UNIT-IV

ORDINARY DIFFERENTIAL EQUATION'S OF FIRST ORDER & FIRST DEGREE

Definition: An equation which involves differentials is called a Differential equation.

Ordinary differential equation: An equation is said to be ordinary if the derivatives have reference to only one independent variable.

Ex . (1) $\frac{dy}{dx} + 7xy = x^2$ (2) $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = e^x$

(1) **Partial Differential equation:** A Differential equation is said to be partial if the derivatives in the equation have reference to two or more independent variables.

E.g: 1. $\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = 4z$

2. $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 2z$

Order of a D.E equation: A Differential equation is said to be of order 'n' if the n^{th} derivative is the highest derivative in that equation.

E.g : (1). $(x^2+1) \cdot \frac{dy}{dx} + 2xy = 4x^2$

Order of this Differential equation is 1.

(2) $x \cdot \frac{d^2y}{dx^2} - (2x-1) \cdot \frac{dy}{dx} + (x-1)y = e^x$

Order of this Differential equation is 2.

(3). $\frac{d^2y}{dx^2}$ + 5 $\left(\frac{dy}{dx}\right)^2$ +2y =0 .

Order=2 , degree=1.

(4). $\frac{\partial^2 u}{\partial x^2}$ + $\frac{\partial^2 u}{\partial y^2}$ =0. Order is 2.

Degree of a Differential equation: Degree of a D .Equation is the degree of the highest derivative in the equation after the equation is made free from radicals and fractions in its derivations.

E.g : 1) y = x . $\frac{dy}{dx}$ + $\sqrt{1+(\frac{dy}{dx})^2}$ on solving we get

(1- x^2) $(\frac{dy}{dx})^2$ +2xy . $\frac{dy}{dx} + (1-y^2)$ = 0 .

Degree = 2

2) a. $\frac{d^2y}{dx^2} = [1+(\frac{dy}{dx})^2]^{3/2}$ on solving . we get

$a^2.(\frac{d^2y}{dx^2})^2 = [1+(\frac{dy}{dx})^2]^3$. Degree = 2

Formation of Differential Equation : In general an O.D Equation is Obtained by eliminating the arbitrary constants c1,c2,c3--------cn from a relation like $\emptyset(x, y, c1, c2, ------cn) = 0.$ ------(1).

Where c1,c2,c3,---------cn are constants.

Differentiating (1) successively w.r.t x n-times and eliminating the n-arbitrary constant c1,c 2,----cn from the above (n+1) equations, we obtain the differential equation f(x , y, $y_1, y_{2,}$-----) =0.

PROBLEMS

1.Obtain the Differential Equation by Eliminating the Arbitrary Constants:

Sol. y= Ae^{-2x} + B e^{5x} --------------------(1).

$y_1 = A(-2)\ e^{-2x}$ + B(5) e^{5x} ----------(2).

y_2 = A (4) . e^{-2x} + B(25) e^{5x} ----------(3).

Eliminating A and B from (1), (2) & (3).

$$=> \begin{bmatrix} e^{-2x} & e^{5x} & -y \\ (-2)\ e^{-2x} & 5e^{5x} & -y_1 \\ (4)\ .e^{-2x} & 25\ e^{5x} & -y_2 \end{bmatrix} = 0$$

$$=> \begin{bmatrix} 1 & 1 & y \\ (-2) & 5 & y_1 \\ 4 & 25 & y_2 \end{bmatrix} = 0$$

⇨ $y_2 - 3y_1$-10y =0.

The required D. Equation obtained by eliminating A & B is

y_2 - 3y_1 -10y = 0

2). Log $(^{y}/_{x}) = cx$

Sol: Log $(^{y}/_{x}) = cx$ -------------(1).

=> log y –log x= cx

=> $\frac{1}{y}\ \frac{dy}{dx} - \frac{1}{x}$ =c ---------------(2).

(2) in (1) => Log $(^{y}/_{x})$ =x[$\frac{1}{y}\ \frac{dy}{dx} - \frac{1}{x}$].

3) $\sin^{-1} x + \sin^{-1} y$ = c.

Sol: $\frac{1}{\sqrt{1-x^2}} + \frac{1}{\sqrt{1-y^2}}\ \frac{dy}{dx}$ =0

⇨ $$\frac{dy}{dx} = \frac{\sqrt{1-y^2}}{\sqrt{1-x^2}}$$

4) y = e^x[Acosx +B sinx]

Sol: => $\frac{dy}{dx}$ = e^x[Acosx +B sinx] + e^x[-Asinx +B cosx]

=> $\frac{dy}{dx}$ = y + e^x(Asinx +B cosx).

=> $\frac{d^2y}{dx^2}$ = $\frac{dy}{dx}+e^x(-A\sin x+B\cos x)+e^x(-A\cos x-R\sin x)$

= $\frac{dy}{dx}+\frac{dy}{dx}-y-y$

= $\frac{d^2y}{dx^2}-2\frac{dy}{dx}+2y=0$ is required equation

5) Y= a $\tan^{-1}x$ + b.

Sol: $\frac{dy}{dx}$ = $\frac{a}{1+x^2}$

=> $(1+x^2)$. $\frac{d^2y}{dx^2}$ + 2x. $\frac{dy}{dx}$ =0

=> $(1+x^2)$. $\frac{d^2y}{dx^2}$ + 2x. $\frac{dy}{dx}$ =0 is the required equation.

6) y=a e^x + be^{-2x}

Sol: $\frac{d^2y}{dx^2}+$ $\frac{dy}{dx}$ -2y =0

7) Find the DEquation of all the circle of radius

Sol. The equation of circles of radius a is $(x-h)^2+(y-k)^2=a^2$ where (h ,k) are the co-ordinates of the center of circle and h,k are orbirtary constants.

Sol: $[1+(\frac{dy}{dx})^2]^3$ = $a^2 .\frac{d^2y}{dx^2}$

8) Find the DEquation of the family of circle passing through the origin and having their center on x-axis.

Ans: Let the general equation of the circle is $X^2+y^2+2gx+2fy+c=0$.

Since the circle passes through origin, so c=0 also the centre (-g,-f) lies on x-axis. So the y-coordinate of the centre i.e, f=0. Hence the system of circle passing through the origin and having their centres on x-axis is $x^2+y^2+2gx=0$.

Ans. $2xy \cdot \frac{dy}{dx} + x^2 - y^2 = 0$.

9) $\sin^{-1}(xy) + 4x = c$.

Ans: $x \cdot \frac{dy}{dx} + y + 4 \cdot \sqrt{1-x^2 y^2} = 0$

10) $y = \frac{a+x}{x^2+1}$

Sol: $(x^2+1) \cdot \frac{dy}{dx} + 2xy - 1 = 0$.

11) $r=a(1+\cos\theta)$

Sol: $r=a(1+\cos\theta)$ -----------(1)

$\frac{dr}{d\theta} = -a\sin\theta$ ----------- (2)

Put a value from (1) in (2).

$$\frac{dr}{d\theta} = \frac{-r}{1+\cos\theta} \cdot \sin\theta$$

$$\frac{dr}{d\theta} = \frac{-r \cdot 2\sin{}^{\theta}/_{2} \cdot \cos{}^{\theta}/_{2}}{2\cos^2 {}^{\theta}/_{2}}$$

$$= -r \tan{}^{\theta}/_{2}$$

Hence $\frac{dr}{d\theta} + r\tan{\theta}/{2} = 0$.

Differential Equations of first order and first degree:

The general form of first order ,first degree DEquation is $\frac{dy}{dx} =$ f(x,y) or [Mdx + Ndy =0 Where M and N are functions of x and y]. There is no general method to solve any first order D.Equation. The equation which belong to one of the following types can be easily solved.

In general the first order D.Equation can be classified as:

(1). Variable separable type

(2). (a) Homogeneous equation and

(b)Non-homogeneous equations which to exact equations.

(3). (a) exact equations and

(b)equations reducible to exact equations.

4) (a) Linear equation &

(b) Bernoulli’s equation.

Type –I : VARIABLE SEPARABLE:

If the D.equation $\frac{dy}{dx}$ =f(x,y) can be expressed of the form $\frac{dy}{dx} = \frac{f(x)}{g(y)}$ or f(x) dx −g(y)dy =0 where f and g are continuous functions of a single variable, then it is said to be of the form variable separable.

General soln of variable saparable is $\int f(x)dx - \int g(y)dy$= c

Where c is any arbitrary constant**.**

PROBLEMS:

1) tan y $\frac{dy}{dx}$ = sin(x+y) + sin(x-y).

Sol: given that sin(x+y) + sin(x-y) = tan y $\frac{dy}{dx}$

⇨ 2sinx.cosx = tan y $\frac{dy}{dx}$ [**Note**: sinC+sinD =2sin($\frac{C+D}{2}$) .cos($\frac{C-D}{2}$)]

⇨ 2sinx = tany secy $\frac{dy}{dx}$

General solution is $2\int \sin x \, dx = \int secy\,.tany\,.dy$

=> -2cosx = secy +c

=> sec y + 2 cos x +c =0 .//

2) Solve (x^2+1) . $\frac{dy}{dx}$+ (y^2+1) =0, y(0) =1.

Sol: Given (x^2+1) . $\frac{dy}{dx}$+ (y^2+1) =0

⇨ $\frac{dx}{x^2+1}$ + $\frac{dy}{y^2+1}$ =0

On Integrations

=> $\int \frac{1}{(1+x)^2}$ dx $\int \frac{1}{(1+y)^2}$ dy =0

=> $\tan^{-1} x$ +$\tan^{-1} y$ =c ---------------(1)

Given y(0)=1 => At x=0 ,y=1 ---------(2)

(2) in (1) => $\tan^{-1}$ 0 +$\tan^{-1}$ 1 =c.

=> 0+$\frac{\pi}{4}$ =c

=> c= $\frac{\pi}{4}$.

Hence the required solution is $\tan^{-1} x$ +$\tan^{-1} y$ = $\frac{\pi}{4}$ .//

Exact Differential Equations:

Def: Let M(x,y)dx +N(x,y) dy =0 be a first order and first degree Differential Equation where M & N are real valued functions of x,y . Then the equation Mdx + Ndy =0 is said to be an exact Differential equation if ∃ a function f ∋.

$$d[f(x,y)] = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

Condition for Exactness: If M(x,y) & N (x,y) are two real functions which have continuous partial derivatives then the necessary and sufficient condition for the Differential equation Mdx+ Ndy =0 is to be exact is that $\frac{\partial M}{\partial y}$

$= \frac{\partial N}{\partial x}$

Hence solution of the exact equation M(x,y)dx +N(x,y) dy =0. Is

$\int M dx \quad + \quad \int N dy \quad = c.$

(y constant) (terms free from x).

------------------********---------------

PROBLEMS:

1)Solve $(1+e^{\frac{x}{y}})$ dx + $e^{\frac{x}{y}}(1-\frac{x}{y})$ .dy =0

Sol: Hence M = $1+e^{\frac{x}{y}}$ & N = $e^{\frac{x}{y}}(1-\frac{x}{y})$

$$\frac{\partial M}{\partial y} = e^{\frac{x}{y}}\left(\frac{-x}{y^2}\right) \quad \& \quad \frac{\partial N}{\partial x} = e^{\frac{x}{y}}\left(\frac{-1}{y}\right) + \left(1-\frac{x}{y}\right)e^{\frac{x}{y}}\left(\frac{1}{y}\right)$$

$$\frac{\partial M}{\partial y} = e^{\frac{x}{y}}\left(\frac{-x}{y^2}\right) \quad \& \quad \frac{\partial N}{\partial x} = e^{\frac{x}{y}}\left(\frac{-x}{y^2}\right)$$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ equation is exact

General solution is

$\int M dx$ + $\int N dy$ = c.

(y constant) (terms free from x)

$\int (1 + e^{\frac{x}{y}})\, dx$ + $\int 0\, dy$ = c.

