**UNIT-2**

**Solution for linear systems**

**Matrix :** A system of mn numbers real (or) complex arranged in the form of an ordered set of ‘m’ rows,, each row consisting of an ordered set of ‘n’ numbers between [ ] (or) ( ) (or) || || is called a matrix of order m xn.

Eg:  [aij ]mxn where 1≤ i≤m, 1≤j≤n.

**some types of matrics :**

1. **square matrix :** A square matrix A of order nxn is sometimes called as a n- rowed matrix A (or) simply a square matrix of order n

eg :is 2nd order matrix

**2. Rectangular matrix :** A matrix which is not a square matrix is called a rectangular matrix,

is a 2x3 matrix

**3. Row matrix :** A matrix of order 1xm is called a row matrix

eg: 

**4. Column matrix :** A matrix of order nx1 is called a column matrix

Eg: 

1. **Unit matrix :** if A= [aij] nxn such that aij = 1 for i = j and aij = 0 for i≠j, then A is called a unit matrx.

Eg:I2 =  I3=

1. **Zero matrix :** it A = [aij] mxn that aij = i, j then A is called a zero matrix (or) null matrix

Eg: O2x3=

1. **Diagonal elements in a matrix** A= [aij]mxn, the elements aij of A for which i = j i.e. (a11, a22….ann) are called the diagonal elements of A

Eg: A= diagonal elements are 1,5,9

Note : the line along which the diagonal elements lie is called the principle diagonal of A

1. **Diagonal matrix :** A square matrix all of whose elements except those in leading diagonal are zero is called diagonal matrix.

If d1, d2….. dn are diagonal elements of a diagonal matrix, A, then A is written as A = diag (d1,d2….dn)

Eg : A = diag (3,1,-2)

1. **Scalar matrix :** A diagonal matrix whose leading diagonal elements are equal is called a scalar matrix. Eg : A= 
2. **equal matrices :** Two matrices A = [aij] and b= [bij] are said to be equal if and only if (i) A and B are of the same type (ii) aij = bij for every i&j
3. **The transpose of a matrix :** The matrix obtained from any given matrix A, by inter changing its rows and columns is called the transpose of A. It is denoted by a1 (or) aT.

If A = [aij] mxn then the transpose of A. is A1 = [bij] nxm, where bji = aij Also (a1)1 = A

Note : A1 and B1 be the transposes of A and B repectively, then

(i) (A1)1 = A

(ii) (A+B)1 = A1+B1

(iii) (KA)1 = KA1, K is a scalar

(iv) (AB)1= B1A1

**12. The conjugate of a matrix :** The matrix obtained from any given matrix A, on replacing its elements by corresponding conjugate complex numbers is called the conjugate of A and is denoted by 

**Note :** if and  be the conjgates of A and B respectively then,

(i)  = A

(ii) (A+B) = A+B

(iii) (KA) = KA, K is a any complex number

(iv) (AB)= B A

Eg ; if A= 

**13. The conjugate Transpose of a matrix**

The conjugate of the transpose of the matrix A is called the conjugate transpose of A and is denoted by Aθ Thus Aθ = is the transpose of A1 now a = [aij]→ Aθ =[bij],

where bij =ij i.e. the (i,j)th element of Aθ conjugate complex of the (j, i)th element of A

Eg: if A =  then Aθ =3x2

Note: Aθ = 

**14.(i) Upper Triangular matrix :** A square matrix all of whose elements below the leading diagonal are zero is called an Upper triangular matrix.

Eg; 



(ii) **Lower triangular matrix ;** A square matrix all of whose elements above the leading diagonal are zero is called a lower triangular matrix

Eg: 

**15. Symmetric matrix :** A square matrix A =[aij] is said to be symmetric if aij = aji for every i and j Thus A is a symmetric matrix iff A1 = A

Eg:  Is a symmetric matrix

**16. Skew – Symmetric matrix :** A square matrix A = [aij] is said to be skew – symmetric

if aij = – aji for every i and j.

Eg :  is a skew – symmetric matrix

Thus A is a skew – symmetric iff A= -A1

**Note:** Every diagonal element of a skew – symmetric matrix is necessarily zero.

Since aii = - aii →aii = 0

**17. Multiplication of a matrix by a scalar:**

Let ‘A’ be a matrix. The matrix obtain by multiplying every element of A by K, a scalar is called the product of A by K and is denoted by KA (or) AK

Thus : Id A + [aij] then KA = [kaij] = k[aij]

**18. Sum of matrices :**

Let A = [aij] B = [bij] be two matrices. The matrix c = [aij] where cij = aij+bij is called the sum of the matrices A and B.

