UNIT-3

Explain the homogeneous transformations as applicable to rotation.

OR

Define and explain about homogeneous transformations.

Answer :

It is a general method for solving the kinematic equation of a robot manipulator with many joints. It is described by a single matrix that combines the effect of translation and rotation.

The rotation transformation operates on homogenous coordinates and perform rotation about a given axis of the reference coordinate system.

The reference co-ordinate system is given as follows.

(a) Rotation 'α' degrees about the x-axis,

$$\text{Rot}(x, \alpha) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha & 0 \\ 0 & \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(b) Rotation 'α' degrees about the y-axis,

$$\text{Rot}(y, \alpha) = \begin{bmatrix} \cos\alpha & 0 & \sin\alpha & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\alpha & 0 & \cos\alpha & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(c) Rotation 'α' degrees about the z-axis,

$$\text{Rot}(z, \alpha) = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 & 0 \\ \sin\alpha & \cos\alpha & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The general form of the homogenous transformation can be partitioned into sub-matrix in tabular form.

$$\left[\begin{array}{l|l} \text{Rotation} & \text{Partition} \\ \text{matrix } (3\times3) & \text{vector } (3\times1) \\ \hline \text{Perspective} & \text{Scaling} \\ \text{transform } (1\times3) & \text{factor } (1\times1) \end{array}\right]$$

The entry (3 × 3) matrix is for rotation, (3 × 1) for translation and other two sub-matrix for perspective transform and scaling factor.

Vector and position nomenclature for homogenous transformation matrix.

$$H = \left[\begin{array}{ccc|c} h_x & o_x & a_x & p_x \\ h_y & o_y & a_y & p_y \\ h_z & o_z & a_z & p_z \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$$

Homogenous transformation is based on mapping an N-dimensional space into $(N+1)$ dimensional space i.e., one more coordinate is added to represent the position of a point.

Example

A 3-dimensional space point has coordinates (x, y, z) is represented by vector (x, y, z, w) in homogenous transformations in which 'w' is a dummy coordinate that on normalization gives $\left(\frac{x}{w}, \frac{y}{w}, \frac{z}{w}, 1\right)$. This additional 1 serves as a tool to accomplish the addition of matrices required in translation by matrix multiplication.

Rotation transformation:-

① Find the rotation of vector $v = 5i + 3j + 8k$ by the angle of $90°$ about x-axis.

Ⓐ

$$H = Rot(x, 90°) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos 90° & -\sin 90° \\ 0 & \sin 90° & \cos 90° \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

Given vector $v = \begin{bmatrix} 5 \\ 3 \\ 8 \end{bmatrix}$.

$$\therefore \quad u = H \cdot v = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 5 \\ 3 \\ 8 \end{bmatrix} = \begin{bmatrix} 5 \\ -8 \\ 3 \end{bmatrix}$$

② The coordinates of point p in frame {1} are $[3.0 \; 2.0 \; 1.0]^T$. The position vector p is rotated about the z-axis by $45°$. Find the coordinates of point Q, the new position of point p.

Ⓐ The $45°$ rotation of p about the z-axis of frame {1}

$$\text{From, } R(z, \theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R(2,45^\circ) = \begin{bmatrix} \cos 45^\circ & -\sin 45^\circ & 0 \\ \sin 45^\circ & \cos 45^\circ & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0.707 & -0.707 & 0 \\ 0.707 & 0.707 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{—(i)}$$

For the rotation of vectors, $Q' = R(2,45^\circ).V$

where, $V = \begin{bmatrix} 3.0 \\ 2.0 \\ 1.0 \end{bmatrix}$

$$\therefore Q' = \underset{3\times3}{\begin{bmatrix} 0.707 & -0.707 & 0 \\ 0.707 & 0.707 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \underset{3\times1}{\begin{bmatrix} 3.0 \\ 2.0 \\ 1.0 \end{bmatrix}} = \underset{3\times1}{\begin{bmatrix} 0.707 \\ 3.535 \\ 1 \end{bmatrix}}$$

Thus, the coordinates of the new point 'Q' relative to frame{1} are $[0.707 \ 3.535 \ 1.0]^T$.

③ A coordinate frame {B} is located initially coincident with a coordinate frame {A}. If the frame {B} is rotated through 30° about Z_B and 60° about current Y_B. Find the rotation matrix that will describe a vector of frame {B} in frame {A}.