$=> x + \frac{e^{\frac{x}{y}}}{\frac{1}{y}}$ = c

$= > x + y\, e^{\frac{x}{y}}$ = C

2. $(e^{y}+1)$ .cosx dx + $e^{y}\, sinx\, dy$ =0.

Ans: $(e^{y}+1)$. sinx =c $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ = e^{x}cosx

3. (r+sinθ −casθ) dr + r (sin$\theta + cos\theta$) d$\theta = 0.$

Ans: $r^2+2r(\sin\theta - cos\theta) = 2c$

$\frac{\partial M}{\partial r} = \frac{\partial N}{\partial \theta}$ = sin$\theta + cos\theta$.

4. Solve [y($1 + \frac{1}{x}$) +cos y] dx+ [x +logx −xsiny]dy =0.

Sol: hence M = y($1 + \frac{1}{x}$) +cos y N = x +logx −xsiny.

$\frac{\partial M}{\partial y}$ = 1+ $\frac{1}{x}$ -siny $\frac{\partial N}{\partial x}$ = 1 +$\frac{1}{x}$ -siny

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ so the equation is exact

General sol $\int M dx$ + $\int N dy$ = c.

(y constant) (terms free from x)

$\int [y + \frac{y}{x} + cosy] dx$ + $\int o\, .dy$ = c.

$\Rightarrow$Y(x+ logx) +x cosy = c.

5. ysin2xdx – (y^2+cosx) .dy =0.

6. (cosx-xcosy)dy – (siny+(ysinx))dx =0

Sol: N = cosx-x cosy & M = -siny-ysinx

$\frac{\partial N}{\partial x}$ = -sinx - cosy $\frac{\partial M}{\partial y}$ = -cosy - sinx

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ $\Rightarrow$ the equation is exact.

General sol $\int M dx$ + $\int N dy$ = c.

(y constant) (terms free from x)

=> $\int(-siny - ysinx).dx$ + $\int o\,.dy$ = c

=> -xsiny+ ycos x =c

=> ycosx – xsiny =c.

7. (sinx . siny - x e^y) dy = (e^y +cosx-cosy) dx

Ans: xe^y +sinx.cosy =c.

8. $(x^2+y^2-a^2)$ x dx +$(x^2-y^2-b^2)$. y .dy =0

Ans: $x^4+2x^2y^2-2a^2x^2-2b^2y^2$ =c .

REDUCTION OF NON-EXACT DIFFERENTIAL EQUATIONS TO EXACT USING INTEGRATING FACTORS

Definition: If the Differential Equation M(x,y) dx + N (x,y) dy = 0. Can be made exact by multiplying with a suitable function u (x,y)$\neq$ 0. Then this function is called an Integrating factor(I.F).

Note: there may exits several integrating factors.

<u>Some methods to find an I.F to a non-exact Differential Equation Mdx+N dy =0</u>

Case -1: Integrating factor by inspection/ (Grouping of terms).

Some useful exact differentials

1 . d (xy) = xdy +y dx
2. d $(\frac{x}{y})$ = $\frac{ydx-xdy}{y^2}$
3. d $(\frac{y}{x})$ = $\frac{xdy-ydx}{x^2}$
4. d$(\frac{x^2+y^2}{2})$ = x dx + y dy
5. d(log$(\frac{y}{x})$) = $\frac{xdy-ydx}{xy}$
6. d(log$(\frac{x}{y})$) = $\frac{ydx-xdy}{xy}$
7. d($tan^{-1}(\frac{x}{y})$) = $\frac{ydx-xdy}{x^2+y^2}$
8. d ($tan^{-1}(\frac{y}{x})$) = $\frac{xdy-ydx}{x^2+y^2}$
9. d(log(xy)) = $\frac{xdy+ydx}{xy}$
10. d(log(x^2+y^2)) = $\frac{2(xdx+ydy)}{x^2+y^2}$
11. d$(\frac{e^x}{y})$ = $\frac{ye^x dx-e^x dy}{y^2}$

PROBLEMS:

1 . Solve xdx +y dy + $\frac{xdy-ydx}{x^2+y^2}$ = 0.

Sol: Given equation x dx + y dy + $\frac{xdy-ydx}{x^2+y^2}$ = 0

$$d\left(\frac{x^2+y^2}{2}\right) + d\left(tan^{-1}\left(\frac{y}{x}\right)\right) = 0$$

on Integrating

$\frac{x^2+y^2}{2}$ + tan^{-1} $(\frac{y}{x})$ = c.

2 . Solve y(x^3. $e^{xy} - y$) dx + x (y + x^3. e^{xy}) dy = 0.

Sol: Given equation is on Regrouping

We get yx^3 e^{xy} dx - y^2dx+ x^2y dy +x^4 e^{xy} dy =0.

$X^3 e^{xy}$(ydx+ xdy)+ y (x dy – ydx) = 0

Dividing by x^3

e^{xy} (ydx + xdy) +$(\frac{y}{x})$. ($\frac{xdy-ydx}{x^2}$) =0

d (e^{xy}) +$(\frac{y}{x})$ .d +$(\frac{y}{x})$ = 0

on Integrating

$e^{xy} + ½\left(\frac{y}{x}\right)^2 = C$ is required G.S.

3. (1+xy) x dy + (1- yx) y dx = 0

Sol: given equation is (1+xy) x dy +(1-yx) y dx =0.

(xdy + y dx) + xy (xdy – y dx) = 0.

Divided by x^2y^2 => $(\frac{xdy+ydx}{x^2y^2})$ + $(\frac{xdy-ydx}{xy})$ =0

$(\frac{d(xy)}{x^2y^2})$ + $\frac{1}{y}$ dy - $\frac{1}{x}$ dx =0.

On integrating => $\frac{1}{xy}$+ log y – log x =log c

- $\frac{1}{xy}$ - logx +log y =log c.

4. Solve ydx –x dy = a (x^2+y^2) dx

Ans: $\frac{\text{ydx} - \text{x dy}}{(x^2+y^2)}$ = a dx

d $(tan^{-1}\frac{y}{x})$ = a dx

integrating on $tan^{-1}\frac{y}{x}$ = ax +c

Method -2: If $M(x,y)\,dx + N(x,y)\,dy = 0$ is a homogeneous differential equation and $Mx + Ny \neq 0$, then $\frac{1}{Mx+Ny}$ is an integrating factor of $Mdx + Ndy = 0$.

1 . Solve $x^2y\,dx - (x^3 + y^3)\,dy = 0$

Sol : $x^2y\,dx - (x^3 + y^3)\,dy = 0$-----------------(1)

Where $M = x^2y$ & $N = (-x^3 - y^3)$

Consider $\frac{\partial M}{\partial y} = x^2$ & $\frac{\partial N}{\partial x} = -3x^2$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ equation is not exact .

But given equation(1) is homogeneous D.Equation then

So $Mx + Ny = x(x^2y) - y(x^3 + y^3) = -y^4 \neq 0$.

$$I.F = \frac{1}{Mx+Ny} = \frac{-1}{y^4}$$

Multiplying equation (1) by $\frac{-1}{y4}$

$$= > \quad \frac{x^2y}{-y^4}\,dx - \frac{x^3+y^3}{-y^4}\,dy = 0 \qquad ----------------------(2)$$

$$= > \quad -\frac{x2}{y3}\,dx - \frac{x3+y3}{-y4}\,dy = 0$$

This is of the form $M_1dx + N_1dy = 0$

For $M_1 = \frac{-x2}{y3}$ & $N_1 = \frac{x^3+y^3}{-y^4}$

$=> \quad \frac{\partial M1}{\partial y} = \frac{3x^2}{y4}$ & $\frac{\partial N1}{\partial x} = \frac{3x^2}{-y^4}$

$=> \quad \frac{\partial M1}{\partial y} = \frac{\partial N1}{\partial x}$ equation (2) is an exact D.equation.

General sol $\int M dx \quad + \quad \int N dy \quad = c.$

(y constant) (terms free from x in N)

$=> \quad \int \frac{-x^2}{y^3} dx \quad + \quad \int \frac{1}{y} dy \quad = c.$

$=> \quad \frac{-x^3}{3y^3}$ + log |y| = c.//

2. Solve y^2 dx + $(x^2 - xy - y^2)dy = 0$

Ans: (x-y) . y^2 = $c1^2$(x+y).

3. Solve y($y^2 - 2x^2) dx$ + x (2 $y^2 - x^2$) dy =0 -----------(1)

Sol: it is the form Mdx +Ndy =0

Where M = y($y^2 - 2x^2$) N= x (2 $y^2 - x^2$)

Consider $\frac{\partial M}{\partial y}$ = $3y^2-2x^2$ & $\frac{\partial N}{\partial x}$ = $2y^2-3x^2$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ equation is not exact .

Since equation(1) is homogeneous D.Equation then

Consider Mx+ N y= x[y($y^2 - 2x^2$)] +y [x (2 $y^2 - x^2$)]

= 3xy ($y^2 - x^2) \neq$ 0.

=> I.F. = $\frac{1}{3xy\,(y^2 - x^2)}$

Multiplying equation (1) by $\frac{1}{3xy\,(y^2 - x^2)}$ we get

$$= > \frac{y\,(y^2 - x^2)}{3xy\,(y^2 - x^2)}\,dx + \; = \frac{x(y^2 - x^2)}{3xy\,(y^2 - x^2)}\,dy = 0$$

=> now it is exact (check)

$$\frac{(y^2 - x^2) - x^2}{3xy\,(y^2 - x^2)}\,dx + \frac{y^2 + (y^2 - x^2)}{3xy\,(y^2 - x^2)}\,dy = 0.$$

$$\frac{dx}{x} - \frac{x\,dx}{y^2 - x^2} + \frac{y\,dy}{y^2 - x^2} + \frac{dy}{y} = 0.$$

$$\left(\frac{dx}{x} + \frac{dy}{y}\right) + \frac{2y\,dy}{2(y^2 - x^2)} \frac{2x\,dx}{2(y^2 - x^2)} = 0$$

Log x +log y $+\frac{1}{2}$ log $(y^2 - x^2)$ - $\frac{1}{2}$ log $(y^2 - x^2) = c \Rightarrow$ xy = c

4. r $(\theta^2 + r^2)$ d θ – θ $(\theta^2 + 2r^2)$ dr =0

Ans: $\frac{\theta^2}{2r^2}$ + logθ + log r^2 =c.

Method- 3: If the equation Mdx + N dy =0 is of the form y. f (xy) .dx + x . g (xy) dy = 0 & Mx- Ny $\neq 0$ then $\frac{1}{\text{Mx} - \text{Ny}}$ is an integrating factor of Mdx+ Ndy =0.

Problems:

1 . solve (xy sinxy +cosxy) ydx + (xy sinxy –cosxy)x dy =0.

Sol: (xy sinxy +cosxy) ydx + (xy sinxy –cosxy)x dy =0 -------(1).

=> this is the form y. f(xy) .dx + x . g (xy) dy =0.

=> consider Mx-Ny

Here M =(xy sinxy + cos xy) y

N= (xy sinxy- cos xy) x

Consider Mx-Ny =2xycosxy

Integrating factor $= \frac{1}{2xy\cos xy}$

So equation (1) x I.F

$\Rightarrow \frac{(xy\sin xy + \cos xy)x}{2xy\ \cos xy}dx + \frac{(xy\sin xy + \cos xy)y}{2xy\ \cos xy}dy = 0.$

$\Rightarrow (y \tan xy + \frac{1}{x}) dx + (y \tan xy - \frac{1}{y})$

dy =0

$\Rightarrow M_1 dx + N_1 dx = 0$

Now the equation is exact.

General sol $\int M_1 dx + \int N_1 dy = c.$

(y constant) (terms free from x in N_1)

=> $\int (y\tan xy + \frac{1}{x})dx + \int \frac{-1}{y}dy = c.$

=> $\frac{y.\log|\sec xy|}{y}$ +logx + (-logy) = log c

=> log|sec(xy)| +log$\frac{x}{y}$ =log c.

=> $\frac{x}{y}$. seexy =c.