The sum of A and B is denoted by A+B thus [aij] + [bij]= [aij+bij]

**19. The difference of two matrices :** If A, B are two matrices of the same type then A+(-B) is taken as A – B

**20. Theorem :** Every square matrix can be expressed as the sum of a symmetric and skew – symmetric matrices in one and only way

**Proof :** let A be any square matrix. We can write

A= ½ (A+A1)+ ½ (A-A1)=P+Q (say).

Where P = ½ (A+A1)

Q = ½ (A-A1)

We have P1 = {½ (A+A1)}1 = ½ (A+A1)1 since [(KA)1 = KA1]

= ½ [A+(A1)1]= ½ [A+A1]=P

P is symmetric matrix.

Now , Q1 = [ ½ (A-A1)]1 = ½ (A-A1)1

= ½ [A1-(A1)1] = ½ (A1-A)

= - ½ (A-A1)= -Q

Q is a skew – symmetric matrix.

Thus square matrix = symmetric + skew – symmetric then to prove the sum is unique.

It possible, let A = R+S be another such representation of A where R is a symmetric one S is a skew – symmetric matrix.

R1 = R and S1 = -S

Now A1 = (R+S)1 = R1+S1 = R-S and

½ (A+A1) = ½ (R+S+R-S) = R

½ (A-A1) = ½ (R+S-R+S) = S

⇒R = P and S=Q

Thus, the representation is unique.

**Theorem2** = Prove that inverse of a non – singular symmetric matrix A = symmetric.

Proof : since A is non – singular symmetric matrix A-1 exists and AT = A

Now, we have to prove that A-1 is symmetric we have (A-1)T = (AT)-1 = A-1 (by (1)) Since (A-1)T = A-1 therefore, A-1 is symmetric.

**Theorem3 :** if A is a symmetric matrix, then prove that adj A is also symmetric

Proof : Since A is symmetric, we have AT = A … (1)

Now, we have (adjA)T = adj AT [ since adj A1 = (AdjA)1]

= adj A [by (1) ]

(adjA)T = adjA therefore, adjA is a symmetric matrix.

**20. Matrix multiplication** : Let A = , B = then the matrix c = [cij]mxp where cij is called the product of the matrices A and B in that order and we write C = AB.

The matrix A is called the pre-factor & B is called the post – factor

**Note :** If the number of columns of A is equal to the number of rows in B then the matrices are said to be conformable for multiplication in that order.

**Theorem 4 :** Matrix multiplication is associative i.e. if A,B,C are matrices then (AB) c= A(BC)

Proof : Let A= [aij]B = [bjk] and C= 

Then AB = [uik] where uik= ------(1)

Also BC = [vjl]where vjl =------(2)

Now, A(BC) is an mxq matrix and (AB)C is also an mxq matrix.

let A(BC) = [*w*il] where *w*il is the (i,j)th element of A(BC)

Then *w*il = 

= 

=

(Since Finite summations can be interchanged)

=  (from (1))

= The (i,j)th element of (AB)C

A(BC) = (AB)C

**21. Positive integral powers of a square matrix:**

Let A be a square matrix. Then A2 is defined A.A

Now, by associative law A3 = A2.A = (AA)A

= A(AA) = AA2

Similarly ­­­­ we have Am-1A = A Am-1 = Am where m is a positive integer

Note : In = I

On = 0

Note 1: multiplication of matrices is distributive w. r .t. addition of matrices.

i.e. A(B+C) = AB + AC

(B+C)A = BA+CA

Note 2: if A is a matrix of order mxn then AI­n = InA = A

22. **Trace of A square matrix :** Let A = [aij] the trace of the square matrix A is defined as . And is denoted by ‘tr A’

Thus trA =  = a11+a22+ …….ann

Eg : A =  then trA = a+b+c

**Properties :** If A and B are square matrices of order n and λ is any scalar, then

1. tr (λ A) = λ tr A
2. tr (A+B) = trA + tr B
3. tr(AB) = tr(BA)

**23. Idempotent matrix :** If A is a square matrix such that A2 = A then ‘A’ is called idempotent matrix

**24. Nilpotent Matrix :** If A is a square matrix such that Am=0 where m is a +ve integer then A is called nilpotent matrix.