Ⓐ θ = Angle made by frame {B} about Z-axis = 30°

ϕ = Angle made by frame {B} about Y-axis = 60°

we know $\quad {}^{A}_{B}R = R(2,30^\circ) \cdot R(y,60^\circ)$

$$\Rightarrow {}^{A}_{B}R = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\phi & 0 & \sin\phi \\ 0 & 1 & 0 \\ -\sin\phi & 0 & \cos\phi \end{bmatrix}$$

$\therefore$ Substitute $\theta = 30^\circ$ and $\phi = 60^\circ$ in the above equation, we get

$$ {}^{A}_{B}R = \begin{bmatrix} \cos 30^\circ & -\sin 30^\circ & 0 \\ \sin 30^\circ & \cos 30^\circ & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos 60^\circ & 0 & \sin 60^\circ \\ 0 & 1 & 0 \\ -\sin 60^\circ & 0 & \cos 60^\circ \end{bmatrix} $$

$$ = \begin{bmatrix} 0.866 & -0.5 & 0 \\ 0.5 & 0.866 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0.5 & 0 & 0.866 \\ 0 & 1 & 0 \\ -0.866 & 0 & 0.5 \end{bmatrix} $$

$$ = \begin{bmatrix} 0.433 & -0.5 & 0.749 \\ 0.25 & 0.866 & 0.433 \\ -0.866 & 0 & 0.5 \end{bmatrix} $$

④ A mobile body reference frame OABC is rotated 60° about OZ axis of the fixed base reference frame OXYZ. If $P_{xyz} = [-1, 2, 4]^T$ and $Q_{xyz} = [2, -3, 3]^T$ are the coordinates with respect to OXYZ frame, determine coordinates of P and Q with respect to the OABC frame.

Ⓐ $P_{xyz} = [-1, 2, 4]^T$, $Q_{xyz} = [2, -3, 3]^T$

Frame OABC is rotated 60° about OZ axis of fixed base OXYZ.

$$ \therefore P_{ABC} = R(z, 60^\circ)^T P_{xyz} $$

$$ Q_{ABC} = R(z, 60^\circ)^T Q_{xyz} $$

$$ \Rightarrow R(z, 60^\circ) = \begin{bmatrix} \cos 60^\circ & -\sin 60^\circ & 0 \\ \sin 60^\circ & \cos 60^\circ & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0.5 & -0.866 & 0 \\ 0.866 & 0.5 & 0 \\ 0 & 0 & 1 \end{bmatrix} $$

$$ R(z, 60^\circ)^T = \begin{bmatrix} 0.5 & 0.866 & 0 \\ -0.866 & 0.5 & 0 \\ 0 & 0 & 1 \end{bmatrix} $$

$$\therefore \quad P_{ABC} = \begin{bmatrix} 0.5 & 0.866 & 0 \\ -0.866 & 0.5 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 1.232 \\ 1.866 \\ 4 \end{bmatrix}$$

$$\text{And, } Q_{ABC} = \begin{bmatrix} 0.5 & 0.866 & 0 \\ -0.866 & 0.5 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ -3 \\ 3 \end{bmatrix} = \begin{bmatrix} -1.598 \\ -3.232 \\ 3 \end{bmatrix}$$

5. For the following rotation matrix determine the axis of rotation and the angle of rotation about the same axis.

$$\begin{bmatrix} \sqrt{3}/2 & 0 & 0.5 \\ 0 & 1 & 0 \\ -0.5 & 0 & \sqrt{3}/2 \end{bmatrix}$$

(A)

$$\text{Rotation matrix, } R = \begin{bmatrix} \sqrt{3}/2 & 0 & 0.5 \\ 0 & 1 & 0 \\ -0.5 & 0 & \sqrt{3}/2 \end{bmatrix} \quad -(i)$$

The above matrix is in the form of $R(y, \theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$ -(ii)

From equations (i) & (ii), we get

$$\cos\theta = \frac{\sqrt{3}}{2} \text{ and } \sin\theta = 0.5$$

$$\boxed{\therefore \theta = 30^\circ}$$

$\therefore$ axis of rotation is y-axis and angle of rotation is 30°.

Composite rotation matrix:-

1. Determine a composite rotation matrix for the following,

(i) Rotation of angle α about x-axis

(ii) Rotation of angle β about y-axis

(iii) Rotation of angle γ about z-axis.