2. Solve (1+xy) y dx + (1-xy) x dy =0

Sol : I.F $=\frac{1}{2x^2y^2}$

=> $\int \frac{1}{2x^2y} + \frac{1}{2x})dx + \int \frac{-1}{2y}dy = c$

=> $\frac{-1}{2xy} + \frac{1}{2}\log x - \frac{1}{2}\log y = c.$

=> $\frac{-1}{xy} + \log(\frac{x}{y}) = c^1$ where $c^1 = 2c.$

3. Solve $(2xy+1) y\, dx + (1+ 2xy-x^3y^3) x\, dy =0$

Ans: $\log y + \frac{1}{x^2y^2} + \frac{1}{3x^3y^3} = c.$

4. solve $(x^2y^2 + xy + 1)\, y\,dx + (x^2y^2 - xy + 1)\, x\,dy = 0$

Ans: $xy - \frac{1}{xy} + \log(\frac{x}{y}) = c$.

Method -4: If there exists a single variable function $\int f(x)\,dx$ such that $\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N}$

$= f(x)$, then I.F. of $Mdx + N\,dy = 0$ is e

PROBLEMS:

1 . Solve $(3xy - 2ay^2)\,dx + (x^2 - 2axy)\,dy = 0$

Sol: given equation is the form $Mdx + Ndy = 0$

$$\Rightarrow \quad M = 3xy - 2ay^2 \qquad \& \; N = x^2 - 2axy$$

$$\frac{\partial M}{\partial y} = 3x-4ay \quad \& \quad \frac{\partial N}{\partial x} = 2x-2ay$$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \quad \text{equation not exact .}$$

Now consider $\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{(3x-4ay)-(2x-2ay)}{(2x-2ay)}$

$$\Rightarrow \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{1}{x} = f(x).$$

$\Rightarrow \quad e^{\int \frac{1}{x} dx} = x$ is an Integrating factor of (1)

$\Rightarrow$ equation (1) x I.F = equation (1) X x

$$\Rightarrow \frac{(3xy-2ay^2)}{1} \, x\,dx + \frac{(x^2-2axy)}{1} \, x\,dy = 0$$

$$\Rightarrow (3x^2y - 2ay^2x)\,dx + (x^3 - 2ax^2y)\,dy = 0$$

It is the form $M_1dx + N_1dy = 0$

General sol $\int M_1 dx + \int N_1 dy = c$.

$= > \int(3x^2 - 2ay^2x)dx + \int o\,dy = c$

$= > x^3y - ax^2y^2 = c$ .//

2 . Solve ydx-xdy+(1+$x^2)dx + x^2 \sin y\, dy = 0$

Sol : given equation is $(y+1+x^2)\,dx + (x^2\,siny - x)$ dy =0.

M= y+1+x^2 & N =$x^2\,siny - x$

$\frac{\partial M}{\partial y} = 1$ $\frac{\partial N}{\partial x} = 2x \sin y - 1$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ = > the equation is not exact.

So consider $\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{M} = \frac{(1-2x\,siny-x)}{x^2 siny - x} = \frac{-2x\,siny - x)}{x^2 siny - x} = \frac{-2}{x}$

I.F = $e^{\int g(y)dy} = e^{-2\int\frac{1}{x}dx} = e^{-2logx} = \frac{1}{x^2}$

Equation (1) X I.F => $\frac{y+1+x^2}{x^2}\,dx + \frac{x^2\,siny - x}{x^2}$ dy =0

It is the form of M1dx+ N1 dy =0.

Gen soln =>$\int(\frac{y}{x^2} + \frac{1}{x^2} + 1)dx + \int siny dy$ =0

=>$\frac{-y}{x}$ - $\frac{1}{x}$ +x- cosy =c.

=> $x^2 - y - 1 - xcosy = cx$.//

----

3. Solve 2xy dy – (x^2+y^2+1)dx =0

Ans: -x + $\frac{y^2}{x}$ + $\frac{1}{x} = c$.

4. Solve (x^2+y^2) dx -2xy dy =0

Ans: $x^2-y^2=cx$.

Method -5: For the equation Mdx + N dy = 0 if $\frac{\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}}{M}$ = g(y) (is a function of y alone) then $e^{\int g(y)dy}$ is the Integrating factor of M dx + N dy =0.

Problems:

1 . Solve $(3x^2y^4+2xy)dx+(2x^3y^3-x^2)\,dy=0$

Sol: $(3x^2y^4+2xy)dx+(2x^3y^3-x^2)\,dy=0$ -----------------(1).

Here M dx + N dy =0.

Where $M=3x^2y^4+2xy$ & $N=2x^3y^3-x^2$

$\frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}$ equation (1) not exact.

So consider $\frac{\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}}{M}=\frac{-2}{y}$ = g(y)

I.F = $e^{\int g(y)dy}=e^{-2\int\frac{1}{y}dy}=e^{-2\log y}=\frac{1}{y^2}$.

Equation (1) x I.F => $\left(\frac{3x2y4+2xy}{y^2}\right)dx+\left(\frac{2x3y3-x2}{y^2}\right)$ dy =0

It is the form M1dx + N1 dy =0

General sol $\int M1dx$ + $\int N1dy$ = c.

(y constant) (terms free from x in N1)

=> $\int(3x^2y^2+\frac{2x}{y})dx+\int o\,dy$ =c.

=> $\frac{3x^3y^2}{3}+\frac{2x^2}{2y}$ =c.

=> $x^3y^2+\frac{x^2}{y}$ =c.//

2 . Solve $(xy^3+y)\,dx+2(x^2y^2+x+y^4)\,dy=0$

Sol: $\frac{(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y})}{M}=\frac{(4xy^2+2)-(3xy^2+1)}{xy^3+y}=\frac{1}{y}$ = g(y).

I.F = $e^{\int g(y)dy}=e^{\int\frac{1}{y}dy}$ =y.

Gen sol: $\int(\mathrm{xy4}+\mathrm{y2})dx$ $+\int(2\mathrm{y5})dy = c$

$\frac{x^2y^4}{2}$ $+y^2$ x $+\frac{2y^6}{6}$ =c .

3 . solve (y^4+2y)dx + (xy^3 +2 y^4 – 4x) dy =0

Sol: $\frac{\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)}{M}$ = $\frac{(y^3-4)-(4y^3+2)}{y^4+2y}$ = $\frac{-3}{y}$ = g(y).

I.F = $e^{\int g(y)dy}$ = $e^{-3\int\frac{1}{y}dy}$ $=\frac{1}{y^3}$

Gen soln : $\int\left(y+\frac{2}{y^2}\right)dx+\int 2ydy=c$.

$\left(y+\frac{2}{y^2}\right)x$ + $y^2 = c.$ //

4 Solve ($3x^2$ y^4 +2xy)dx + ($2x^3y^3$ $-x^2$) dy = 0

Ans : $x^3y^3+x^2$=cy

5. Solve (y+ y^2)dx + xy dy =0

Ans: x + xy =c .

6. Solve (xy^3+y) dx + 2(x^2y^2+x+y^4)dy =0.

Ans: (x^2+y^4-1) e^{x^2} =c.

LINEAR DIFFERENTIAL EQUATION'S OF FIRST ORDER:

Def: An equation of the form $\frac{dy}{dx}+P(x).y=Q(x)$ is called a linear differential equation of first order in y.

Working Rule: To solve the liner equation $\frac{dy}{dx}+P(x).y=Q(x)$

first find the Integrating factor I.F =$e^{\int p(x)dx}$

General solution is y x I.F = $\int Q(x)\times I.F.dx+c$

Note: An equation of the form Q(y) $(y).x = \emptyset(y)$ is called a linear Differential equation of first order in x.

Then Integrating factor =$e^{\int p(y)dy}$

Gen soln is = x X I.F = $\int Q(y) \times I.F.dy + c$

PROBLEMS:

1 . Solve (1+ y^2) dx=(tan^{-1}y −x) dy

Sol: (1+ y^2) $\frac{dx}{dy}$= (tan^{-1}y −x)

$\frac{dx}{dy}$ +($\frac{1}{1+y^2}$) . x = $\frac{tan^{-1}}{1+y^2}$

It is the form of $\frac{dx}{dy}$ + p(y).x = Q(y)

I.F =$e^{\int p(x)dx}$ = $e^{\int \frac{1}{1+y^2}dy}$ = $e^{tan^{-1} y}$

=> Gen sol is x. $e^{tan^{-1} y}$ =$\int \frac{tan^{-1}}{1+y^2}$. $e^{tan^{-1} y}$ dy+ c.

= > x. $e^{tan^{-1} y}$ = $\int$ t . e^t $dt + c$

[put tan^{-1} y = t

$\Rightarrow$ $\frac{1}{1+y^2}dy = dt$]

$\Rightarrow$x. $e^{tan^{-1} y}$ = t . e^t -e^t +c

= > x. $e^{tan^{-1} y}$ = tan^{-1} y. $e^{tan^{-1} y}$ -$e^{tan^{-1} y}$ +c

= > x = $tan^{-1} y - 1$ + c/ $e^{tan^{-1} y}$ is the required solution

2. Solve (x+y+1) $\frac{dy}{dx}$ = 1.

Sol: g iven equation is (x+y+1) $\frac{dy}{dx}$ = 1.

= > $\frac{dx}{dy} - x$ = y+1.

It is of the form $\frac{dx}{dy}$ + p(y).x = Q(y)

Where p(y) = -1 ; Q(y) = 1+y

= > I.F =$e^{\int p(y)dy}$ = $e^{-\int dy}$ = e^{-y}

Gen soln = x X I.F = $\int Q(y) \times I.F.dy + c$

$= > x \,.\, e^{-y} = \int (1+y)\, e^{-y} dy + c$

$= > x \,.\, e^{-y} = \int e^{-y} dy + \int y e^{-y} dy + \ c$

$= > \quad x e^{-y} = - \ e^{-y} - \ y \, x e^{-y} - e^{-y} \ + c$

$= > \quad x e^{-y} = - \ e^{-y} (2+y) \ + c$ .//

3. Solve $y^1 + y = e^{e^x}$

Sol: this is of the form $\frac{dy}{dx}$ Q(x) $y =$

Where p(x) =1 Q(x) = e^{e^x}

=> I.F = $e^{\int p(x)dx} = e^{\int dx} = e^x$

Gen soln is is y x I.F = $\int Q(x) \times I.F.dx + c$

=> y. $e^x = \int e^{e^x} e^x dx + c$

=> y. $e^x = \int e^t t \, dt + c$ put $e^x = t$

=> y. e^x = t $e^t - \ e^t$ +c $e^x dx = dt$

=> y. $e^x = e^{e^x} (e^x - 1) + c.$

4. Solve x.$\frac{dy}{dx}$ + y =log x

Sol : this is of the form $\frac{dy}{dx}$ +p(x)y = $\emptyset(x).$

Where p(x) = $\frac{1}{x}$ & $\emptyset(x) = \frac{\log x}{x}$

i.e , $\frac{dy}{dx} + \frac{1}{x}$. y = $\frac{\log x}{x}$

=> I.F = $e^{\int p(x)dx} = e^{\int \frac{1}{x} dx} = e^{\log x}$ =x.

Gen soln is is y x I.F = $\int Q(y) \times I.F.dy + c$

=> y.x = $\int \frac{\log x}{x} \, x \, dx + c$

=> y .x = x (logx-1) +c.//

5 . Solve $(1+y^2) + (x- e^{tan^{-1} y})\frac{dy}{dx} = 0.$

Sol : Given equation is $\frac{dx}{dy} + \frac{x}{1+y2} = \frac{e^{tan^{-1} y}}{1+y2}$

It is of the form $\frac{d}{d}$ Q(y) (y) .x = $\emptyset(x)$.

Where p (y) Q(y) ; $\emptyset(x) = \frac{e^{tan^{-1} y}}{1+y2}$.

I.F $= e^{\int p(y)dy} = e^{\int \frac{1}{1+y2} dy} = e^{tan^{-1} y}$.