Note : If m is least positive integer such that Am = 0 then A is called nilpotent of index m

**25. Involutary :** If A is a square matrix such that A2 = I then A is called involuntary matrix.

**26. Orthogonal Matrix :** A square matrix A is said to be orthogonal if AA1 = A1A = I

**Theorem 5 :** If A, B are orthogonal matrices, each of order n then AB and BA are orthogonal matrices.

**Proof :** Since A and B are both orthogonal matrices.

AAT = ATA =I -------- 1

BBT = BTB = I -------- 2

Now (AB)T = BTAT

Consider (AB)T (AB) = (BTAT) (AB)

= BT(ATA)B

= BTIB (by 1)

= BTB

= I (by 2)

AB is orthogonal

Similarly we can prove that BA is also orthogonal

**Theorem 6 :** Prove that the inverse of an orthogonal matrix is orthogonal and its transpose is also orthogonal.

Proof : let A be an orthogonal matrix

Then AT­.A = AAT= I

Consider AT­A = I

Taking inverse on both sides (AT­.A)-1 = I -1

A-1(AT) -1 = I

A-1(A-1) T = I

A-1 is orthogonal

Again AT.A = I

Taking transpose on both sides (AT.A) T = IT

AT(AT) T = I

Hence AT is orthogonal

**Examples:**

**1. show that A =  is orthogonal.**

Sol: Given A = 

AT = 

Consider A.AT =  

= 



A is orthogonal matrix.

**2. Prove that the matrix is orthogonal.**

Sol: Given A = 

Then AT = 

Consider AAT =  

==

AAT = I

Similarly AT A = I

Hence A is orthogonal Matrix

**3. Determine the values of a, b, c when  is orthogonal.**

Sol: - For orthogonal matrix AAT =I

So AAT = 

= I

Solving 2b2-c2 =0, a2-b2-c2 =0

We get c =  a2 =b2+2b2 =3b2

⇨ a = 

**From the diagonal elements of I**

4b2+c2= 1 ⇨ 4b2+2b2=1 (c2=2b2)

⇨ b = 

a= 

= 

b= 

c = 

= 

**27. Determinant of a square matrix:**

If A =  then  

**28. Minors and cofactors of a square matrix**

Let A =[aij] be a square matrix when form A the elements of ith row and jth column are deleted the determinant of (n-1) rowed matrix [mij] is called the minor of aij of A and is denoted by |mij|

The signed minor (-1) i+j |mij| is called the cofactor of aij and is denoted by Aij..

If A =  then

| A | = a11 |m11| + a12 |m12 | +a13 |m13| (or)

= a11 A11 +a12 A12 +a13 A13

**Eg: Find the Determinant of  by using minors and co-factors.**

Sol: A = 

detA = 1

=1(-12-12)-1(-4-6)+3(-4+6)

= -24+10+6 = -8

similarly we find det A by using co-factors also.

Note 1: If A is a square matrix of order n then , where k is a scalar.

Note 2: If A is a square matrix of order n, then 

Note 3: If A and B be two square matrices of the same order, then 

**29. Inverse of a Matrix:** let A be any square matrix, then a matrix B, if exists such that AB = BA =I then B is called inverse of A and is denoted by A-1.

**Theorem 7:** The inverse of a Matrix if exists is Unique

Proof: Let if possible B and C be the inverses of ‘A’.

Then AB = BA =I

AC = CA= I

consider B = BI

=B(AC)

=(BA)C

=IC

⇨B=C

Hence inverse of a Matrix is &Unique

Note:1 (A-1)-1 = A

Note 2: I-1 = I

**30. Adjoint of a matrix:**

Let A be a square matrix of order n. The transpose of the matrix got from A

By replacing the elements of A by the corresponding co-factors is called the adjoint of A and is denoted by Adj A.

Note: For any scalar k, Adj(kA) = kn-1 adj A

**Note :** The necessary and sufficient condition for a square matrix to posses inverse is that 

Note: if  then 

**3. Singular and Non-singular Matrices:**

A square matrix A is said to be singular if .

If  then A is said to be non-singular.

Note: 1. only non-singular matrices posses inverses.

2. The product of non-singular matrices is also non-singular.

**Theorem 9**: If A, B are invertible matrices of the same order, then

(i). (AB)-1 = B-1A-1

(ii). (A1)-1 = (A-1)1

Proof: (i). we have (B-1A-1) (AB) = B-1(A-1A)B

= B-1(I B)

= B-1B

= I

(AB)-1 = B-1A-1

(ii). A-1A = AA-1 = I

Consider A-1A =I

→(A-1 A)1 = I1

→ A1. (A-1)1 = I

→(A1)-1 = (A-1)1

**Examples:**

**1). Express the matrix A as sum of symmetric and skew – symmetric matrices. Where**

A = 

Sol: Given A = 

Then AT = 

Matrix A can be written as A = ½ (A+AT)+ ½ (A-AT)

⇨P = ½ (A+AT) = 



Q= ½ (A-AT)

= 

=

A = P+Q where ‘P’ is symmetric matrix

‘Q’ is skew-symmetric matrix.

