frac{\omega_1}{RC} - \frac{1}{LC} = 0 \quad \text{--- (6)}$$

If we simplify Eq. (6), we get

$$\omega_1 = \frac{-1}{2RC} + \sqrt{\left(\frac{1}{2RC}\right)^2 + \frac{1}{LC}} \quad \text{--- (7)}$$

Similarly, to obtain the upper half-power freq.

$$\omega_2 C - \frac{1}{\omega_2 L} = \frac{1}{R} \quad \text{--- (8)}$$

From Eq. (8), we have

$$\omega_2 = \frac{1}{2RC} + \sqrt{\left(\frac{1}{2RC}\right)^2 + \frac{1}{LC}} \quad \text{--- (9)}$$

Bandwidth, $\boxed{BW = \omega_2 - \omega_1 = \frac{1}{RC}}$

The quality factor is defined as $Q_r = \dfrac{\omega_r}{\omega_2 - \omega_1}$

$$Q_r = \frac{\omega_r}{1/RC} = \omega_r RC.$$

In other words,

$$Q = 2\pi \times \frac{\text{max. Energy stored}}{\text{Energy dissipated / cycle}}$$

In case of an inductor,

The max. Energy stored $= \frac{1}{2} L I^2$

Energy dissipated per cycle $= \left(\frac{I}{\sqrt{2}}\right)^2 \times R \times T$

The quality factor, $Q = 2\pi \times \dfrac{1/2\,(LI^2)}{\frac{I^2}{2} R \times \frac{1}{f}}$.

$$\therefore\ Q = 2\pi \times \frac{\frac{1}{2} L \left(\frac{V}{\omega L}\right)^2 R}{\frac{V^2}{2} \times \frac{1}{f}}$$

$$= \frac{2\pi f LR}{\omega^2 L^2} = \frac{R}{\omega L}.$$

For a capacitor, max. Energy stored $= 1/2\,(CV^2)$

Energy dissipated per cycle $= P \times T = \dfrac{V^2}{2 \times R} \times \dfrac{1}{f}$.

The quality factor $Q = 2\pi \times \dfrac{1/2\,(CV^2)}{\frac{V^2}{2R} \times \frac{1}{f}}$

$$= 2\pi f CR = \omega CR$$

$$\boxed{Q = \omega CR}$$

Magnification :-

Current magnification occurs in a parallel resonant circuit. The voltage applied to the parallel circuit, $V = IR$.

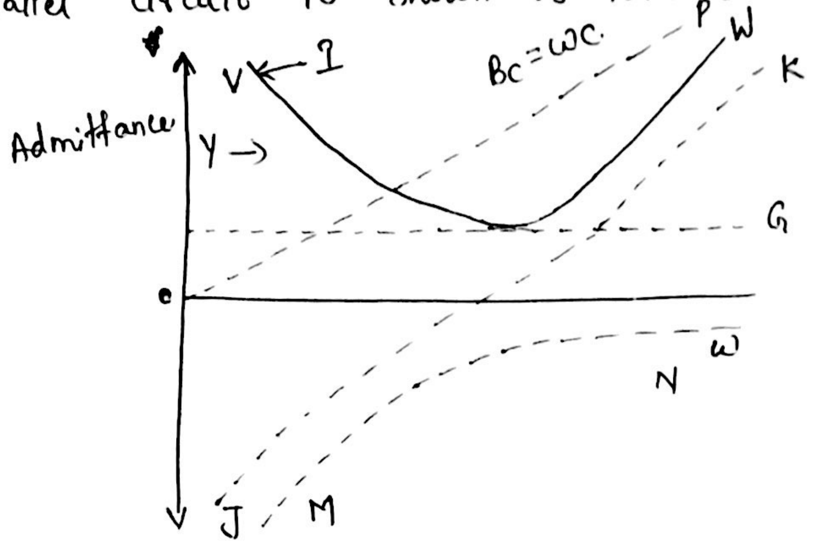
Since $I_L = \dfrac{V}{\omega_r L} = \dfrac{IR}{\omega_r L} = I Q_r$.

For the capacitor, $I_C = \dfrac{V}{1/\omega_r C} = IR\,\omega_r C = I Q_r$.

Therefore, the quality factor $Q_r = I_L / I$ or I_C / I.