General solution is is x x I.F = $\int Q(y) \times I.F.dy + c$.

$=> x . e^{tan^{-1} y}. = \int \frac{e^{tan^{-1} y}}{1+y2} \quad e^{tan^{-1} y}.$ dy +c

$=> x . e^{tan^{-1} y} = \int e^t \quad e^t.$ dt +c

[**Note: put** $tan^{-1} y$ = t

=> $\frac{1}{1+y2} dy$ = dt]

$=> x . e^{tan^{-1} y} = \int e^{2t}$. dt +c

$=> x . e^{tan^{-1} y} = \frac{e^{2t}}{2} + c$

$=> x . e^{tan^{-1} y} = \frac{e^{2 tan^{-1} y}}{2} + c$ //

6. solve $\frac{dy}{dx} + \frac{y}{x \log x} = \frac{sin2x}{log x}$

Ans: ylogx = $\frac{-\cos 2x}{2}$ + c.

7 . $\frac{dy}{dx}$ + (y-1). Cox = $e^{-sin\, y} \quad cos^2$ x

Ans: y. $e^{-sin\, y} = \frac{x}{2} + \frac{sin2x}{4} + e^{-sin\, y}$ +c //

8. $\frac{dy}{dx} + \frac{2x}{1+x2}$. y= $\frac{1}{(1+x^2)^2}$ given y = 0 , where x= 1.

Ans : y(1 + x2) = $tan^{-1} x \; -\frac{\pi}{4}$

9. Solve $\frac{dy}{dx} - \frac{tany}{1+x}$ = (1+x) e^x . sec y

Sol : the above equation can be written as

Divided by sec y => cos y $\frac{dy}{dx}$ - $\frac{siny}{1+x}$ = (1+x) e^x -----
-------(1)

Put sin y = u

= > cos y $\frac{dy}{dx}$ =$\frac{du}{dx}$

D. Equation (1) is $\frac{du}{dx}$ – $\frac{1}{1+x}$. u = (1+x) e^x

this is of the form $\frac{d}{d_{\sim}}$ Q(x) $).u = \emptyset(x)$

Where p(x) =$\frac{-1}{1+x}$ Q(x) =(1+x) e^x

=> I.F = $e^{\int p(x)dx}$ = $e^{\int \frac{-1}{1+x}dx}$ =$e^{-\log(1+x)}$ =$\frac{1}{1+x}$

Gen soln is is u x I.F = $\int Q(y) \times I.F.dy + c$

=> u. $\frac{1}{1+x}$ = $\int (1+x)\, e^x \frac{1}{1+x} dx \, + c$

=> u. $\frac{1}{1+x}$ = $\int \, e^x dx \, + c$

=> (sin y) $\frac{1}{1+x}$ = e^x +c

(Or)

= > sin y = (1+x) e^x +c . (1+x) is required solution.

10. Solve $\frac{dy}{dx}$ - ytan = $\frac{sinx \,.\, cos^2 x}{y^2}$

Ans : $y^3 \, cos^3 x$ = $\frac{-\,cos^6 x}{2}$ +c .

11 .Solve $\frac{dy}{dx}$ – yx = $y^2 \, e^{\frac{x^2}{2}}$ $.sinx$

Ans: $\frac{1}{y}\ e^{\frac{-x^2}{2}}$ = cosx + c.

12. $e^{x} . \frac{dy}{dx}$ = 2xy^2 + y e^{x}

Ans : $\frac{1}{y}\ e^{x}$ = x^2 + c.

13. $\frac{dy}{dx}$ + ycos x = y^3 sin x

Ans : : $\frac{1}{y^2}$ = (1+ 2 sinx) + c e^{2sinx}

(or)

$\frac{-1}{y^2}\ e^{-2sinx}$= -(1+ 2 sinx) e^{-2sinx}+ c .

14. $\frac{dy}{dx}$ + y cot x = $y^2\ sin^2x\ cos^2\ x$

Ans: ysinx (c + $cos^3\ x)$ =3.

15. Solve $\frac{dy}{dx}$ = e^{x-y} ($e^{x} - e^{y}$)

Ans: $e^{x}.\ e^{e^{x}} = e^{e^{x}}(e^{x} - 1) + c$

BERNOULI'S EQUATION :

(EQUATION'S REDUCIBLE TO LINEAR EQUATION)

Def: An equation of the form $\frac{Q(x)}{dx}$ p(x) .y = $\emptyset(x).\ y^n$ **----- ------ --(1)**

Is called Bernoulli's Equation, where p & Q are function of x and n is a real constant.

Working Rule:

Case -1 : if n=1 then the above equation becomes $\frac{dy}{dx}$ + p. y = Q.

=> Gen soln of $\frac{dy}{dx}+(p-Q)y=0$ is

$\int \frac{dy}{dx} + \int (p - Q)dx = c$ by variable separation method.

Case -2: if $n \neq 1$ then divide the given equation (1) by y^n

$$\Rightarrow \quad y^{-n} \cdot \frac{dy}{dx} + p(x) \cdot y^{1-n} = Q \text{ ---------(2)}$$

Then take $y^{1-n} = u$

$$(1-n)\, y^{-n} . \frac{dy}{dx} = \frac{du}{dx}$$

$$\Rightarrow \quad y^{-n} . \frac{dy}{dx} = \frac{1}{1-n} \frac{du}{dx}$$

Then equation (2) becomes

$$\frac{1}{1-n} \frac{du}{dx} + p(x) \cdot u = Q$$

$\frac{du}{dx}$ + (1-n) p.u = (1-n)Q which is linear and hence we can solve it.

Problems:

1 . Solve $x \frac{dy}{dx} + y = x^3 y^6$

Sol: given equation can be written as $\frac{dy}{dx} + \left(\frac{1}{x}\right) y = x^2 + y^6$

Which is of the form $\frac{dy}{dx} + p(x).y = Q\ y^n$

Where p(x) $= \frac{1}{x}$ $Q(x) = x^2$ & $n=6$

Divided by y^2 $\Rightarrow$ $\frac{1}{y^6} \cdot \frac{dy}{dx} + \frac{1}{x}\frac{1}{y^5} = x^2$ ---------------(2)

Take $\frac{1}{y^5} = u$

$$\Rightarrow \frac{-5}{y^6} \frac{dy}{dx} = \frac{du}{dx} \quad \}\text{---------------(3)}$$

$$\Rightarrow \frac{1}{y^6} \frac{dy}{dx} = \frac{-1}{5} \frac{du}{dx} \quad \}\text{---------------(3)}$$

(3) in (2) $\Rightarrow$ $\frac{du}{dx} - \frac{5}{x} u = -5x^2$

Which is a L.D equation in u

I.F = $e^{\int p(x)dx}$ $= e^{-5\int \frac{1}{x}dx}$ $= e^{-5\log x}$ $= \frac{1}{x^5}$

Gensol $\Rightarrow$ u .I.F $= \int Q(y) \times I.F.dy + c$

u. $\frac{1}{x^5}$ = $\int -5x2.\frac{1}{x^5}\, dx$ +c

$\frac{1}{y^5 x^5}$ = $\frac{5}{2x^2}$ +c (or) $\frac{1}{y^5} = \frac{5\,x^3}{2}$ + cx^5

2. Solve $\frac{dy}{dx}$ $(x^2y^3 + $xy) =1

Sol: $\frac{dx}{dy}$ -x .y= x^2y^3 => $\frac{1}{x^2}$. $\frac{dx}{dy}$ - $\frac{1}{x}$ .y = y^3 --------------------

-(1)

Put $\frac{1}{x}$ = u

$\Rightarrow$ $\frac{-1}{x^2}$. $\frac{dx}{dy}$ = $\frac{du}{dx}$ ---------------(2).

(2) in (1) $\Rightarrow$ $-\frac{du}{dx}-$ u .y = y^3

(Or) $\frac{du}{dx}$ + u .y = -y^3.

Is a L.D Equation in � u’

I.F = $e^{\int P(y)dy}$ $= e^{\int y dy}$ $= e^{-\frac{y^2}{2}}$

Gensol $\Rightarrow$ u .I.F $= \int Q(y) \times I.F.dy + c$

$\Rightarrow$ u . $e^{-\frac{y^2}{2}}$ = $\int y^3 . e^{-\frac{y^2}{2}} dy$ + c

$\Rightarrow$ $\frac{e^{-\frac{y^2}{2}}}{x}$ = -2$(\frac{y^2}{2} - 1)$. $e^{-\frac{y^2}{2}}$ +c

(or)

X(2-y^2)+ cx$e^{-\frac{y^2}{2}}$ =1.

3. Solve $\frac{dy}{dx}$ -y tanx = y^2 sec x

Ans: I.F = $e^{-\int tanx dx}$ = $e^{\int \log cox}$ = $cosx$

Gen sol $\frac{1}{y}$ cos x= -x +c .

4. (1-x^2) $\frac{dy}{dx}$ + xy = y^3 sin^{-1}x

Sol: given equation can be written as

$\frac{dy}{dx}$ +$\frac{x}{1-x^2}$ y = $\frac{y^3}{1-x^2}$ sin^{-1}x

Which is a Bernoulli's equation in ' y '

Divided by y^3 $\Rightarrow \frac{1}{y^3}$. $\frac{dy}{dx}$+ $\frac{1}{y^2}$ $\frac{x}{1-x^2}$ = $\frac{sin^{-1}x}{1-x^2}$ -----------(1).

Let $\frac{1}{y^2}$ = u

$\Rightarrow$ $\frac{-2}{y^3}$ $\frac{dy}{dx}$ = $\frac{du}{dx}$ = > $\frac{1}{y^3}$ $\frac{dy}{dx}$ = $-\frac{1}{2}\frac{du}{dx}$-------------

(2)

(2) in (1) $\Rightarrow -\frac{1}{2}\frac{du}{dx}$ + $\frac{x}{1-x^2}$ $.u$= $\frac{sin^{-1}x}{1-x^2}$

Which is a L.D equation in u

$\Rightarrow$ I.F = $e^{\int p(x)dx}$ = $e^{-\int \frac{2x}{1-x^2}dx}$ =$e^{\log(1-x^2)}$ = $(1-x^2)$

Gensol $\Rightarrow$ u .I.F =$\int Q(x) \times I.F.dx + c$

$\Rightarrow$ $\frac{1}{y^2}$ $(1-x^2)$ = -$\int \frac{2sin^{-1}x}{1-x^2}(1-x^2)dx + c$

= > $\frac{(1-x^2)}{y^2}$ = -2 [xsin^{-1}x +$\sqrt{1-x^2}$] +c

5 . e^x $\frac{dy}{dx}$ = 2xy^2 + y . e^x

Ans: $\frac{e^x}{y}$ = $x^2 + c$.

APPLICATION OF DIFFERENTIAL EQUATIONS OF FIRST ORDER

ORTHOGONAL TRAJECTORIES (O.T)

Def: A curve which cuts every member of a given family of curves at a right angle is an orthogonal trajectory of the given family.

Orthogonal trajectories in Cartesian co-ordinates:

Working rule: **<u>To find the family of O.T is Cartesian form</u>** . let f(x,y,c) =0 be the given equation of family of curves in Cartesian form.

Step: (1) . D . w. r .t ' x ' and obtain F(x, y, y^1) = 0 ----------(2)

Of the given family of curves.

(2) .replace $\frac{dy}{dx}$ by $-\frac{dx}{dy}$ is (2)

Then the D. Equation of family of O.T is

F(x, y, $-\frac{dx}{dy}$) =0 ----------------(3).

(3) solve equation(3) to get the equation of family of O.T's of equation(1).

PROBLEMS:

1 . Find the O.T's of family of semi-cubical parabolas $ay^2=x^3$ where a is a parameters.