**2. find the Adjoint and inverse of a matrix A = **

Sol: Adj( A) = 

Where Aij are the cofactors of the elements of aij.

Thus minors of aij are

  

  

  

Cofactors Aij = (-1)i+j Mij

Adjoint of A =  

 -4-2(-1) +3(14) = 40





**MATRIX INVERSE METHOD**

**3). Solve the equations 3x+4y+5z = 18, 2x-y+8z =13 and 5x-2y+7z =20**

Sol: The given equations in matrix form is AX = B



det A = 3(-7+16)-4(14-40)+5(-4+5) = 136

co-factor matrix is D = 

D = 

Adj A = DT = 

A-1 = 1/det A adj A = 

A x = B => x = A-1 B







Soln is x =3 , y=1, z=1

**Sub – Matrix:** Any matrix obtained by deleting some rows or columns or both of a given matrix is called is sub matrix.

e.g: let A =  then  is a sub matrix of A obtained by deleting third row and 4th column of A.

**Minor of a Matrix:** let A be an mxn matrix. The determinant of a square sub matrix of A is called a minor of the matrix.

Note: If the order of the square sub matrix is ‘t’ then its determinant is called a minor of order ‘t’.

Eg: A = 

 is a sub-matrix of order ‘2’

 = 2-3 = -1 is a minor of order ‘2’

 is a sub-matrix of order ‘3’

det C = 2(7-12)-1(21-10)+(18-5)

= 2(-5)-1(11)+1(13)

= -10-11+13 = -8 is a minor of order ‘3’

**\*Rank of a Matrix:**

Let A be mxn matrix. If A is a null matrix, we define its rank to be ‘o’. if A is a non-null matrix, we say that r is the rank of A if

1. Every (r+1) th order minor of A is ‘o’ (zero) &
2. At least one r th order minor of A which is not zero.

**Note:** 1. it is denoted by ρ(A)

2. Rank of a matrix is unique.

3. Every matrix will have a rank.

4. if A is a matrix of order mxn,

Rank of A ≤ min(m,n)

5. if ρ(A) = r then every minor of A of order r+1, or more is zero.

6. Rank of the Identity matrix In is n.

7. If A is a matrix of order n and A is non-singular then ρ(A) = n

**Important Note:**

1. The rank of a matrix is ≤r if all minors of (r+1)th order vanish.
2. The rank of a matrix is ≥r, if there is at least one minor of rth order which is not equal to zero.

**Examples:**

1. **find the rank of the given matrix **

sol: Given matrix A = 

→ det A = 1(48-40)-2(36-28)+3(30-28)

= 8-16+6 = -2 ≠ 0

We have minor of order 3 ≠ 0

Ρ(A) =3

**2. Find the rank of the matrix **

Sol: Given the matrix is of order 3x4

Its Rank ≤ min(3,4) = 3

Highest order of the minor will be 3.

Let us consider the minor 

Determinant of minor is 1(-49)-2(-56)+3(35-48)

= -49+112-39 = 24 ≠ 0.

Hence rank of the given matrix is ‘3’.

**\* Elementary Transformations on a Matrix:**

i). Interchange of ith row and jth row is denoted by Ri ↔ Rj

(ii). If ith row is multiplied with k then it is denoted by Ri →K Ri

(iii). If all the elements of ith row are multiplied with k and added to the corresponding elements of jth row then it is denoted by Rj → Rj +KRi

**Note:** 1. The corresponding column transformations will be denoted by writing ‘c’. i.e

ci ↔cj, ci ↔ k cj cj ↔cj + kci

2. The elementary operations on a matrix do not change its rank.

**Equivalance of Matrices:** If B is obtained from A after a finite chain of elementary transformations then B is said to be equivalent to A.

It is denoted as B~A.

**Note :** 1. If A and B are two equivalent matrices, then rank A = rank B.

2. If A and B have the same size and the same rank, then the two matrices are equivalent.

**Echelon form of a matrix:**

A matrix is said to be in Echelon form, if

(i). Zero rows, if any exists, they should be below the non-zero row.

(ii). The first non-zero entry in each non-zero row is equal to ‘1’.

(iii). The number of zeros before the first non-zero element in a row is less than the number of such zeros in the next row.

**Note :** 1. The number of non-zero rows in the row echelon form of A is the rank of ‘A’.

2. The rank of the transpose of a matrix is the same as that of original matrix.

Eg: 1.  is a row echelon form.

