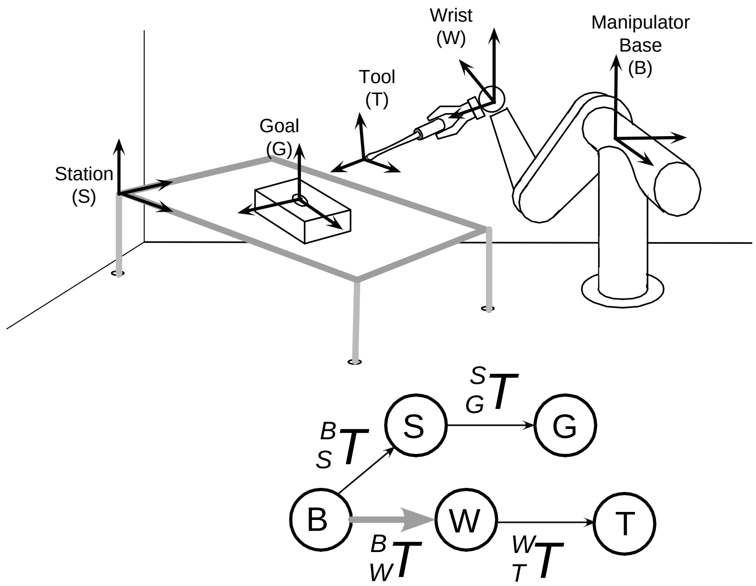
1 Manipulator Kinematics

1.1 Standard Frames



There are several important frames relevant to robot kinematics:

{B}	*Base Frame*
{S}	*Station Frame* (Specified relative to the base frame, $^B_S T$)
{W}	*Wrist Frame* (Specified relative to the base frame, $^B_W T$)
{T}	*Tool Frame* (Specified relative to the wrist frame, $^W_T T$)
{G}	*Goal Frame* (Specified relative to the station frame, $^S_G T$)

While the position and orientation of the wrist frame depends on the manipulator configuration, ${}^{B}_{W}T = T(\mathbf{q})$, the other frames are usually constant. There are two fundamental problems associated with this representation:

1. *Where is the tool?*
   Given is manipulator configuration $\mathbf{q}$,
   find the pose of the tool frame,
   with respect to the goal frame ${}^{G}_{T}T$.

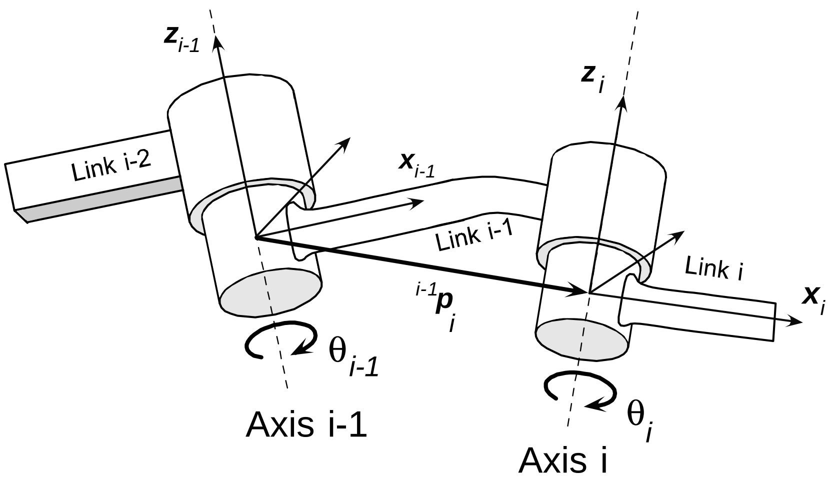
2. *Where should the joints go?*
   Given is desired pose of the tool frame
   with respect to the goal frame ${}^{G}_{T}T$, find
   the corresponding manipulator configuration $\mathbf{q}$.

The first problem can be reduced to the problem of finding the variable part of the environment, ${}^{B}_{W}T(\mathbf{q})$ for a given $\mathbf{q}$. The rest of the problem can then be solved easily form the local definition of other frames: ${}^{G}_{T}T = {}^{S}_{G}T^{-1}\ {}^{B}_{S}T^{-1}\ {}^{B}_{W}T(\mathbf{q})\ {}^{W}_{T}T$. Computing the matrix ${}^{B}_{W}T(\mathbf{q})$ is called: *forward (direct kinematics)*.