Sol : the given family of semi-cubical parabola is $ay^2=x^3$ ---------------(1)

D. w. r. t 'x ' => a 2y $\frac{-dy}{dx}$ = 3 x^2 -------------(2)

Eliminating 'a ' => $\frac{x^3}{y^2}$ .2y . $\frac{dy}{dx}$ =3 x^2

=> 2 x^3y - $\frac{dy}{dx}$ = 3 x^2

Replace $\frac{dy}{dx}$ by -$\frac{dx}{dy}$ => 2 x^3y (-$\frac{dx}{dy}$) = 3 x^2.

=> -2x $\frac{dx}{dy}$ =y

=> $\int -2x dx - \int y\, dy$ = -c

=> x^2 + $\frac{y^2}{2}$ =c

=> $\frac{x^2}{c}+\frac{y^2}{2c}$ =1.

2. Find the O.T of the family of circles $x^2+y^2+2gx+c=0$.

Sol: $x^2+y^2+2gx+c=0$. ---------(1) is represents a system of co- axial circle with g as parameter

D . w. r.t 'x ' => 2x+ 2y$\frac{dy}{dx}$ + 2g =0--------------(2)

Substituting eq from (2) in (1)

=> x^2+y^2 -(2x+ 2y $\frac{dy}{dx}$) x +c =0.

=> y^2-x^2-2xy $\frac{dy}{dx}$ + c=0

Replace $\frac{dy}{dx}$ by $-\frac{dx}{dy}$

=> y^2-x^2-2xy $(-\frac{dx}{dy})$ + c=0

=> y^2-x^2-2xy $(\frac{dx}{dy})$ + c=0

This can be written as

$2x \cdot \frac{dx}{dy}$ $- \cdot \frac{1}{y}$ x^2 $=$ $\frac{-(c+y2)}{y}$

This is a Bernoulies equation in x

So put $x^2 = u \Rightarrow 2x \cdot \frac{dx}{dy} = \frac{du}{dx}$

$\Rightarrow \frac{du}{dy} \quad -\frac{1}{y} \quad u = \frac{-(c+y2)}{y}$

Which is a liner equation in ' u '

$\Rightarrow$ I.F = $e^{\int \frac{1}{y} dy} = e^{-\log y} = \frac{1}{y}$

Gensol $\Rightarrow$ u .I .F $= \int Q(y).I.F.dy + K$

$\Rightarrow x^2 \cdot \frac{1}{y} = \int \frac{-(c+y2)}{y} \frac{1}{y} dy + k$

$= -c\left(\frac{-1}{y}\right) - y + k$

$\Rightarrow \frac{x2}{y} = \frac{c}{y} - y + K$

3. Find the O.T's of the family of parabolas through origin and for an y –axis.

Sol : equation is x2=4ay $\Rightarrow$ 2x =4a $\cdot \frac{dy}{dx}$

$\Rightarrow \frac{dy}{dx} = \frac{2y}{x}$

Ans: $\frac{x2}{2c} + \frac{y2}{c} = 1.$

4. find the O.T of the one parameter family of curves

$e^x + e^{-y}$=c.

Sol: D. W. r. t 'x ' $\Rightarrow$ $e^x + e^{-y}(\frac{dy}{dx})$=c

O.T $\Rightarrow$ $e^x + e^{-y}(\frac{-dx}{dy})$=c

Ans: $e^y - e^{-y} = k$.

5. find the O.T of the family of circle passing through origin and center on x-axis. Equation is $x^2 + y^2 + 2gx = 0$.

   Ans: $\frac{x2}{y}$ = - y +c.

6. Prove that the system of parabolas $y^2=4a(x+a)$ is self orthogonal

ORTHOGONAL TRAJECTORIES IN POLAR FORM

Working Rule: to find the O.T of a given family of cuves in polar-co ordinates. Let $f(r, \theta, c) = 0$ -----(1) be the given family of curves in polar form.

1 .) D. w .r .t θ *and* obtain F [r, $\theta, \frac{dr}{d\theta}$] =0 by eliminating the parameter

2.) replace $\frac{dr}{d\theta}$ by $-$ r^2 $\frac{d\theta}{dr}$ then the D.Equation of family of O.T

$$F [r, \theta, -r^2 \frac{d\theta}{dr}] =0$$

3.) solve the above equation to get the equation of O.T of (1)

Problems:

1 . Find the O.T of family of

a) $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ where a is a parameter.

b) $\frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} = 1$ λ is a parameter is self –orthogonal

c) $r\sin 2\theta = \lambda$, λ is a parameter

Ans: $r^4 \cos 2\theta = C^4$.

2 . Find the O.T of family of curves $r^n = a^n \cos n\theta$

Ans: $r^n = c \sin n\theta$

3. find the O.T of family of curves $r=2a(\cos\theta + \sin\theta)$

ans : $r = (\sin\theta - \cos\theta) \cdot c$

4. find the O.T of family of curves $r^n \sin n\theta = a^n$

Ans: $r^n = c \sec n\theta$

NEWTON'S LAW OF COOLING

STATEMENT: The rate of change of the temp of a body is proportional to the difference of the temp of the body and that of the surroundings medium.

Let 'θ' be the temp of the body at time 't' and θo be the temp of its surroundings medium(usually air). By the Newton's low of cooling , we have

$$\frac{d\theta}{dt} \,\square\, (\theta - \theta o) \Rightarrow -\frac{d\theta}{dt} = k(\theta - \theta o)$$ k is +ve constant

$$\Rightarrow \int \frac{d\theta}{(\theta - \theta o)} = -k \int dt$$

$$\Rightarrow \log(\theta - \theta o) = -kt + c.$$

If initially $\theta = \theta 1$ is the temp of the body at time t=0 then

$$c = \log(\theta 1 - \theta o) \Rightarrow \log(\theta - \theta o) = -kt + \log(\theta 1 - \theta o)$$

$$\Rightarrow \log\left(\frac{(\theta - \theta o)}{(\theta 1 - \theta o)}\right) = -kt.$$

$$\Rightarrow \frac{(\theta-\theta o)}{(\theta 1-\theta o)} = e^{-kt}$$

$$\theta = \theta o + (\theta 1 - \theta o).\, e^{-kt}$$

Which gives the temp of the body at time 't' .

1. Find the O.T of the co focal and coaxial parabolas $r = \frac{2a}{1+\cos\theta}$

Ans: $r = \frac{c}{1-\cos\theta}$

Problems:

1 A body is originally at 80^0 and cools dowm to 60^0 c in 20 min . if the temp of the air is 40^0 c. Find the temp of body after 40 min.

Sol: By Newton's low of cooling we have

$\frac{d\theta}{dt} = k(\theta - \theta o)$ θo is the temp of the air.

$\Rightarrow \int \frac{d\theta}{(\theta-40)}. = -k \int dt$ $\theta o = 40^0$ c

$\Rightarrow \log(\theta - 40) = -kt + \log c$

$\Rightarrow \log(\frac{\theta-40}{c}) = -kt$

$\Rightarrow \frac{\theta-40}{c} = e^{-kt}$

$\Rightarrow \theta = 40 + c\, e^{-kt}$ --------------(1)

When t=0 , $\theta = 80^0$ c $\Rightarrow$ 80 = 40 +c ------------(2).

When t=20 , $\theta = 60^0$ c $\Rightarrow$ 60 = 40 +ce^{-20k} ------------(3).

Solving (2) & (3) $\Rightarrow$ c$e^{-20k} = 20$

C=40 $\Rightarrow$ 40e^{-2k} =20

=> $k = \frac{1}{20}\log 2$

When t= 40^0 c => equation (1) is $\theta = 40 + 40\, e^{-(\frac{1}{20}\log 2)40}$

$= 40 + 40\, e^{-2\log 2}$

$= 40 + (40 \times \frac{1}{4})$

$\Rightarrow \theta = 50^0$ c

2 . An object when temp is 75^0c cools in an atmosphere of constant temp. 25^0 c, at the rate $k\theta, \theta$ being the excess temp of the body over that of the temp. If after 10min , the temp of the object falls to 66^0 c , find its temp after 20 min . also find the time required to cool down to 55^0c .

Sol : we will take one as unit of time.

It is given that $\frac{d\theta}{dt} = -k\theta$

$\Rightarrow$ sol is $\theta = c\, e^{-kt}$ ------------(1).

Initially when t=0 $\Rightarrow \theta = 75^0 - 25^0 = 50^0$

$\Rightarrow$ c= 50^0 -----------------(2)

When t= 10 min $\Rightarrow \theta = 65^0 - 25^0 = 40^0$

$\Rightarrow 40 = 50\, e^{-10k}$

$\Rightarrow e^{-10k} = \frac{4}{5}$ ---------------------(3).

The value of θ when t=20 $\Rightarrow \theta = c\, e^{-kt}$

$\theta = 50 e^{-20k}$

$\theta = 50(e^{-10k})^2$

$\theta = 50(\frac{4}{5})^2$

when $t=20 \Rightarrow \theta = 32^0$ c.

3. A body kept in air with temp $25^0 c$ cools from 140^0 c to 80^0 in 20 min. Find when the body cools down in 35^0.

Sol : here $\theta o = 25^0 c$ $\Rightarrow \frac{d\theta}{(\theta-25)} = -k\,dt$

$\Rightarrow \log(\theta - 25) = -kt + c$ ------------(1).

When t=0 , $\theta = 140^0$ c $\Rightarrow \log(115) = c$

$\Rightarrow c = \log(115)$.

$\Rightarrow kt = -\log(\theta - 25) + \log 115$--------(2)

When t=20 , $\theta = 80^0$ c

$\Rightarrow \log(80^0\ c) = -20k + \log 115$

$\Rightarrow 20k = \log(115) - \log(55)$ ---------(3)

(2)/ (3) => $\frac{kt}{20k} = \frac{\log 115 - \log(\theta - 25)}{\log 115 - \log 55}$

$\frac{t}{20} = \frac{\log 115 - \log(\theta - 25)}{\log 115 - \log 55}$

When $\theta = 35^0$ c $\Rightarrow \frac{t}{20} = \frac{\log 115 - \log(10)}{\log 115 - \log 55}$

$\Rightarrow \frac{t}{20} = \frac{\log(11.5)}{\log(\frac{23}{11})} = 3.31$

$\Rightarrow t = 20 \times 3.31 = 66.2$

The temp will be 35^0 c after 66.2 min.

4 . If the temp of the air is 20^0 c and the temp of the body drops from 100^0 c to 80^0 c in 10 min. What will be the its temp after 20min. When will be the temp 40^0 c .

Sol: log $(\theta - 20)$ = -kt + log c

c =80^0 c and e^{-10k} = $\frac{3}{4}$.

t =$\frac{10 \log(\frac{1}{4})}{\log(\frac{3}{4})}$.

5. the temp of the body drops from 100^0 c to 75^0 c is temp in 10 min. When the surrounding air is at 20^0 c temp. What will be its temp after half an hour, when will the temp be 25^0 c .