2.  is a row echelon form.

**Examples :**

1. **find the rank of the matrix A =  by reducing it to Echelon form.**

soln: Given A = 

Applying row transformations on A.

A ~  R1 ↔ R3

~ R2 → R2 –3R1

R3→ R3 -2R1

~R31→ R3 – R2

~

This is the Echelon form of matrix A.

The rank of a matrix A.

= Number of non – zero rows =2

1. **For what values of k the matrix  has rank ‘3’.**

Sol: The given matrix is of the order 4x4

If its rank is 3 ⇨ det A =0

A = 

Applying R2 → 4R2-R1, R3 →4R3 – kR1, R4 → 4R4 – 9R1

We get A ~ 

Since Rank A = 3 🡺 det A =0

🡺 4 

🡺 1[(8-4k)3]-1(8-4k)(4k+27)] = 0

🡺 (8-4k) (3-4k-27) = 0

🡺(8-4k)(-24-4k) =0

🡺 (2-k)(6+k)=0

🡺 k =2 or k = -6

**Normal Form:**

Every mxn matrix of rank r can be reduced by a finite number of elementary transformations to the form , where Ir is the r – rowed unit matrix.

Note: 1. If A is an mxn matrix of rank r, there exists non-singular matrices P and Q such that PAQ = 

2. Normal form another name is “canonical form”

Eg: By reducing the matrix into normal form, find its rank.

Sol: Given A =

A ~  R2 → R2 – 2R1

R3 → R3 – 3R1

A ~  R3 → R3/-2

A ~  R3 → R3+R2

A ~  c2 - 2c1

c3-3c1

c4-4c1

A ~ 3c3 -2c2

3c4-5c2

A ~  c2/-3

c4/18

A~ c4 ↔ c3

This is in normal form [I3 0]

Hence Rank of A is ‘3’.

**Note:** we know – non –singular matrices P and Q such that P A Q = 

Suppose we want to find P and Q we have procedure.

Let order of matrix ‘A’ is ‘3’

🡺 A = I3 A I3

A =  A 

Now we go on applying elementary row operations and column operations on the matrix A (L.H.S) until it is reduced to the normal form 

Every row operations will also be applied to the pre-factor of A on R.H.S

Every column operation will also be applied to the post –factor of A on R.H.S.

**Examples:**

1. find the non-singular matrices P and Q is of the normal form where A = 

sol: Write A = I3 A I3

 =  A 

=  A  R2 → R2-2R1

R3→R3-3R1

  A  R3 → R3-R2

R2 →1/3 R2

=  A  c3 → c3-2c1

I2 = PAQ where P = Q = 

1. **Find the non-singular matrices P and Q such that the normal form of A is P A Q**.

Where A = . Hence find its rank.

Sol: we write A = I3A I4

→ = A 

→= A R2 →R2- R1

R3→R3-R1

= A R3 →R3-2R2

**Applying c2-3c1, c3-6c1, and c4+c1, we get.**

= A 

Applying c3+c2 and c4-2c2, we get

= A 

→ = P A Q

Where P =  Q= 

Here A ~, Hence ρ(A) =2

The inverse of a matrix by elementary Transformations: **(Gauss – Jordan method)**

1. suppose A is a non-singular matrix of order ‘n’ then we write A = In A
2. Now we apply elementary row-operations only to the matrix A and the pre-factor In of the R.H.S
3. We will do this till we get In = BA then obviously B is the inverse of A.

1. **Find the inverse of the matrix A using elementary operations**.

**Given A = **

We can write A = I3 A

=  A

Applying R3 →2R3-R2, we get

=  A

Applying R1→R1-3R2, we get

=  A

Applying R1 → R1+5R3, R2 → R2-3R3/2 , we get

= A ⇨I3 = BA

B is the inverse of A.

**System of linear simultaneous equations:**

**1.Linear Equation:**

An Equation is of the form a1x1+a2x2+a3x3+…..+….anxn =b where x1,x2,xn are unknown and a1,a2,…an, b are constants is called a linear equation in ‘n’ unknowns.

Consider the system of m linear equations in n unknowns x1,x2,…..xn as given below

a11x1+a12x2+a13x3+…..+….a1nxn =b1

a21x1+a22x2+a23x3+…..+….a2nxn =b2

………………………………….. ----------------(1)

am1x1+am2x2+am3x3+…..+….amn xn =bm

where aij’s and b1,b2---bm are constants.

\*An ordered n-tuple (x1 x2….xn) satisfying all the equations in (1) simultaneously is called a solution of the system (1).