Reactance curves in parallel Resonance :-

The effect of variation of frequency on the reactance of the parallel circuit is shown as follows:

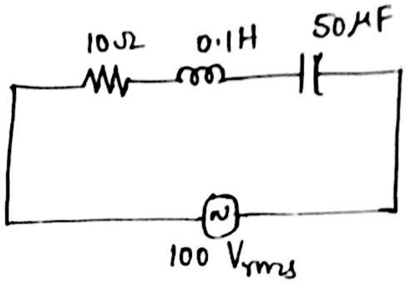


The effect of inductive susceptance,

$$B_L = \frac{-1}{2\pi f L}$$

Inductive susceptance is inversely proportional to the freq. or ω. Hence, it is represented by a rectangular hyperbola, MN. It is drawn in fourth quadrant, since B_L is $-$ve. Capacitive susceptance, $B_C = 2\pi f C$. It is directly proportional to the freq. f or ω. Hence it is represented by OP, passing through the origin. Net susceptance $B = B_C - B_L$. It is represented by ~~op~~ the curve JK, which is hyperbola. At the point ω_r, the total susceptance is zero, and resonance takes place. The variation of the admittance Y and the current I is represented by curve VW. The current will be minimum at resonant frequency.

Problem:- For the circuit shown, determine the freq. at which the circuit resonates. Also find the vol. across the inductor at resonance and the Q-factor of the circuit.



Sol:- The freq. of resonance occurs when $X_L = X_C$.

$$\omega L = \frac{1}{\omega C}.$$

$$\therefore \omega = \frac{1}{\sqrt{LC}} \text{ rad/sec}.$$

$$= \frac{1}{\sqrt{0.1 \times 50 \times 10^{-6}}} = 447.2 \text{ rad/sec}.$$

$$f_r = \frac{1}{2\pi}(447.2) = 71.17 \text{ HZ}.$$

The current passing through the circuit at resonance,

$$I = \frac{V}{R} = \frac{100}{10} = 10A.$$

The vol. drop across the inductor

$$V_L = IX_L = I\omega L$$

$$= 10 \times 447.2 \times 0.1$$

$$= 447.2 V$$

The quality factor $Q = \dfrac{\omega L}{R} = \dfrac{447.2 \times 0.1}{10} = 4.47$.

Problem:- A series RLC circuit has a quality factor of 5 at 50 rad/sec. The current flowing through the circuit at resonance is 10A and the supply voltage is 100V. The total impedance of the circuit is 20Ω. Find the circuit constants,

Sol:- The quality factor $Q = 5$

At resonance, the impedance becomes resistance.

The current at resonance is $I = \dfrac{V}{R}$

$$10 = \frac{100}{R}$$

$$\therefore R = 10\Omega$$

$$Q = \frac{\omega L}{R}$$

Since $Q = 5$, $R = 10$

$$\omega L = 50$$

$$\therefore L = \frac{50}{\omega} = 1H$$

$$Q = \frac{1}{\omega CR}$$

$$\therefore C = \frac{1}{QR}$$

$$= \frac{1}{5 \times 50 \times 10} = 400\mu F.$$

Problem :- A voltage $v(t) = 10 \sin \omega t$ is applied to a series RLC circuit. At the resonant freq. of the circuit, the max. vol. across the capacitor is found to be 500V. Moreover, the bandwidth is known to be 400 rad/s and the impedance at resonance is 100Ω. Find the resonant frequency. Also find the values of L and C of the circuit.

Sol :- The applied voltage to the circuit is

$$V_{max} = 10V$$

$$V_{rms} = \frac{10}{\sqrt{2}} = 7.07V.$$

The vol. across capacitor $V_c = 500V$.

The magnification factor $Q = \frac{V_c}{V} = \frac{500}{7.07} = 70.7$.