The second problem can be reduced to the inversion of the matrix function ${}^{B}_{W}T(\mathbf{q}) = T$, where T is the desired pose of the wrist with respect to the manipulator base, which can be expressed in terms of other known local definitions: $T = {}^{B}_{S}T\ {}^{S}_{G}T\ {}^{G}_{T}T\ {}^{W}_{T}T^{-1}$. Inversion of the matrix function ${}^{B}_{W}T(\mathbf{q})$ is called *inverse kinematics*.

1.2 Assignment of Frames to Links

As a first step in solving the problems above is to assign a frame to each manipulator link. A straightforward way to do this is to place the frames to some "geometrically meaningful place" on the link. For example, in the figure below the z-axes of the frames are aligned with the joint axes thus putting in a direct correspondence the frame orientation with the joint angle θ_i. The relative pose of the frame $\{i\}$ with respect to the frame $\{i-1\}$ is thus uniquely defined by six independent parameters, one of which is θ_i.

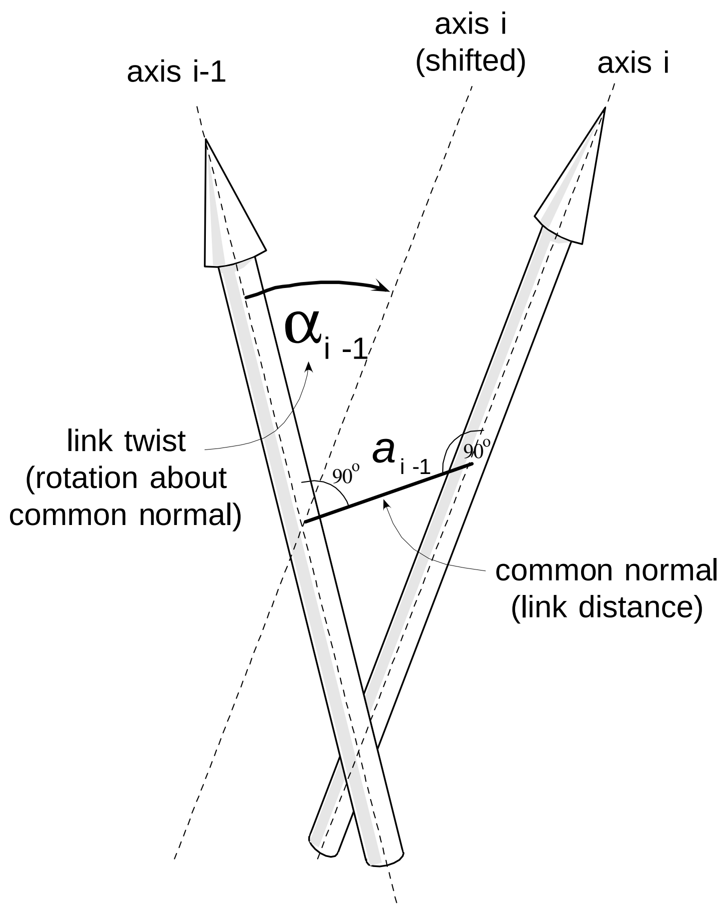


Can we define the relative poses of frames by fewer parameters? An approach which requires only four parameters is proposed by Denavit and Hartenberg[1]. Such reduction of required parameters is possible if we freely choose the location of the frame's origins, rejecting the "geometrically meaningful place" on the link.

[1] J.Denavit and R.S. Hartenberg: "A Kinematic Notation for Lower-Pair Mechanisms Based on Matrices," Journal of Applied Mechanics, pp. 215-221, June 1955.