**Consistent:** Above system (1) have at least one solution, then the system is called consistent.

If (1) does not have any solution, then the system is called inconsistent.

The system of equations in (1) can be written in matrix form as A x =B ----(2)

Where A = [aij] *X*= (x1,x2…..xn)T and B = (b1,b2….bm)T

Note: The matrix [A / B] is called the augmented matrix of the system (1).

**Homogeneous system:** In AX =B, if B = 0 then the system is called homogeneous, otherwise the system is called non-homogeneous.

Note: 1. The system AX =0 is always consistent since x =0 (i.e x1= x2 …..xn =0) is always a solution of AX =0

→This solution is called a trivial solution of the system.

2. Given AX = 0, we try to decide wheather it has a solution x ≠0, such a solution is called a non-trivial soln.

**Examples:**

1). 2x+3y =0 , 3x-2y=0 here x =0, y =0 is a solution

We cannot find any other solution.

About system has only trivial –soln.

2). 3x+5y=0, 6x+10y =0 here x =0, y =0 is a solution and (5k,-3k)is also solution. Where k any constant i.e (5,-3) taking k=1 is solution the system.

Above Homo’s system have both trivial and non-trivial solutions.

3). 3x+5y = 4, 4x+3y = 4

This system has a unique solution.

It is .

4). 3x+5y =2, 6x+10y =4

Using 3x+5y =2 we get x = 2-5y/3= 2/3 -5y/3

For any y = k. x= 2-5k/3

x=2-5k/3, y =k is a sol of the system.

The system is consistent and has infinitely many solutions.

Procedures to solve AX = B

Let us first consider n equations in n unknowns ie. m=n then the system will be of the form

a11x1+a12x2+a13x3+…..+….a1nxn =b1

a21x1+a22x2+a23x3+…..+….a2nxn =b2

…………………………………..

an1x1+an2x2+an3x3+…..+….annxn =bn

The above system can be written as AX = B ---------(1)

Where A is an mn matrix.

**Cramer’s Rule (Determinant Method)**

The solution of the system of linear equations

a1*x*+b1*y*+c1*z* =d1

a2*x*+b2*y*+c2*z* =d2

a3*x*+b3*y*+c3*z* =d3

x=  y = z = 

  



**Eg: Solve the following equations by cramer’s rule.**

Sol: x+y+z = 6 x-y+z = 2 2x-y+3z = 9

 =1(-2)-1(1)+1(1)

= -2-1+1 = -2

 = 6(-2)-1(-3)+1(7)  =1(-3)-6(1)+1(5)

= -12+3+7 = -2 =-3-6+5 = -4

 = 1(-7)-1(5)+6(1)

= -7-5+6 = -6

x =  = -2/-2 = 1 y= = -4/-2 = 2 z = = -6/-2 = 3

**1). Solution by using A-1**

Pre-multiplying the above equation (1) by A-1

We get A-1(Ax) = A-1B

🡺(A-1A)x = A-1B

🡺 Ix = A-1B

X = A-1B

Thus a solution is obtained.

**2). Solution by Cramer’s rule:**

If , let us define Ai = matrix obtained by replacing the ith column of A by B.

The solution is given by xi = det Ai/det A where i=1 to n

Note : The above two-methods are applicable, if .

3**). Solving Ax = B using Echelon form:**

Consider the system of m equations in n unknowns given by

a11x1+a12x2+a13x3+…..+….a1nxn =b1

a21x1+a22x2+a23x3+…..+….a2nxn =b2

…………………………………..

am1x1+am2x2+am3x3+…..+….amnxn =bm

we know this system can we write Ax = B

→ The augmented matrix of the above system is [A / B]



The system Ax = B is consistent if ρ(A) = ρ[A/B]

i). ρ(A) = ρ[A/B]= r < n(no. of unknowns).

Then there are infinite no of solutions.

ii). ρ(A) = ρ[A/B] = number of unknowns then the system will have unique solution.

iii). ρ(A) ≠ ρ[A/B] the system has no solution.

**Examples:**

**1). Show that the equations x+y+z = 4, 2x+5y-2z =3, x+7y-7z =5 are not consistent.**

Sol: write given equations is of the form Ax = B

i.e 

consider Augment matrix i.e [A /B]

🡺 [A/B] = 

Applying R2 →R2-2R1 and R3 → R3-R1, we get

[A/B] ~ 

Applying R3→ R3-2R1, we get

[A/B] ~ 

ρ (A) =2 and ρ(A/B) =3

The given system is inconsistent since ρ(A) ≠ ρ[A/B].