The bandwidth $BW = 400 \text{ rad/s}$

$$\omega_2 - \omega_1 = 400 \text{ rad/s}.$$

The impedance at resonance $Z = R = 100\Omega$

Since $Q = \frac{\omega_r}{\omega_2 - \omega_1}$

$$\omega_r = Q(\omega_2 - \omega_1) = 28280 \text{ rad/sec}.$$

$$f_r = \frac{28280}{2\pi} = 4499 \text{ Hz}.$$

The bandwidth $\omega_2 - \omega_1 = \frac{R}{L}$

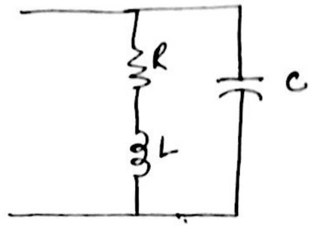
$$\therefore L = \frac{R}{\omega_2 - \omega_1} = \frac{100}{400} = 0.25H$$

Since $f_r = \frac{1}{2\pi\sqrt{LC}}$

$$C = \frac{1}{(2\pi f_r)^2 \times L} = \frac{1}{2\pi \times (4499)^2 \times 0.25} = 5nF.$$

Problem:-
In the circuit shown, an inductance of 0.1H having a Q of 5 is in parallel with a capacitor. Determine the value of Capacitance and coil resistance at resonant freq. of 500 rad/s.

Sol:- The Quality factor $Q = \frac{\omega_r L}{R}$



Since $L = 0.1H$, $Q = 5$ and

$\omega_r = 500$ rad/s

$$Q = \frac{500 \times 0.1}{R}$$

$$\therefore R = \frac{500 \times 0.1}{5} = 10\,\Omega$$

Since $\omega_r^2 = \frac{1}{LC}$

$$(500)^2 = \frac{1}{0.1 \times C}$$

$\therefore$ The capacitance value $C = \frac{1}{0.1 \times (500)^2} = 40\mu F$.

Problem:- A series circuit has a resonance freq. of 120KHZ, a bandwidth of 50KHZ, and $Q = 2$. Determine the cut-off frequencies

Sol:- Since $Q < 10$, we cannot use the formulas $f_2 = f_r + \frac{B.W}{2}$

$$f_1 = f_r - \frac{B.W}{2}$$

$$B.W = f_2 - f_1$$

$$50,000 = f_2 - f_1$$

$$f_2 = 50,000 + f_1$$

also $f_r = \sqrt{f_1 f_2}$

$$1,20,000 = \sqrt{f_1 f_2}$$

$$14.4 \times 10^9 = f_1 f_2$$

$$14.4 \times 10^9 = f_1(50,000 + f_1)$$

$$f_1^2 + 50,000 f_1 - 14.4 \times 10^9 = 0$$

$$f_1 = \frac{-50,000 \pm \sqrt{(50,000)^2 + 4(14.4 \times 10^9)}}{2} = \frac{-50,000 \pm 24.5 \times 10^1}{2}$$

$$f_1 = -147.5\text{ KHZ} ;\ 195\text{ KHZ}.$$

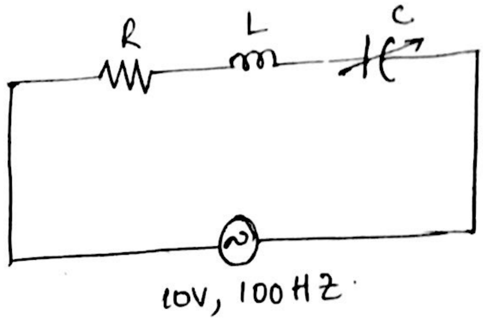
Since -ve freq. does not exist.

$$\boxed{f_1 = 195\text{ KHZ}}$$

$$50 = f_2 - 195$$

$$\therefore \boxed{f_2 = 245\text{ KHZ}}$$

problem :- In the circuit shown, determine the circuit constant when the circuit draws a max. current at 10 μF with a 10V, 100 HZ supply. when the capacitance is changed to 12 μF, the current that flows through the circuit becomes 0.707 times its maximum value. Determine Q of the coil at 900 rad/sec. Also find the maximum current that flows through the circuit.



Sol :- At resonant freq, the circuit draws maximum current. So, the resonant freq. $f_r = 100\,Hz$.

$$f_r = \frac{1}{2\pi\sqrt{LC}}$$

$$L = \frac{1}{C(2\pi f_r)^2}$$

$$= \frac{1}{10\times10^{-6}(2\pi\times100)^2} = 0.25\,H$$

we have $\omega L - \frac{1}{\omega C} = R$

$$900\times0.25 - \frac{1}{900\times12\times10^{-6}} = R$$

$$\therefore R = 132.4\,\Omega$$

The Quality factor $Q = \frac{\omega L}{R} = \frac{900\times0.25}{132.4} = 1.69$

the max. current in the circuit is $I = \frac{10}{132.4} = 0.075\,A$