1.3 Denavit-Hartenberg Parametrization

Two fixed axes:

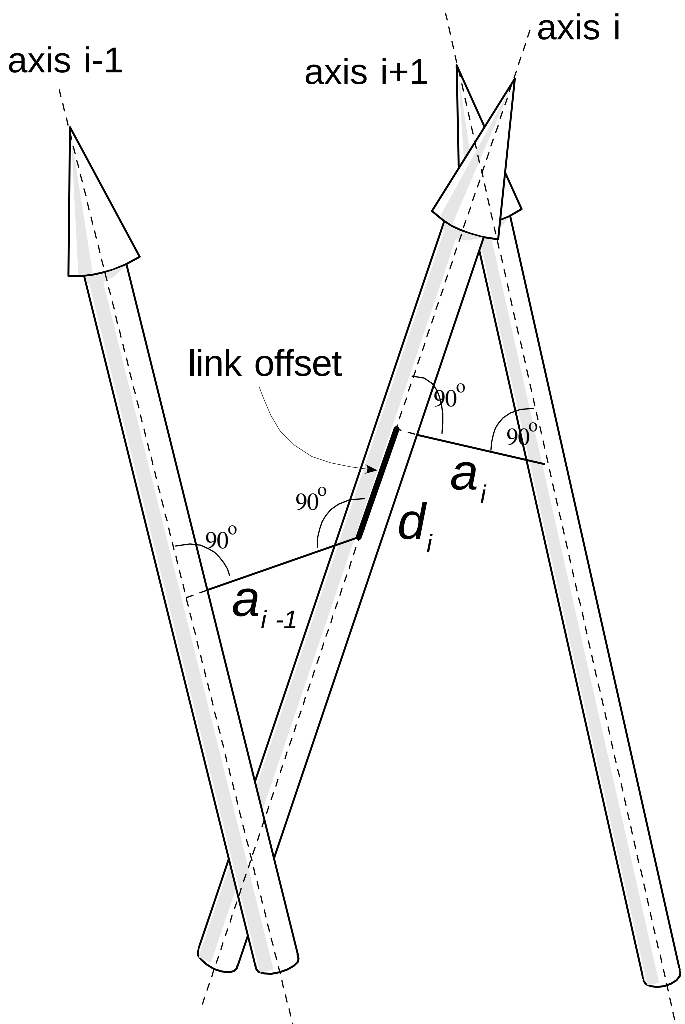


In order to define a relative position and orientation of two fixed axes (axes which don't move) only two parameters are required: *link distance* (also called *common normal*) and the *link twist*.

Link distance a_{i-1} is defined as distance between two axes, $i-1$, and i, along the line which is perpendicular to both axes (if axes are intersecting, then $a_{i-1} = 0$). Link twist α_{i-1} is defined as the angle between the two axes, $i-1$, and i, about the common normal (if axes are parallel, then $\alpha_{i-1} = 0$).

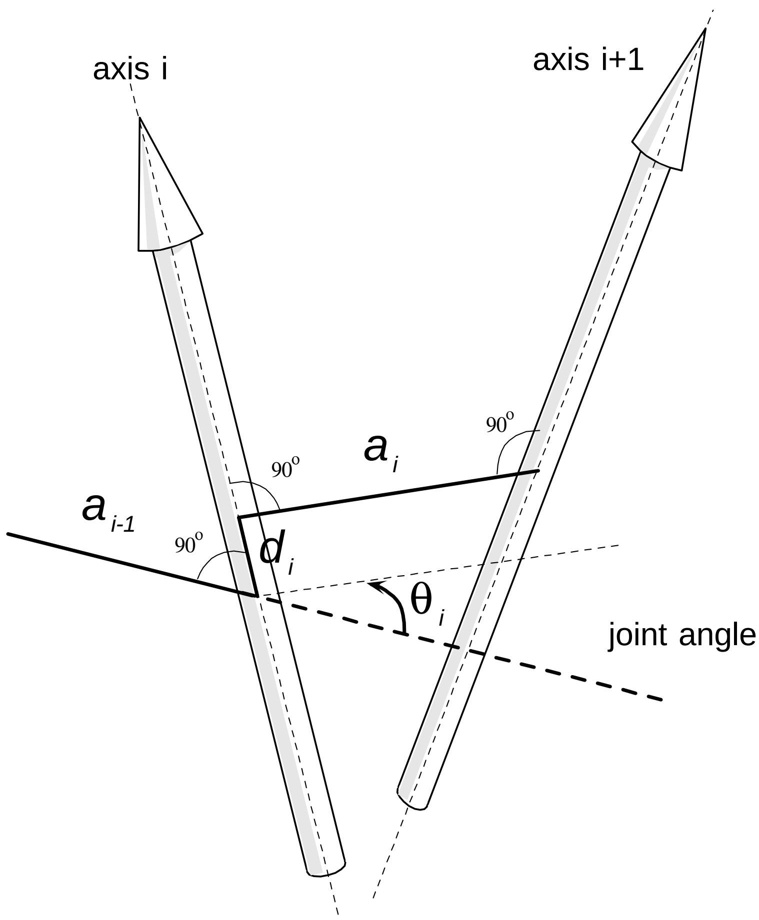
Three fixed axes:

If there are more than two fixed axes the neighboring common normals in general case will not intersect the common axes at the same point. Therefore a new parameter is needed: *link offset*. Link offset d_i is distance between the common normals a_{i-1} and a_i along the axis i. For prismatic joints the d_i are variable parameters, and for revolute joints the d_i are constant parameters.



Moving axes:

If the axes are not fixed (i.e. they rotate), then one more parameter is needed: *the joint angle*. The joint angle θ_i is the angle between the common normals a_{i-1} and a_i about axis i. For revolute joints the θ_i are variable parameters, and for prismatic joints the θ_i are constant parameters.



1.4 Convention for Frame Assignments

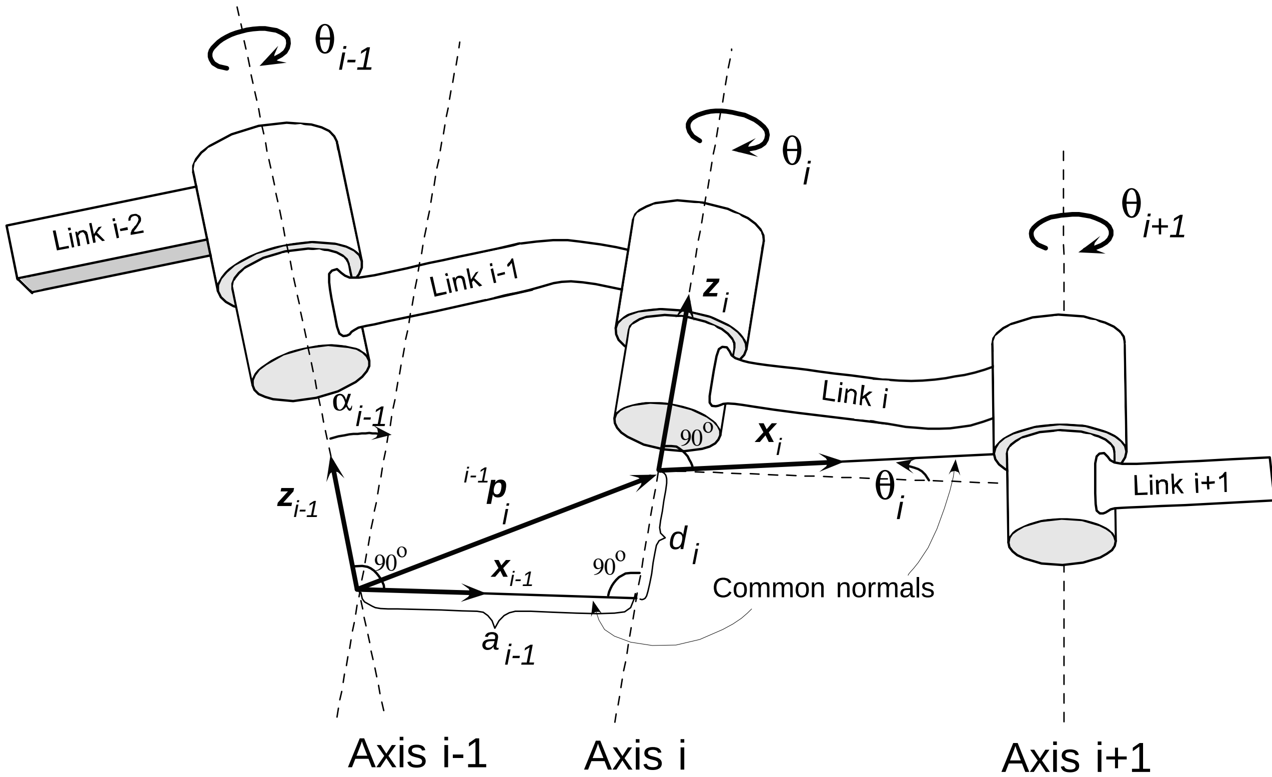
Following the Denavit-Hartenberg parametrization we use the following rules for the frame assignment to links:

	Intermediate links:
(1)	To each link i is assigned a frame $\{i\}$.
(2)	z-axis of the frame $\{i\}$, $\mathbf{z}_i$ is coincident with the joint axis i.
(3)	The origin of the frame $\{i\}$ is located at the intersection of the joint axis i with the common normal a_i, or at the intersection of joint axes i and $i+1$ if $a_i = 0$.
(4)	The x-axis of the frame $\{i\}$, $\mathbf{x}_i$, is placed along the common normal a_i, pointing to the joint $i+1$. If $a_i = 0$, the axis $\mathbf{x}_i$ is perpendicular to both $\mathbf{z}_i$ and $\mathbf{z}_{i+1}$ (direction is arbitrary)
	First link:
(5)	Frame $\{0\}$ is coincident with the frame $\{1\}$. (Consequently: $a_0 = 0, \alpha_0 = 0, \theta_1 = 0$ -prismatic, $d_1 = 0$ -revolute.)
	Last link:
(6)	Axis $\mathbf{x}_n$ of the frame $\{n\}$ is colinear with $\mathbf{x}_{n-1}$ at $\theta_n = 0$. The origin of $\{n\}$ is at the intersect. of $\mathbf{x}_{n-1}$ with joint axis n. Consequently $d_n = 0$ (home position for prismatic joints)

Remark 1 *Older robotic literature uses original convention in which the z-axis of the frame $\{i\}$ is coincident with the joint axis $i+1$ Here we accept more modern approach proposed by J. Craig*[2]

[2] J. Craig: "Introduction to Robotics," Addison-Wesley, 1986.

1.5 Relative Link Pose



We can now summarize the DH parameters in terms of coordinate axes:

a_{i-1}	= distance between	$\mathbf{z}_{i-1}$ and $\mathbf{z}_i$	along $\mathbf{x}_{i-1}$
α_{i-1}	= angle between	$\mathbf{z}_{i-1}$ and $\mathbf{z}_i$	about $\mathbf{x}_{i-1}$
d_i	= distance between	$\mathbf{x}_{i-1}$ and $\mathbf{x}_i$	along $\mathbf{z}_i$
θ_i	= angle between	$\mathbf{x}_{i-1}$ and $\mathbf{x}_i$	about $\mathbf{z}_i$

From the definitions above follows the local definition of the frame $\{i\}$ with respect to the frame $\{i-1\}$:

1.	$\{i\}$ is initially coincident with $\{i-1\}$	
2.	Rotate about $\{i\}$ $\mathbf{x}_{i-1}$ by α_{i-1}	$\rightarrow Rot(\mathbf{e}_1, \alpha_{i-1})$
3.	Translate new $\{i\}$ along $\mathbf{x}_{i-1}$ by a_{i-1}	$\rightarrow Trans(\mathbf{e}_1, a_{i-1})$
4.	Rotate new $\{i\}$ about $\mathbf{z}_i$ by θ_i	$\rightarrow Rot(\mathbf{e}_3, \theta_i)$
5.	Translate new $\{i\}$ along $\mathbf{z}_i$ by d_i	$\rightarrow Trans(\mathbf{e}_3, d_i)$

This results in the following HT matrix:

$$ {}^{i-1}_{\ \ i}T = Rot(\mathbf{e}_1, \alpha_{i-1})\, Trans(\mathbf{e}_1, a_{i-1})\, Rot(\mathbf{e}_3, \theta_i)\, Trans(\mathbf{e}_3, d_i) $$

or

$$ \boxed{{}^{i-1}_{\ \ i}T = Screw(\mathbf{e}_1, \alpha_{i-1}, a_{i-1})\, Screw(\mathbf{e}_3, \theta_i, d_i)} \tag{1} $$