Sol : $\frac{d\theta}{dt}$ = -k$(\theta - \theta o)$

log $(\theta - 20)$ = -kt + log c

when t=0 , θ =100^0 => c=80

when t=10 , θ =75^0 = > e^{-10k} = $\frac{11}{16}$.

when t =30min => θ = 20 +80 ($\frac{1331}{4096}$) = 46^{O}C

when θ =25^0c = > t = 10 ($\frac{log5-log80}{log11-log6}$) = 74.86 min

LAW OF NATURAL GROWTH OR DECAY

(STATEMENT: Let x(t) or x be the amount of a substance at time ‘ t’ and let the substance be getting converted chemically . A law of chemical conversion states that the rate of change of amount x(t) of a chemically changed substance is proportional to the amount of the substance available at that time

$$\frac{dx}{dt} \quad \alpha \quad x \quad \text{(or)} \quad \frac{dx}{dt} \quad = \text{ - kt } \;; \; (k > 0)$$

Where k is a constant of proportionality

Note: In case of Natural growth we take

$$\frac{dx}{dt} \quad = k .x \text{)}$$

PROBLEMS

1 The number N of bacteria in a culture grew at a rate proportional to N . The value of N was initially 100 and increased to 332 in one hour. What was the value of N after 1hrs

Sol: The D. Equation to be solved is $\frac{dN}{dt}$ = kN

$$\Rightarrow \quad \frac{dN}{N} \quad = \text{ k dt}$$

$$\Rightarrow \quad \int \frac{dN}{N} .= \int k dt$$

$$\Rightarrow \quad \log N = kt + \log e$$

$$\Rightarrow \quad N = c\, e^{kt} \text{ ------------(1).}$$

When t= 0sec , N =100 $\Rightarrow$ 100 = c $\Rightarrow$ c =100

When t =3600 sec , N =332 $\Rightarrow$ 332 =100 e^{3600k}

$$\Rightarrow \quad e^{3600k} \quad = \frac{332}{100}$$

Now when t =$\frac{3}{2}$ hors = 5400 sec then N=?

$$\Rightarrow \quad N = 100\, e^{5400k}$$

$$\Rightarrow \quad N = 100[\, e^{3600k} \,]^{\frac{3}{2}}$$

$$\Rightarrow \quad N = 100 \quad [\frac{332}{100}]^{\frac{3}{2}} \quad = 605.$$

$$\Rightarrow \quad N = 605.$$

2 . In a chemical reaction a given substance is being converted into another at a

rate proportional to the amount of substance converted. If $\frac{1}{5}$ of the original

amount has been transformed in 4 min, how much time will be required to

transform one half.

Ans: t= 13 mins.

3. The temp of cup of coffie is 92^0c . in which freshly period the room temp being 24^0c . in one min it was cooled to 80^0c . how long a period must elspse , before the temp of the cup becomes 65^0c .

Sol: : By Newton's Law of Cooling,

$\frac{d\theta}{dt}$ = -k$(\theta - \theta o)$; k>0

θo = 24^0c $\Rightarrow$ log ($\theta - 24$) = -kt + log c-------------(1).

When t=0 ; θ = 92 $\Rightarrow$ c =68

When t =1 ; $\theta =$ 80^0c $\Rightarrow$ e^k = $\frac{68}{56}$

$\Rightarrow$ k = log ($\frac{68}{56}$).

When $\theta =$ 65^0c , t =?

Ans: t = $\frac{41}{56}$ min.

RATE OF DECAY OR RADIO ACTIVE MATERIALS

STATEMENT:

The disintegration at any instance is propositional to the amount of material present in it.

If u is the amount of the material at any time 't' , then $\frac{du}{dt}$ = - ku , where k is any constant (k >0).

Problems:

1). if 30% of a radioactive substance disappears in 10days flow long will it take for 90% of it to disappear.

Ans: 64.5 days

2). In a chemical reaction a gives substance is being converted into another at a rate proportional to the amount of substance unconverted. If $\frac{1}{5}$ Of the original amount has been transformed to required to transform one-half.

Ans:

3 The radioactive material disintegrator at a rate proportional to its mass. When mass is 10 mgm , the rate of disintegration is 0.051 mg per day . how long will it take for the mass to be reduced from 10 mg to 5 mg.

Ans: 136 days.

4. uranium disintegrates at a rate proportional to the amount present at any instant . if m1 and M2 are grms of uranium that are present at times T1 and T2 respectively find the half=cube of uranium.

Ans: $T = \frac{(T2-T1)\,log2}{\log\left(\frac{M1}{M2}\right)}$.

5. The rate at which bacteria multiply is proportional to the instance us number

present. If the original number double in 2 hrs, in how many hours will it be triple.

Ans: $\frac{2log3}{log2}$ hrs.

6. a) if the air is maintained at 30^0c and the temp of the body cools from 80^0c to

60^0c in 12 min . find the temp of the body after 24 min.

Ans: 48^0c

b) If the air is maintained at 150^0c and the temp of the body cools from 70^0c

to 40^0c in 10 min. Find the temp after 30 min.

Ans:

ASSIGNMENT PROBLEMS:

UNIT V

DIFFERENTIAL EQUATIONS OF FIRST ORDER AND FIRST DEGREE

1) Find the differential equation of the family of curves $y=e^x\,(acosx + bsinx)$ where a and b are arbitrary constants.

2) Form the differential equation of the family of curves log $(y+a)=x^2+c$, c is the parameter.

3) Solve : $(x^2+1)y_1+y^2+1=0$, $y(0)=1$

4) Solve : $(1+ e^{\frac{x}{y}})\,dx + e^{\frac{x}{y}}\,(1- \frac{x}{y})\,dy = 0$

5) Solve the differential equation $\frac{y+x-2}{y-x-4} = \frac{dy}{dx}$

6) Solve : $\frac{dy}{dx} + \frac{ycosx+siny+y}{sinx+x\,cosy+x} = 0$

7) Solve : (xy sinxy + cosxy) ydx + (xysinxy – cosxy) x dy=0

8) Solve : $2xy\,dy - (x^2+y^2+1)\,dx = 0$

9) Solve : $(y^4 + 2y)\,dx + (xy^3 + 2y^4 - 4x)\,dy=0$

10) Find the equation of the curve satisfying the differential equation

$$(1+x^2)\frac{dy}{dx} + 2xy = 4\,x^2$$

11) Solve : $(1+ y^2)\,dx = (\tan^{-1} y - x)\,dy$

12) Solve : $x\frac{dy}{dx} + y = x^3 y^6$

13) Solve : $\frac{dy}{dx}(x^2y^3 + xy) = 1$

14) Solve the differential equation $3\frac{dy}{dx}$ - y cosx = y^4(sin2x – cosx)

15) Solve $\frac{dy}{dx} - \frac{\tan y}{1+x} = (1+x)\, e^x \sec y$

16) Find the orthogonal trajectories of the family of circles.

$$x^2 + y^2 + 2gx+c = 0$$

17) Find the equation of the system of orthogonal trajectories of the family

of curves $r^n \sin n\theta = a^n$ where a is the parameter.

18) Find the orthogonal trajectories of the family of curves $r^n = a^n \cos n\theta$

19) If the temperature of a body is changing from 100^0c to 70^0c in 15.

minutes, find when the temperature will be 40^0c, if the temperature of

air is 30^0c.

20) In a chemical reaction a given substance is being converted into

another at a rate proportional to the amount of substance unconverted. If

$(1/5)^{th}$ of the original amount has been transformed in 4 minutes. How

much time will be required to transform one half.

21) Solve : $x \frac{dy}{dx} + y = x^3 y^6$

22) If the temperature of air is 20^0c and the temperature of the body drops from 100^0c to80^0c in 10 minutes . what will be its temperature after 30 minutes. When will be the temperature 40^0c

[Objective type Questions]

Differential Equation's of First Order and First Degree:

1) The order of $x^3 \frac{d^3 y}{dx^3} + 2 x^2 \frac{d^2 y}{dx^2}$ - 3y = x is

   a) 2 b) 3 c) 1 d) None

2) The order of $(\frac{d^2 y}{dx^2})^2 = [1 + (\frac{dy}{dx})^2]^{\frac{3}{2}}$ is

   a) 2 b) 1 c) 3 d)None

3) The degree of Differential Equation $[\frac{d^2 y}{dx^2} + (\frac{dy}{dx})^2]^{\frac{3}{2}} = a \frac{d^2 y}{dx^2}$ is

   a) 3 b) 2 c) 1 d) 9

4) The degree of Differential Equation $(\frac{d^2 y}{dx^2})^4 = [1 + (\frac{dy}{dx})^2]^3$ is

   a) 4 b) 3 c) 2 d) None

5) The general solution of $\frac{dy}{dx} = e^{x+y}$ is

   a) $e^x + e^y = c$ b) $e^x + e^{-y} = c$ c) $e^{-x} + e^y = c$ d) $e^{-x} + e^{-y} = c$

6) Find the differential equation corresponding to $y = a e^x + b e^{2x} + c e^{3x}$

a) $y^{111} - 6y^{11} + 11y^{1} - 6y = 0$ b) $y^{111} + y^{11} - 3y^{1} = 0$

c) $y^{11} + 2y^{1} + y = 0$ d) $y^{111} - 2y^{11} + 3y^{1} + y = 0$

7) Find the differential equation of the family of curves $y = e^{x}$ (Acosx + Bsinx)

a) $y^{11} - 2y^{1} + 3y = 0$ b) $y^{11} - 3y^{1} + y = 0$

c) $y^{11} - 2y^{1} + 3y = 0$ d) None

8) Form the differential equation by eliminating the arbitary constant : $y^2 = (x - c)^{2}$

a) $(y^1)^{2} = 1$ b) $y^{11} + 2y^{1} = 2$ c) $(y^1)^{2} = 0$ d) None

9) Find the differential equation of the family of parabolas having vertex at

the origin and foci on y -axis

a) $xy^1 = 2x$ b) $xy^1 = 2y$ c) $xy^1 = 4y$ d) None

10) Form the differential equation by eliminating the arbitary constant

tanx + tany = c

a) $y_1(\tan y + \sec^2 x) = 0$ b) $y_1(\tan y \sec^2 y) + \tan y \sec^2 x = 0$

c) $y_1(\tan x \sec^2 x) + \tan y \sec^2 y = 0$ d) None

11) Obtain the differential equation of the family of ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

a) $xyy^{11} + xy^{1} = 0$ b) $xy^{11} + xy = 0$

c) $xyy^{11} + x (y^{1})^{2} - yy^{1} = 0$ d) None

12) The solution of the differential equation $\frac{dy}{dx} + \frac{y}{x} = x^2$ under the condition that y = 1 when x=1 is

a) $4xy= x^3+3$ b) $4xy= x^4+3$ c) $4xy= y^4+3$ d) None

13) The family of straight lines passing through the origin is represented by

the differential equation

a) $ydx + xdy=0$ b) $xdy - ydx = 0$ c) $xdx + ydx = 0$

d) $ydy - xdx = 0$

14) The differential equation of a family of circles having the radius 'r' and

centre on the x – axis is

a) $y^2[1 + (\frac{dy}{dx})^2] = r^2$ b) $x^2[1 + (\frac{dy}{dx})^2] = r^2$

c) $(x^2 + y^2)[1 + (\frac{dy}{dx})^2] = r^2$ d) $r^2[1 + (\frac{dy}{dx})^2] = x^2$

15) The differential equation satisfying the relation $x = A\cos(mt - \propto)$ is

a) $\frac{dx}{dt} = 1 - x^2$ b) $\frac{d^2x}{dt^2} = -\propto^2 x$

c) $\frac{d^2x}{dt^2} = -m^2x$ d) $\frac{dx}{dt} = -m^2x$

16) The equation $\frac{dy}{dx} + \frac{ax+hy+g}{hx+by+f} = 0$ is

a) Homogeneous b) Variable separable c) Exact d) None

17) Find the differential equation of the family of cardioids r = a(1+cosθ)

a) $\frac{dr}{d\theta}$ + rsinx = 0 b) $\frac{dr}{d\theta}$ + r tan($\frac{\theta}{2}$) = 0

c) $\frac{dr}{d\theta}$ + r sin($\frac{\theta}{2}$) = 0 d) $None$

18) The equation $\frac{dy}{dx} + \sqrt{\frac{1+ y^2}{1+ x^2}} = 0$ is

a) Variable separable b) Exact c) Homogeneous d) None

19) The solution of the differential equation is $\frac{dy}{dx} = e^{(x - y)} + x^2e^{-y}$

a) $e^y = \frac{x^3}{3} + e^x + c$ b) $e^y = e^x + 3x + c$ c) $e^x = \frac{x^3}{3} + e^y + c$ d) None