(A) (i) Rotation of Angle α about X-axis

$$R(x,\alpha) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{bmatrix}$$

(ii)

$$R(y,\beta) = \begin{bmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{bmatrix}, \quad \text{(iii) } R(z,\gamma) = \begin{bmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The composite rotation matrix (R) following the sequence of rotations can be obtained by,

$$R = R(x,\alpha) \cdot R(y,\beta) \cdot R(z,\gamma)$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{bmatrix} \begin{bmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} \cos\beta\cos\gamma & -\cos\beta\sin\gamma & \sin\beta \\ \sin\gamma & \cos\gamma & 0 \\ -\sin\beta\cos\gamma & \sin\beta\sin\gamma & \cos\beta \end{bmatrix}$$

$$= \begin{bmatrix} \cos\beta\cos\gamma & -\cos\beta\sin\gamma & \sin\beta \\ \cos\alpha\sin\gamma + \sin\alpha\sin\beta\cos\gamma & \cos\alpha\cos\gamma - \sin\alpha\sin\beta\sin\gamma & -\sin\alpha\cos\beta \\ \sin\alpha\sin\gamma - \cos\alpha\sin\beta\cos\gamma & \sin\alpha\cos\gamma + \cos\alpha\sin\beta\sin\gamma & \cos\alpha\cos\beta \end{bmatrix}$$

Traslation:-

(1) For the vector $v = 25I + 10J + 20K$ perform of 8 in x-direction, 5 in y-direction and 0 in z-direction.

(A)

$$H = Tras(8,5,0) = \begin{bmatrix} 1 & 0 & 0 & 8 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$V = \begin{bmatrix} 25 \\ 10 \\ 20 \\ 1 \end{bmatrix}$$

$$\therefore u = H \cdot V = \underset{4\times4}{\begin{bmatrix} 1 & 0 & 0 & 8 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}} \underset{4\times1}{\begin{bmatrix} 25 \\ 10 \\ 20 \\ 1 \end{bmatrix}} = \underset{4\times1}{\begin{bmatrix} 33 \\ 25 \\ 20 \\ 1 \end{bmatrix}}$$

② A vector v is $3i + 2J + 5K$ is translated 3 units along y and x axis, and 2 units in z-direction. Find the final vector.

Ⓐ

$$H = \text{Tran}(3, 3, 2) = \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$V = \begin{bmatrix} 3 \\ 2 \\ 5 \\ 1 \end{bmatrix}$$

$$u = H \cdot V = \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 5 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 5 \\ 7 \\ 1 \end{bmatrix}$$

Rotation and translation transformation :-

① Find the transformation matrices for the following operations on the point $4i + 5J - 2K$.

(i) Rotate $60°$ about x-axis and then translate 3 units along y-axis.

(ii) Translate 6 units along y-axis and rotate $30°$ about x-axis.

Ⓐ The vector matrix for the given vector $4i + 5J - 2K$ is

$$u = \begin{bmatrix} 4 \\ 5 \\ -2 \\ 1 \end{bmatrix}$$

(i) The Homogeneous transformation matrix (T) to perform rotation of 60° about x-axis and then translation of −3 units along y-axis is given by $T = Tran(y,-3) \times Rot(x, 60°)$.

$$T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos 60° & -\sin 60° & 0 \\ 0 & \sin 60° & \cos 60° & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0.5 & -0.866 & 0 \\ 0 & 0.866 & 0.5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0.5 & -0.866 & -3 \\ 0 & 0.866 & 0.5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The transformed vector, $V = T \cdot u$

$$V = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0.5 & -0.866 & -3 \\ 0 & 0.866 & 0.5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 5 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 1.232 \\ 3.330 \\ 1 \end{bmatrix}$$

(ii)

$$T = Rot(x, 30°) \times Tran(y, 6)$$

$$T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos 30° & -\sin 30° & 0 \\ 0 & \sin 30° & \cos 30° & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0.866 & -0.5 & 0 \\ 0 & 0.5 & 0.866 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0.866 & -0.5 & 5.196 \\ 0 & 0.5 & 0.866 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The transformed vector, $V = T.u$

$$V = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0.866 & -0.5 & 5.196 \\ 0 & 0.5 & 0.866 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 5 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 10.526 \\ 3.768 \\ 1 \end{bmatrix}.$$

(2) Find the transformation matrices for the following operations on the point $2i-5J+4K$.

(i) Rotate 60° about x-axis and then translate 4 units along Y-axis.

(ii) Translate -6 units along y-axis and rotate 30° about x-axis.

(A) given vector = $2i-5J+4K$.