**2). Show that the equations given below are consistent and hence solve them**

x-3y-8z = -10, 3x+y-4z =0, 2x+5y+6z =3

**sol:** matrix notation is



Augmented matrix [A/B] is

[A/B] = 

~R2 → R2-3R1

R3 → R3 -2R1

~R2 →1/10 R2

R3 →1/11R3

~~R3 →R3-R2

This is the Echelon form of [AB]

Ρ(A) =ρ(A/B) = 2<3 (no. of unknown)

The system has infinite number of soln.

The given system of equations is equal to



x-3y-8z = -10

y+2z =3

Give arbitary value to z. i.e say z =k then y = 3-2k and x = ++2k

For different values of k, we have an infinite number of solutions

.

**3). Discuss for what values of λ, μ the simultaneous equations x+y+z = 6,x+2y+3z =10** **x+2y+λz = μ** **have**

**(i). no solution**

**(ii). A unique solution**

**(iii). An infinite number of solutions.**

Sol: The matrix form of given system of Equations is

A x = = B

The augmented matrix is [A/B] = 

[A/B] ~R2 → R2 – R1

R3 → R3-R1

~R3 →R3 – R2

Case (i): let λ ≠ 3 the rank of A = 3 and rank [AB] = 3

Here the no. of unknowns is ‘3’

Here ρ (A) =ρ(A/B)= No. of unknows

The system has unique solution if λ≠3 and for any value of ‘μ’.

Case (ii). Suppose λ =3 and μ≠10.

We have ρ(A)= 2 ρ(AB) = 3

The system have no solution.

Case (iii): let λ =3 and μ=10.

We have ρ(A)= 2 ρ(AB) = 2

Here ρ(A)= ρ(AB) ≠ No. of unknowns =3

The system has infinitely many solutions.

**Linearly dependent set of vectors:**

A set { x1,x2,--------------xr} of r vectors is said to be a linearly dependent set, if there exist r scalars k1,k2---kr not all zero, such that k1x1+k2x2+--------krxr = 0

**Linearly independent set of vectors:**

A set { x1, x2,-------------xr} of r vectors is said to be a linearly independent set, if k1x1+k2x2+--------------+krxr =0 then k1 =0, k2 = 0------kr =0.

**Linear combination of vectors:**

A vector x which can be expressed in the form x = k1x1+k2x2+--------------+knxn is said to be a linear combination of x1,x2------xn here k1,k2-----------kn are any scalars.

**Consistency of system of Homogeneous linear equations:**

A system of m homogeneous linear equations in n unknowns, namely

a11x1+a12x2+a13x3+…..+….a1nxn =0

a21x1+a22x2+a23x3+…..+….a2nxn =0

………………………………….. -------------(1)

am1x1+am2x2+am3x3+…..+….amnxn =0

we write above system is of the form AX = 0

i.e 

here A is called Co –efficient matrix.

**Note:**

1. Here x1= x2 = --------- xn = 0 is called trivial solution or zero solution of AX = 0
2. A zero solution always linearly dependent.

**Theorem:** The number of linearly independent solutions of the linear system Ax = 0 is

(n-r) , r being the rank of the matrix A and n being the number of variables.

**Note:**

1. if A is a non-singular matrix then the linear system Ax = 0 has only the zero solution.
2. The system Ax =0 possesses a non-zero soln. if and only if A is a singular matrix.

Working rule for finding the solutions of the equation Ax =0

If (i).Rank of A = No. of unknowns i.e r = n then the system has zero solution.

(ii). Rank of A < No of unknowns (r<n) and No. of equations < No. of unknowns (m<n) then the system has infinite no. of solutions.

Note: If Ax =0 has more unknowns than equations the system always has infinite solutions.

**Examples:**

1). Solve the system of equations x+3y-2z =0 2x-y-4z =0 , x-11y+4z =0

Sol: we write the given system is Ax = 0

i.e 

**Examples:**

**1). Solve the system of equations x+3y-2z = 0, 2x-y+4z = 0, x-11y+14z = 0**

Sol: We write the given system is Ax = 0

ie. 

A ~ R2 →R2 -2R1

R3→R3-R1

~R3 →R3 -2R2

The Rank of the A = 2 ie. ρ(A)

No of unknowns is ‘3’

We have infinite No. of solution

Above matrix can we write as

x+3y-2z =0 -7y+8z =0, 0=0

say z = k then y=8/7k & x= -10/7 k

giving different values to k, we get infinite no. of values of x,y,z.

**2). Show that the only real number λ for which the system x+2y+3z = λx, 3x+3y+z = λz, has non-zero solution is 6 and solve them.**

Sol: Above system can we expressed as Ax = 0

ie. 

given system of equations possess a non –zero solution 🡺 i.e ρ(A) < no. of unknowns.