20) The general solution of $\frac{dy}{dx} = (4x + y+1)^2$ is

a) $\tan^{-1}(\frac{4x+y+1}{2}) = c$ b) $\frac{1}{2}\tan^{-1}(\frac{4x+y+1}{2}) = y + c$

c) $\frac{1}{2}\tan^{-1}(\frac{4x+y+1}{2}) = x + c$ d) None

21) The solution of of the Differential equation $(x^2+1)\, y_1 + y^2+ 1 = 0$, y(0) = 1 is

a) $\frac{\pi}{4}$ b) $\frac{\pi}{6}$ c) $\frac{\pi}{2}$ d) $\frac{\pi}{8}$

22) The solution of $\frac{ydx-xdy}{y^2}$ = 0 is

a) xy = c b) y = cx c) x = cy d) x = cy^2

23) The general solution of $\frac{xdx+ydy}{x^2+y^2}$ = 0 is

a) log(x+y) = c b) log($x^2 + y^2$) = c c)log(xy) = c d) None

24) The equation of the form $\frac{dy}{dx}$ + p(x)y = q(x) is

a) Homogeneous b) Exact c) Linear d) None

25) Integral factor of $\frac{dy}{dx}$ + p(x)y = q(x) is

a) $e^{\int pdx}$ b) $e^{\int pdy}$ c) $e^{\int qdx}$ d) $e^{\int qdy}$

26) The general solution of $\frac{dy}{dx}$ + ycotx = cosx is

a) y = $\frac{1}{2}$ sinx + c cosx b) y = $\frac{1}{2}$ cosx + c sinx

c) y = $\frac{1}{2}$ sinx + c cosecx d) None

27) The form of Bernoulli's Equation is

a) $\frac{dy}{dx}$ + px = Qy^n b) $\frac{dy}{dx}$ + py = Qx^n

c) $\frac{dy}{dx}$ + Qy^n = px d) $\frac{dy}{dx}$ + py = Qy^n

28) The equation of the form M(x,y)dx + N(x,y)dy = 0 is called

if $\frac{\partial m}{\partial y} = \frac{\partial n}{\partial x}$

a) Linear b) Bernoulli's c) Exact d) Homogeneous

29) Integrating factor of the homogenous de Mdx + Ndy = 0 is

a) $\frac{1}{Mx - Ny}$ b) $\frac{1}{Mx + Ny}$ c) $\frac{1}{Nx - My}$ d) None

30) If $\frac{1}{N}(\frac{\partial M}{\partial y}$ - $\frac{\partial N}{\partial x})$ is a function of x alone say f(x) then the

integrating factor of Mdx + Ndy = 0 is

a) $e^{\int f(x)dy}$ b) $e^{\int f(y)dy}$ c) $e^{\int f(x)dx}$ d) $e^{\int f(x)dy}$

31) The integrating factor of $(x^2 - 3xy + 2y^2)dx + x(3x-2y)dy =$

c is

a) $\frac{1}{x^2}$ b) $\frac{1}{x^5}$ c) $\frac{1}{x}$ d) $\frac{1}{x^3}$

32) The given differential equation y(x+y)dx + (x+2y-1)dy = 0

is

a) Exact b) Not Exact c) We can't say d)

None

33) The integrating factor of the equation y $f_1(xy)$dx + x

$f_2(xy)$dy is

a) $\frac{1}{Mx + Ny}$ b) $\frac{1}{My + Nx}$ c) $\frac{1}{Mx - Ny}$ d) $\frac{1}{My - Nx}$

34) If $\frac{1}{M}(\frac{\partial N}{\partial x}$ - $\frac{\partial M}{\partial y})$ is a function of y alone then the integrating

factor is

a) $e^{\int g(x)dy}$ b) $e^{\int g(x)dy}$ c) $e^{\int g(y)dy}$ d) *None*

35) The general solution of $(1 + x^2)$ dy – $(1+y^2)$ dx = 0 is

a) $\tan^{-1}y - \tan^{-1}x = c$ b) $\tan^{-1}y + \tan^{-1}x = c$

c) $\sin^{-1}x - \sin^{-1}y = c$ d) None

36) The general solution of $\frac{dy}{dx} + xy = x$ is

a) $y \frac{dy}{dx} = 2x$ b) $x \frac{dy}{dx} = 2y$ c) $y \frac{dy}{dx} = -2x$

d) None

37) The differential equation of orthogonal trajectories of the family of curves

$y^2 = 4ax$, where a is the parameter is

a) $y \frac{dy}{dx} = x$ b) $x \frac{dy}{dx} + y = 0$ c) $y \frac{dy}{dx} + x = 0$

d) None

38) The general solution of $(1+y^2)\,dx = (\tan^{-1}y - x)\,dy$ is

a) $x = \tan^{-1}y - 1 + ce^{-\tan^{-1}y}$ b) $x = \tan^{-1}y - 1 + c$

$e^{-\tan^{-1}y}$

c) $x = \tan^{-1}y$ d) None

39) The differential equation of the orthogonal trajectories of the family of

curves $xy = a^2$ where a is the parameter is

a) $y \frac{dy}{dx} = x$ b) $y \frac{dy}{dx} + x = 0$ c) $x \frac{dy}{dx} + y = 0$

d) None

40) The differential equation of the orthogonal trajectories of the family of

curves $r = a\theta$ where a is the parameter is

a) $\frac{dr}{d\theta}$ + rθ= 0 b) $\frac{dr}{d\theta}$ = - rθ c) $\frac{dr}{d\theta}$ = θ

d) $\frac{dr}{d\theta}$ = r

41) The equation y - 2x = c represents the orthogonal trajectories of the family of

a) y = a e^{-2x} b) $x^2 + y^2 = a$ c) x y = c d) x + 2y = c

42) The general solution of the differential equation $\frac{dy}{dx} = \frac{y}{x}$ + tan $(\frac{y}{x})$ is

a) sin $(\frac{y}{x})$ = c b) sin $(\frac{y}{x})$ = cx c) cos $(\frac{y}{x})$ = cx d) None

43) The integrating factor of x $\frac{dy}{dx}$ - y = $2x^2$ cosec2x is

a) x b) $\frac{1}{x}$ c) e^{-x} d) None

44) The integrating factor of $(1 - x^2)$ y + xy = ax is

a) $\frac{1}{x^2-1}$ b) $\frac{1}{\sqrt{x^2-1}}$ c) $\frac{1}{1-x^2}$ d) $\frac{1}{\sqrt{1-x^2}}$

45) The orthogonal trajectories of the family of ellipses is

a) $\frac{x^2}{a^2} + \frac{y^2}{\lambda} = 1$ b) $\frac{x^2}{a} + \frac{y^2}{\lambda^2} = 1$ c) $\frac{x^2}{a^2} + \frac{y^2}{a^2+\lambda} = 1$ d) $None$

46) The differential equation $\frac{dy}{dx} + \frac{y}{x} = y^2$x sinx is

a) Bernoulli's equation b) Exact Equation

c) Linear Equation d) None

47) A differential equation of the form $\frac{dy}{dx}$ + p(x)y = Q(x) then the general

solution of this differential equation is (first order in y)

a) y(I.F) = Q(x) * (I.F) + c b) y(I.F) = $\int Q(y) * (I.F)dx + c$

c) y(I.F) = $\int Q(x) * (I.F)dx + c$ d) None

48) The differential equation $y^2dx + (x^2 - xy - y^2)\,dy = 0$ is

a) Exact b) Not Exact c) We can't say d) None

49) d(xy) =

a) xdx+ydy b) xdy+ydx c) xdy+ydy d) None

50) d($tan^{-1}(\frac{x}{y})$) =

a) $\frac{-xdx + ydy}{x^2 + y^2}$ b) $\frac{xdx + ydy}{x^2 + y^2}$ c) $\frac{-xdy + ydx}{x^2 + y^2}$ d) None

Linear differential equations of higher order with the constant coefficient

The general form of linear differential equation of n^{th} order is

$\frac{d^n y}{dx^n} + P_1\frac{d^{n-1}y}{dx^{n-1}} + P_2\frac{d^{n-2}y}{dx^{n-2}} + ------ + P_n y = X$ where P_1, P_2, P_3-----P_n are constants and X is a function of x

If we put $\frac{d}{dx} = D$ then the linear equation is reduced to f(D)y =X where

$f(D)= D^n+P_1D^{n-1}+------P_{n-1}D+P_n$

If y_1, y_2 are two solutions of the equation f(D)y=0 then $c_1y_1+c_2y_2$(=u) is also its solution.

Since the general solution of a D.E of the nth order contains n arbitrary constants so that if $y_1,y_2, y_3,\ldots\ldots y_n$ are n independent solutions of f(D)y=0 then $c_1y_1+ c_2y_2 + \ldots\ldots+ c_ny_n(= u)$ is its complete solution.
If y=v is any particular solution of equation f(D)y=X.then y=u+v is the complete solution of f(D)y=X where u is the solution of f(D)y=0.
The part u is called complementary function (C.F.)and the part v is called the particular integral(P.I.) of f(D)y=X.
If f(D)y=0,the equation obtained by replacing D with 'm ' is called auxiliary equation (A.E)
To find the C.F write the A.E.
$m^n+P_1 m^{n-1}+\text{------}P_{n-1}m+P_n=0$ and solve it for m

Write the C.F as follows.

S.No	Roots of A.E.	Complimentary Function
1.	$m_1,m_2,m_3,\ldots$..(real and different roots)	$c_1e^{m_1x}+ c_2e^{m_2x} + c_3e^{m_3x} +\ldots\ldots$
2.	$m_1,m_1,m_3,\ldots$..(two real and equal roots)	$(c_1+ c_2x) e^{m_1x} + c_3e^{m_3x} +\ldots\ldots$
3.	$m_1,m_1,m_1,m_4,\ldots$..(three real and equal roots)	$(c_1+ c_2x+ c_3x^2) e^{m_1x} + c_4e^{m_4x} +\ldots\ldots$
4.	$a+ib,a-ib,m_3,\ldots\ldots$(a pair of imaginary roots)	$e^{ax}(c_1\cos bx+ c_2\sin bx) + c_3e^{m_3x} +\ldots\ldots$
5.	$a\pm ib,a\pm ib,m_5,\ldots\ldots$(two pairs of equal imaginary roots)	$e^{ax}[(c_1+ c_2x)\cos bx+ (c_3+ c_4x)\sin bx] + c_5e^{m_5x} +\ldots\ldots$
6. .	$a+\sqrt{b}, a-\sqrt{b}, m_3,\ldots\ldots$(a pair of surd roots)	$e^{ax}(c_1\cosh\sqrt{b}\, x+ c_2\sinh\sqrt{b}\, x) + c_3e^{m_3x} +\ldots\ldots$
7	$a\pm\sqrt{b}$, $a\pm\sqrt{b}$,$m_5,\ldots$..(two pairs of equal surd roots)	$e^{ax}[(c_1+ c_2x)\cosh\sqrt{b}\, x+ (c_3+ c_4x)\sinh\sqrt{b}\, x] + c_5e^{m_5x} +\ldots\ldots$

Computation of Particular Integral(P.I):

Let the D.E. be f(D)y = X .The inverse operator $\frac{1}{f(D)}$ is defined as $\frac{1}{f(D)}U = V$ if

f(D)V = U where U,V are functions of x.