∴ The vector matrix for the given vector is $u = \begin{bmatrix} 2 \\ -5 \\ 4 \\ 1 \end{bmatrix}$

(i) $T = \text{Tran}(y, 4), \times \text{Rot}(x, 60°)$

$$T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0.5 & -0.866 & 0 \\ 0 & 0.866 & 0.5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0.5 & -0.866 & 4 \\ 0 & 0.866 & 0.5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The transformed vector, $V = T.u$

$$V = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0.5 & -0.866 & 4 \\ 0 & 0.866 & 0.5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ -5 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -1.964 \\ -2.33 \\ 1 \end{bmatrix}$$

(ii) $T = \text{Rot}(\cancel{y}x, 30°) \times \text{Tran}(y, -6)$.

$$T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0.866 & -0.5 & 0 \\ 0 & 0.5 & 0.866 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -6 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0.866 & -0.5 & -5.196 \\ 0 & 0.5 & 0.866 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

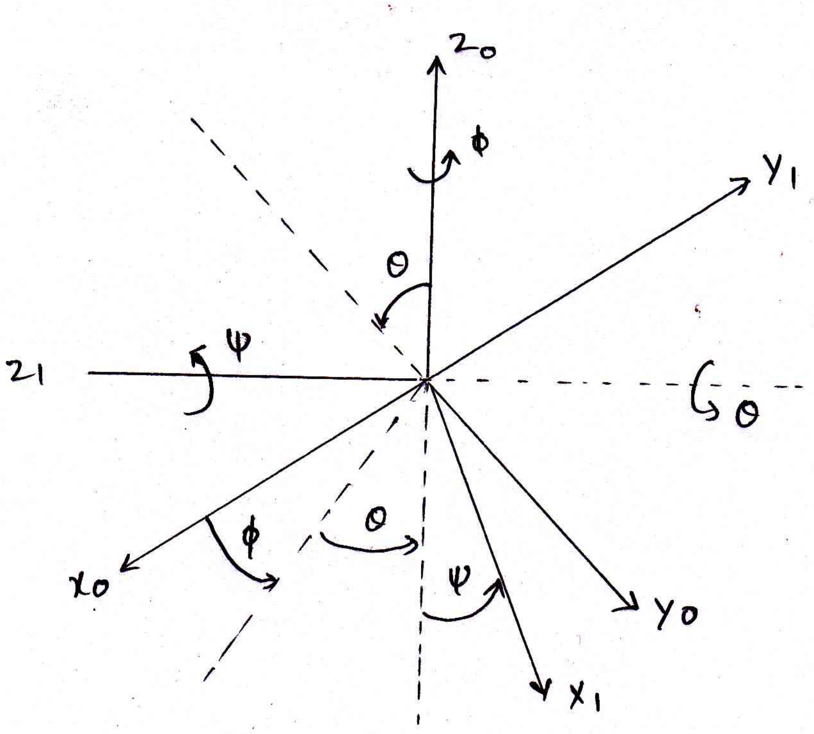
The transformed vector, $v = T \cdot u$

$$v = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0.866 & -0.5 & -5.196 \\ 0 & 0.5 & 0.866 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ -5 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -11.526 \\ -2.036 \\ 1 \end{bmatrix}$$

Euler angles :-

A body possesses 3-rotational DOF. Three independent quantities are required to represent "Euler Angles".

* A method to specify a rotation matrix in terms of 3 independent quantities is known as "Euler angles".
* Figure shows the fixed co-ordinate frame x_0, y_0 and z_0 and the rotated frame $ox_0y_0z_0$ by angles (θ, ϕ, ψ) known as "Euler angles".



① obtain the rotation matrix coorespoding to the set of Euler angles with respect to fixed XZX axes.

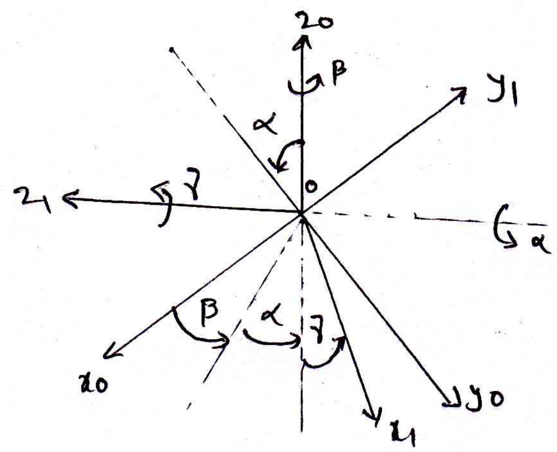
Ⓐ Rotation matrices must be multiplied together to represent a sequence of rotations about principle axes of the OXYZ coordinate system.