🡺 For this we must have det A = 0

🡺 

🡺  R1 →R1+R2+R3

🡺(6-λ) 

🡺 (6-λ)  c2 → c2-c1

c3→ c3 –c1

🡺 (6-λ)[(-2- λ)(-1- λ)+1] =0

🡺 (6- λ) (λ2+3 λ+3) = 0

🡺 λ = 6 only real values.

When λ = 6, the given system becomes



Sol : 🡺 R2 → 5R2+3R1

R3→5R3+2R1

🡺  R3 →R3+R2

-5x+2y+3z = 0 and -19y+19z = 0

🡺 y =z

Say z = k 🡺 y = k and x =k.

Solution is x =y =z=k.

**System of linear equations – Triangular systems:**

Consider the system of n linear algebraic equations in n unknowns

a11 x1+a12x2+………+a1n =b1

a21 x1+a22x2+………+a2n =b2

---------------------------------------

---------------------------------------

an1 x1+an2x2+………+ann =bn

The given system we can write Ax =B

i.e 

**Lower Triangular system:**

Suppose the co-efficient matrix A is such that all the elements above the leading diagonal are zero. That is , A is a lower triangular matrix of the form.

A = 

In this case the system will be of the form

a11 x1 =b1

a21 x1+a22x2+………+a2n =b2

---------------------------------------

---------------------------------------

an1 x1+ an2x2+………+ann =bn

from above equations, we get

x1 = b1/a11

x2 = 

The method of constructing the exact solution is called method of forward substitution.

**Upper triangular system:**

Suppose the co-efficient matrix A is such that all the elements below the leading diagonal are all zero. i.e A is an upper triangular matrix of the form.



Above system can be of the form

a11 x1+a12x2+………+a1n =b1

a22x2+………+a2n =b2

---------------------------------------

---------------------------------------

an1 x1+an2x2+………+ann =bn

from the above equations, we get

xn = bn



= and so on.

The method of constructing the exact solution is called method of backward substitution.

**Solution of linear systems – Direct methods**

**1. Gauss Elimination method**

Consider the system of equations for n=3

a11 x1+a12x2+a13x3 =b1

a21 x1+a22x2+a23x3 =b2

a31 x1+a32x2+a33x3 =b3

the Augmented matrix of this system is

‘

Applying R2 → R2 –a21/a11 R1 R3 →R3-a31/a11 R1, we get



Where α22 = a22 – (a21/a11) a12 α23= a23 – (a21/a11)a13

α32 = a32-(a31/a11) a12 α33 = a33-(a31/a11)a13

β2 = b2-(a21/a11)b1 β3 = b3-(a31/a11)b1

where a11 ≠ 0 and a11 is called first pivot.

-a21/a11, -a31/a11 are multipliers for the first stage.

Now Applying R3→R3 – α32/α22(R1) we get

[AB] ~

Where γ33 = α33 – (α32/α22) α23

∆3 = β3 -(α32/α22) β2

Here we assume α22 ≠ 0 and New Pivot is α22 & -(α32/α22) is the multipliler.

The augmented matrix becomes an upper triangular matrix, which can be solved by backward substitution.

**Note:-** if one of the elements a11,a22, a33 are zero the method is modified by rearranging the rows, so that the pivot is non-zero. This process is called partial pivoting.

**Example :** Solve the system of equations 3x+y-z = 3, 2x-8y+z = -5, x-2y+9z = 8 using Gauss elimination method.

Sol: The augmented Matrix is [A B] = 

Performing R2 →R2 – 2/3 R1

R3 → R3 – 1/3 R1, we get

[A B] ~

[A B]~ R3 →R3 -7/26 R2

From above we get

3x+y-z =3

-26/3 y +5/3 x = -7

693/78 x = 231/26

🡺 x =1, y=1, z=1

**2. Gauss – Jordan method:**

This method is a modification o f the Gauss elimination method. In this method unknowns are eliminated so that the system is in diagonal form. This can be done with or without using pivoting.

**Example:** Solve the equations 10x+y+z =12, 2x+10y+z =13 and x+y+5z = 7 by Gauss –jordan method.

Sol: The augmented matrix [A, B] = 

🡺[A / B] ~R1 →R1 -9R3

🡺[A / B] ~R2 →R2-2R1

R3 →R3-R1

🡺[A / B] ~R1 →8R1+8R2

R3 →R3-9R2

🡺[A / B] ~R3 →R3/-473

R1→R1-420R3

R2 →R2-583R3

Solution is x=1, y=1, z=1.