Results

$\frac{1}{f(D)}(X)$ is P.I. of f(D)y=X since $f(D)\left[\frac{1}{f(D)}X\right] = X$

$$\frac{1}{D-a}(X) = e^{ax}\int X\, e^{-ax}\,dx$$

$$\frac{1}{D+a}(X) = e^{-ax}\int X\, e^{ax}\,dx$$

The following table summarizes the formula which we provided to compute the P.I. of f(D)y = X

S.No	X	Formula to compute $\frac{1}{f(D)}$ X
1	e^{ax} when $f(a) \neq 0$ when $f(a)=0$ and $f^1(a) \neq 0$ when $f(a)=f^1(a)=0$ and $f^{11}(a) \neq 0$ and so on….	$\frac{1}{f(D)} e^{ax} = \frac{1}{f(a)} e^{ax}$ (Replacing D with ‘a’) $\frac{1}{f(D)} e^{ax} = x \frac{1}{f^1(a)} e^{ax}$ $\frac{1}{f(D)} e^{ax} = x^2 \frac{1}{f''(a)} e^{ax}$
2	Sin bx or cos bx when $f(-b^2) \neq 0$ When $f(-b^2)=0$ and $f^1(-b^2) \neq 0$ when $f(-b^2)= f^1(-b^2)=0$ and $f^{11}(-b^2) \neq 0$ and so on…	Replace D^2 by $-b^2$ $\frac{1}{f(D)}\sin bx = \frac{1}{f(-b^2)}\sin bx$ $\frac{1}{f(D)}\cos bx = \frac{1}{f(-b^2)}\cos bx$ $\frac{1}{f(D)}\sin bx = x\frac{1}{f^1(-b^2)}\sin bx$ $\frac{1}{f(D)}\cos bx = x\frac{1}{f^1(-b^2)}\cos bx$ $\frac{1}{f(D)}\sin bx = x^2\frac{1}{f''(-b^2)}\sin bx$ $\frac{1}{f(D)}\cos bx = x^2\frac{1}{f''(-b^2)}\cos bx$

5	x^m, m is a positive integer	Write f(D) in the form $(1+\phi(D))^{-1}$ and expand in ascending powers of D by binomial theorem as per as the term D^m and operate on x^m term by term.
6	$e^{ax}V$, V is a function of x.	Applying $\frac{1}{f(D)} e^{ax}V = e^{ax} \frac{1}{f(D+a)} V$
7	x V , V is a function of x	$\frac{1}{f(D)} xV = x \frac{1}{f(D)} V - \frac{1}{f(D)} f^1(D) \frac{1}{f(D)} V$

Note: $\frac{1}{D^2+b^2}\cos bx = \frac{x}{2b}\sin bx$ $\qquad \frac{1}{D^2+b^2}\sin bx = -\frac{x}{2b}\cos bx$

- **Equations reducible to linear equations with constant coefficients:**

Cauchy's Homogeneous Linear Equations:

The general equation of n^{th} orD.E.r is of the form

$$x^n \frac{d^n y}{dx^n} + P_1 x^{n-1}\frac{d^{n-1}y}{dx^{n-1}} + P_2 x^{n-2}\frac{d^{n-2}y}{dx^{n-2}} + ------ + P_n y = X$$

where P_1,P_2,P_3-----P_n are constants and X is a function of x.

- Procedure:

Take $x= e^t$ so that log x=t which gives

$x^nD^n= \theta(\theta-1)\dots\dots(\theta-n+1)$ where $\theta = d/dt$,n=1,2,3,…….

The above equation reduces to Linear equations with constant coefficients.

Legendre's Equation: The general equation of n^{th} orD.E.r is of the form

$(ax+b)^n \frac{d^n y}{dx^n} + P_1(ax+b)^{n-1}\frac{d^{n-1}y}{dx^{n-1}} + P_2(ax+b)^{n-2}\frac{d^{n-2}y}{dx^{n-2}} + ------ + P_n y = X$ where P_1,P_2,P_3-----P_n ,a,b are constants and X is a function of x.

- Procedure:

Take $ax+b= e^t$ so that log(ax+b) = t which gives

$(ax+b)^nD^n=a^n \theta(\theta-1)\dots\dots(\theta-n+1)$ where $\theta = d/dt$,n=1,2,3,…….

The above equation reduces to Linear equations with constant coefficients.

- **Method of Variation of parameters:**

This method is applicable to equations with constant coefficients but the C.F. must be known before the method is applied.

Procedure to solve $y^{11}+py^1+qy=X$ where p,q are constants and X is function of x

Find C.F. $c_1y_1+c_2y_2$

Assume P.I. as uy_1+vy_2 so that $u^1y_1^{\,1}+v^1y_2^{\,1}=X$

To find u,v as functions of x assume that $u^1y_1+v^1y_2=0$

Compute $u=-\int \frac{y_2X}{W}dx$ and $v = \int \frac{y_1X}{W}dx$ where $W=\begin{vmatrix} y_1 & y_2 \\ y_1^1 & y_2^1 \end{vmatrix}$ is called the **Wronskian** of y_1 ,y_2

The general solution is $y = c_1y_1+c_2y_2+uy_1+vy_2$.

Simultaneous Linear differential equations:

_ A set of D.E.'s that arise from a se of two or more dependent variables and one independent variable is known as simultaneous linear equations with constant coefficients.

To solve we use the process of elimination (Similar approach to that of solving two linear equations).

Bending of beams

The goal of mathematical modeling is to represent natural processes by mathematical equations, to analyze the mathematical equations, and then to use the mathematical model to better understand and predict the natural process. In this module, we are interested in predicting and understanding the deflection of loaded beams by mathematically modeling their deflection curves.

Before beginning our mathematical analysis, let us review the ingredients that will go into it. As we have already seen, the shape of the deflection curve will depend on several factors. The four factors that enter into our mathematical model are:

- The material properties of the beam as measured by the modulus of elasticity.
- The beam's cross section as measured by its centroidal moment of inertia.
- The load on the beam, described as a function of the position along the beam.
- The way the beam is supported, which is captured by the boundary conditions of the differential equation in our model.

Experiments show that the deflection curve depends inversely on the modulus of elasticity, *E* , and also depends inversely on the centroidal moment of inertia of the beam's cross section, *I* . The way the deflection depends on the applied load, *q(x)* , and on the manner in which the beam is supported is more complicated, as we shall see.

Under the assumption that the deflections of the beam are small, an understanding of certain physical principles and geometric concepts allows one to derive a fourth-order differential equation that the deflection function *w(x)* must satisfy:

$$w''''(x) = \frac{q(x)}{EI} \qquad (1)$$

We will call this equation the *static beam equation* . In words, the beam equation tells us that the deflection function is a function whose fourth derivative at *every* point, *x* , is equal to the load at that point divided by a constant quantity that depends only on the material properties and shape of the beam.

The importance of this equation is that in principle, we can determine the quantities on the right-hand side of the equation. Thus, the static beam equation enables us to mathematically solve for the deflection function!

Simple Harmonic Motion (SHM)

Introduction

In addition to linear motion and rotational motion there is another kind of motion that is common in physics. This is the to and fro motion of oscillations or vibrations.

When something oscillates, it moves back and forth with time. It is helpful to trace out the position of an oscillating particle with time so we can define some terminology.

Period, Amplitude and Frequency

The time taken for the particle to complete one oscilation, that is, the time taken for the particle to move from its starting position and return to its original position is known as the period. and is generally given the symbol T. The frequency ν is related to the period, it is defined as how many oscillations occur in one second. Since the period is the time taken for one oscillation, the frequency is given by

$f = 1/T$(1)

The frequency is measured in [s^{-1}]. This unit is known as the Hertz (Hz) in honour of the physicist Heinrich Hertz. The maximum displacement of the particle from its resting position is known as the amplitude. The frequency is also given the symbol f.

Simple Harmonic Motion

The definition of simple harmonic motion is simply that the acceleration causing the motion a of the particle or object is proportional and in opposition to its displacement x from its equilibrium position.

$a(t) \propto -x(t)$

Where k is a constant of proportionality. This remembering that the acceleration is the second derivative of position, also leads us to the differential equation

$x''(t) = - k\, x(t)$

Simple Harmonic Motion is closely related to circular motion as can be seen if we take an object that moves in a circular path, like a ball stuck on a turntable. If we consider just the y-component of the motion the path with time we can see that it traces out a wave as shown in Flash 2.

$y(t) = A \sin(\omega t)$(2)

We set this out mathematically, using a differential equation as in equation (4). We specify the equation in terms of the forces acting on the object. The acceleration is the second derivative of the position with respect to time and this is proportional to the position with respect to time. The minus sign indicates that the position is in the opposite direction to the acceleration.

$m\, y''(t) = -\, k\, y(t)$(4)

The derivation of the solution can be found here

For which the general solution is a wave like solution.

$y(t) = c_1 \cos(\omega t) + c_2 \sin(\omega t)$

Where, ω is the angular frequency. ($\omega=2\pi f$) The values of c_1 and c_2 are determined by the initial conditions. Specifically, $c_1 = y_0$ and $c_2 = v_0/\omega$ These two initial conditions specify the starting position and the initial velocity.

The general solution can also be written more compactly as

$y(t)= A \cos(\omega t - \varphi)$(5)

Where $\varphi = \tan^{-1}(\omega y_0/v_0)$, $A = (y_0^2 + (v_0/\omega)^2)^{1/2}$

Differentiating once with respect to time, we obtain the velocity. (The derivative of $\cos x = -\sin x$)

$v = y'(t)= -\, \omega A \sin(\omega t - \varphi)$(6)

Finally, the acceleration is the derivative of the velocity with respect to time. (The derivitive of $-\sin x = -\cos x$)

$a = y''(t) = -\, \omega^2 A \cos(\omega t - \varphi)$(7)

Substituting equations (5) and (7) into equation (4) we verify that this does indeed satisfy the equation for simple harmonic motion. With the constant of proportionality $k = \omega^2$

Thus

$a(t) = -\, \omega^2 y(t)$

The time for the maximum velocity and acceleration can be determined from these equations. From equation (6) the maximum magnitude of the velocity occurs when $\sin(\omega t - \varphi)$ is 1 or -1. Therefore the maximum velocity is $\pm\omega A$. Intuitively, we can imagine that this velocity occurs when the oscillating system has reached the equilibrium position and is about to overshoot. The minus sign indicates the direction of travel is in the opposite direction.

The maximum acceleration occurs where the argument of cosine in eqn (7) is also -1 or 1. Thus the maximum acceleration is $\pm\omega^2 A$ which occurs at the ends of the oscillations, as this is where the direction changes.

Examples of SHM

Spring Mass System

Consider a spring of spring constant *k* conected to a mass *m*. If the mass is displaced from its equilibrium position by a distance *x* a force *F* will act in the opposite direction to the displacement. From Hooke's Law the magnitude of the force is given by

$F = -k\,x$

When the mass is released it the force will act on the mass to bring it back to its equilibrium position. However, if there is no friction the inertia of the mass will cause it to overshoot the equilibrium position and the force will act in the opposite direction slowing it down and pulling it back.

1. Solve $(D^3+1)y = \cos(2x-1)$
2 Solve $(D^3+1)y = 3 + 5e^x$ where $D=d/dx$
3. Solve $(D^3-D)\,y = 2x+1+4\cos x+2e^x$
4 Solve $(D^2-3D+2)y = xe^{3x}+\sin 2x$
5 Solve $(D^2+4D+3)y = e^{-x}\sin x + x$
6 Solve $(D^2+3D+2)y = x \sin 2x$
7 Solve $(D^2-1)y = x\sin x + e^x(1+x^2)$
8 Solve $(D^2-1)y = 2e^x+3x$
9 Solve $(D^2-1)y = x \sin 3x+\cos x$
10 Solve $(x^4D^3+2x^3D^2-x^2D+x)y = 1/x$
11 Solve $[(1+x)^2D^2+(1+x)D+1]y = \sin 2(\log(1+x))$
12 Solve $(x^2D^2-xD+2)y = x\log x$
13 Solve $(x^2D^2-2xD-4)y = x^2+2\log x$
14 Solve $(x^2D^2-xD+1)y = \log x$
15 Solve $(D^4+4)y=0$
16 Solve $(D-2)^2y = 8(e^{2x}+\sin 2x+x^2)$
17 Solve $(D^4-1)y = \cos x \cosh x$
18 Solve $(D^2-2D+1)y = xe^x\sin x$
19 Solve $(x^3D^3+2x^2D^2+2)y = 10(x+1/x)$

20 Without using variation of parameters solve $(D^2+a^2)y = \tan ax$

Solve by method of variation of parameters:

21 $(D^2+4)y = \tan 2x$

22 $(D^2-1)y = \dfrac{1}{e^x-1}$

23 $(D^2 + n^2)y = \sec nx$

24 $(D^2-3D+2)y = e^{2x}+x^2$

25 Solve the simultaneous Differential equations:

$$\frac{dx}{dt}+2y=e^t \text{ and } \frac{dy}{dt}-2x= e^{-t}$$