(i) Rotation of angle 'α' about x-axis i.e., $R(x,\alpha)$

(ii) Rotation of angle 'β' about z-axis i.e., $R(z,\beta)$

(iii) Rotation of angle 'γ' about x-axis i.e., $R(x,\gamma)$

$$R(x,\alpha) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{bmatrix}$$



$$R(z,\beta) = \begin{bmatrix} \cos\beta & -\sin\beta & 0 \\ \sin\beta & \cos\beta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R(x,\gamma) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\gamma & -\sin\gamma \\ 0 & \sin\gamma & \cos\gamma \end{bmatrix}$$

Now, the resultant rotation matrix can be obtained by,

$$R = R(x,\alpha)\, R(z,\beta)\, R(x,\gamma)$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} \cos\beta & -\sin\beta & 0 \\ \sin\beta & \cos\beta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\gamma & -\sin\gamma \\ 0 & \sin\gamma & \cos\gamma \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} \cos\beta & -\sin\beta\cos\gamma & \sin\beta\sin\gamma \\ \sin\beta & \cos\beta\cos\gamma & -\cos\beta\sin\gamma \\ 0 & \sin\gamma & \cos\gamma \end{bmatrix}$$

$$R = \begin{bmatrix} \cos\beta & -\sin\beta\cos\gamma & \sin\beta\sin\gamma \\ \cos\alpha\sin\beta & \cos\alpha\cos\beta - \sin\alpha\sin\gamma & -\cos\alpha\cos\beta\sin\gamma - \sin\alpha\cos\gamma \\ \sin\alpha\sin\beta & \sin\alpha\cos\beta\cos\gamma + \cos\alpha\sin\gamma & -\sin\alpha\cos\beta\sin\gamma + \cos\alpha\cos\gamma \end{bmatrix}$$

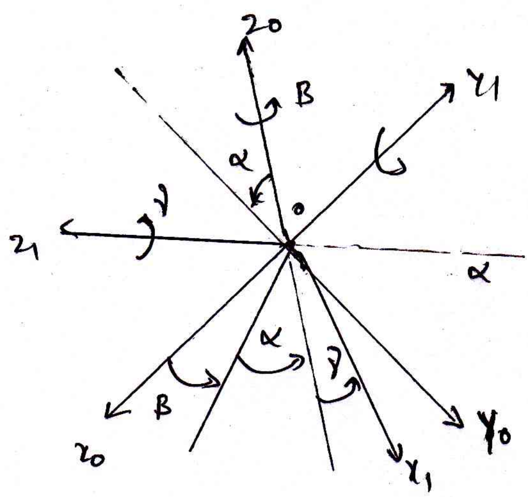
② obtain the rotation matrix corresponding to the set of Euler angles with respect to the fixed zyx axes.

Ⓐ (i) Rotation of angle 'α' about z-axis i.e., $R(z,\alpha)$

(ii) Rotation of angle 'β' about y-axis i.e., $R(y,\beta)$

(iii) Rotation of angle 'γ' about x-axis i.e., $R(x,\gamma)$

$$R(z,\alpha) = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



$$R(y,\beta) = \begin{bmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{bmatrix}$$

$$R(x,\gamma) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\gamma & -\sin\gamma \\ 0 & \sin\gamma & \cos\gamma \end{bmatrix}$$

Now, the resultant rotation matrix can be obtained by,

$$R = R(z,\alpha) \cdot R(y,\beta) \cdot R(x,\gamma)$$

$$= \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\gamma & -\sin\gamma \\ 0 & \sin\gamma & \cos\gamma \end{bmatrix}$$

$$= \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\beta & \sin\beta\sin\gamma & \sin\beta\cos\gamma \\ 0 & \cos\gamma & -\sin\gamma \\ -\sin\beta & \cos\beta\sin\gamma & \cos\beta\cos\gamma \end{bmatrix}$$

$$= \begin{bmatrix} \cos\alpha\cos\beta & \cos\alpha\sin\beta\sin\gamma - \sin\alpha\cos\gamma & \cos\alpha\sin\beta\cos\gamma + \sin\alpha\sin\gamma \\ \sin\alpha\cos\beta & \sin\alpha\sin\beta\sin\gamma + \cos\alpha\cos\gamma & \sin\alpha\sin\beta\cos\gamma - \cos\alpha\sin\gamma \\ -\sin\beta & \cos\beta\sin\gamma & \cos\beta\cos\gamma \end{bmatrix}$$

Different problem :-

① A Frame B is initially coincident with coordinate frame A. The frame B is then rotated about the vector $0.707i + 0.707j$, defined in frame A and passing through a point (1,2,3) through an angle 60°. Give description of the frame B' with respect to A.

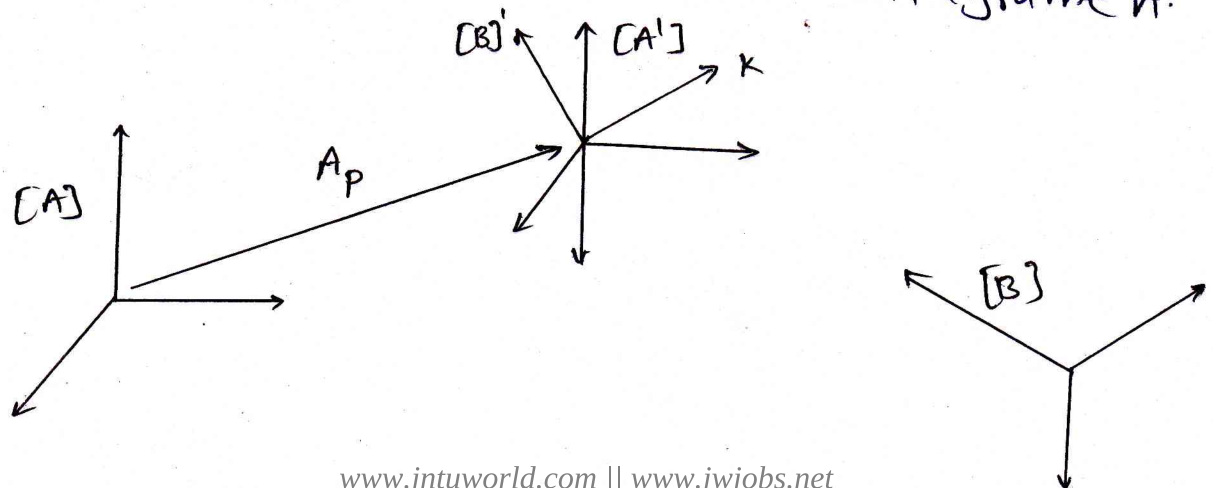
(A). Given vector $= 0.707i + 0.707j + 0k$.

∴ The vector matrix for the given vector is

$$\hat{k} = [0.707\ 0.707\ 0]^T$$

Point $P = [1,2,3]$, and $\theta = 60°$.

Case (i) :: Frame B' is initially coincident with frame A.



Considering two frames [A'], [B'] as shown in fig. Before the rotation of [B] about the vector, [A'] and [B'] are coincident with same orientation (i.e., [A] and [B]) to each other.

Description of [A] in terms of [A']

$$\therefore {}^{A}_{A'}[T] = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \text{ and } {}^{B'}_{B}[T] = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Case (ii):- Rotation of B about a vector through an Angle 60°.

Now, considering the equivalent rotation of matrix

$${}^{A'}_{B'}[T] = \begin{bmatrix} K_x^2 V\theta + \cos\theta & K_x K_y V\theta - K_z \sin\theta & K_x K_z V\theta + K_y \sin\theta \\ K_x K_y V\theta + K_z \sin\theta & K_y^2 V\theta + \cos\theta & K_y K_z V\theta - K_x \sin\theta \\ K_x K_z V\theta - K_y \sin\theta & K_y K_z V\theta + K_x \sin\theta & K_z^2 V\theta + \cos\theta \end{bmatrix}$$

where $V\theta = (1-\cos\theta)$, $K_x = 0.707$, $K_y = 0.707$, $K_z = 0$, we get

$$\therefore {}^{A'}_{B'}[T] = \begin{bmatrix} 0.7499 & 0.2499 & 0.6122 & 0 \\ 0.2499 & 0.7499 & -0.6122 & 0 \\ -0.6122 & 0.6122 & 0.5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Desired frame is obtained by writing a transformed equation i.e.,

$${}^{A}_{B}[T] = {}^{A}_{A'}[T] \cdot {}^{A'}_{B'}[T] \; {}^{B'}_{B}[T]$$

$${}^{A}_{B}[T] = \begin{bmatrix} 0.750 & 0.250 & 0.612 & -2.086 \\ 0.250 & 0.750 & -0.612 & 2.086 \\ -0.612 & 0.612 & 0.5 & 0.888 \\ 0 & 0 & 0 & 1 \end{bmatrix}_{4\times4}.$$