Expanded form:

$$ {}^{i-1}_{\ \ i}T = \begin{bmatrix} 1 & 0 & 0 & a_{i-1} \\ 0 & c_{\alpha_{i-1}} & -s_{\alpha_{i-1}} & 0 \\ 0 & s_{\alpha_{i-1}} & c_{\alpha_{i-1}} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_i & -s_i & 0 & 0 \\ s_i & ci & 0 & 0 \\ 0 & 0 & 1 & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ \boxed{{}^{i-1}_{\ \ i}T = \begin{bmatrix} c_i & -s_i & 0 & a_{i-1} \\ c_{\alpha_{i-1}}\, s_i & c_{\alpha_{i-1}}\, c_i & -s_{\alpha_{i-1}} & -s_{\alpha_{i-1}}\, d_i \\ s_{\alpha_{i-1}}\, s_i & s_{\alpha_{i-1}}\, c_i & c_{\alpha_{i-1}} & c_{\alpha_{i-1}}\, d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}} \tag{2} $$

1.6 DH Table

In order to describe kinematically the entire manipulator, all DH parameters can be put in a compact table:

i	α_{i-1}	a_{i-1}	θ_i	d_i	σ_i
1	α_0	a_0	θ_1	d_1	σ_1
2	α_1	a_1	θ_2	d_2	σ_2
...	...	...	...	...	...
i	α_{i-1}	a_{i-1}	θ_i	d_i	σ_i
...	...	...	...	...	...
n	α_{n-1}	a_{n-1}	θ_n	d_n	σ_n

The parameter σ_i gives information about the joint type ($\sigma_i = 0$ for revolute joint, $\sigma_i = 1$ for prismatic joint). Usually some parameters are predefined by the convention: $\alpha_0 = a_0 = 0$, $\theta_1 = 0$, $\theta_n = 0$ (for prismatic joints), or $d_1 = d_n = 0$ (for revolute joints)

Note that the screws of the relative HT matrix for the i-th link (1)

$$\boxed{{}^{i-1}_{\ i}T = Screw(\mathbf{e}_1, \alpha_{i-1}, a_{i-1})\, Screw(\mathbf{e}_3, \theta_i, d_i)}$$

directly map the i-th row of the DH table.

Sometimes it is convenient to consider the DH table as a constant quantity which does not change with the manipulator configuration. In other words, the values for θ_i and d_i are constant offsets given by the manufacturer for the *initial configuration* (*home configuration*) of the manipulator. The relative HT matrices can then be expressed in the following way for any configuration given by the joint vector $\mathbf{q} = (q_1, q_2, ..., q_n)$:

$$\boxed{{}^{i-1}_{\ i}T = \begin{cases} Screw(\mathbf{e}_1, \alpha_{i-1}, a_{i-1})\, Screw(\mathbf{e}_3, \theta_i + q_i, d_i) & \text{if } \sigma_i = 0 \\ Screw(\mathbf{e}_1, \alpha_{i-1}, a_{i-1})\, Screw(\mathbf{e}_3, \theta_i, d_i + q_i) & \text{if } \sigma_i = 1 \end{cases}} \quad (3)$$

1.7 Forward Kinematics

1.7.1 Recurrent Formula

The HT matrix of the manipulator's wrist with respect to the base can be expressed now as the product:

$$ {}^{B}_{W}T = {}^{0}_{n}T = \prod_{i=1}^{n} {}^{i-1}_{i}T \tag{4} $$

where ${}^{i-1}_{i}T$ are known matrices computed by (3).

More convenient form of (4) is its *recurrent* version:

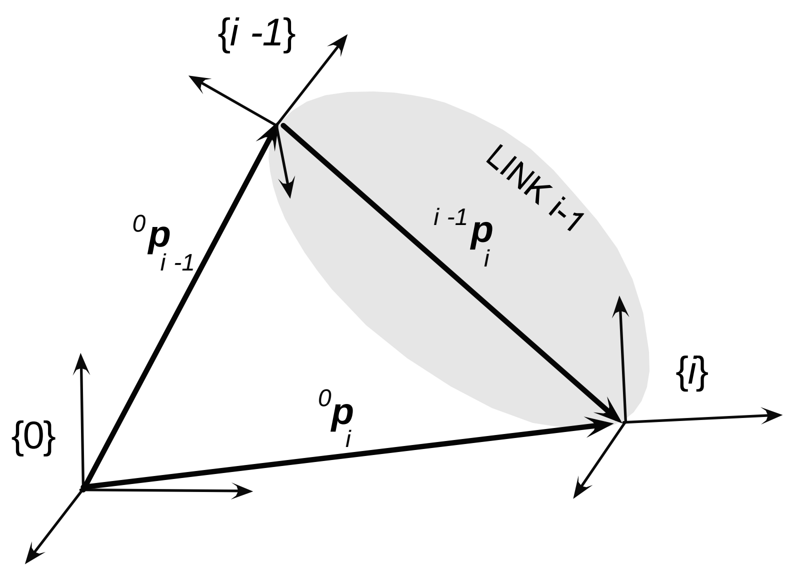
$$ \begin{aligned} {}^{0}_{0}T &= I_4 \\ {}^{0}_{i}T &= {}^{0}_{i-1}T \; {}^{i-1}_{i}T \quad i = 1, 2, ..., n \end{aligned} \tag{5} $$

This form is suitable for computer programming.

The equation (5) can also be expressed in a de-blocked (non-homogeneous) form:

$$\boxed{\left.\begin{array}{l} {}^0_iR = {}^{\ 0}_{i-1}R \ {}^{i-1}_{\ \ i}R, \\[2ex] {}^0\mathbf{p}_i = {}^0\mathbf{p}_{i-1} + {}^{\ 0}_{i-1}R \ {}^{i-1}\mathbf{p}_i \end{array}\right\} \quad i = 1, 2, ..., n} \tag{6}$$

which has an obvious geometrical interpretation shown in the following figure:



The iterative process (6) can be used for two purposes: to numerically evaluate the matrix ${}^0_nT$, or to derive symbolically the closed-form solution for ${}^0_nT$ (by using packages like MATHEMATICA, MAPLE, MACSYMA and the like).

1.7.2 Recursive Procedure

There is an even more suitable form of the forward kinematics equations for computer programming if the implementation language supports recursive procedures (like C/C^{++}, Pascal, Matlab etc.)

For example, using the matlab-style the recursive version would look like this:

```
%          FKIN()   – Compute forward kinematics
%          DH       – Denavit-Hartenberg table
%          q        – Manipulator configuration
%          i        – Joint index

function   T = fkin(DH, q, i)
           T = ^{i-1}_{i}T  (equation (3))
           if  i > 1
               T = fkin(DH, q, i-1)*T
           end
```

Apparently, the recursion hides the iteration construct which is explicitly visible in recurrence (6).

1.7.3 Efficient Recurrent Formula

Let us rewrite the expression for the relative orientation matrices from (2):

$$ {}^{i-1}_{i}R = \begin{bmatrix} c_i & -s_i & 0 \\ c_{\alpha_{i-1}}\, s_i & c_{\alpha_{i-1}}\, c_i & -s_{\alpha_{i-1}} \\ s_{\alpha_{i-1}}\, s_i & s_{\alpha_{i-1}}\, c_i & c_{\alpha_{i-1}} \end{bmatrix} $$

In most of the real manipulator designs, the joint axes are either parallel ($\alpha_{i-1} = 0^0, 180^0$), or they are perpendicular ($\alpha_{i-1} = \pm 90^0$). This considerably simplifies the matrices ${}^{i-1}_{i}R$, which then become:

$$ {}^{i-1}_{i}R = \begin{bmatrix} c_i & -s_i & 0 \\ \pm\, s_i & \pm\, c_i & 0 \\ 0 & 0 & \pm 1 \end{bmatrix} \qquad \text{for} \quad \alpha_{i-1} = 0^0/180^0 $$

$$ {}^{i-1}_{i}R = \begin{bmatrix} c_i & -s_i & 0 \\ 0 & 0 & \mp 1 \\ \pm\, s_i & \pm\, c_i & 0 \end{bmatrix} \qquad \text{for} \quad \alpha_{i-1} = \pm 90^0 $$

The recurrent form (6) would then not be very efficient since there are unnecessary multiplications and additions due to the presence of zeroes and ones in the matrices ${}^{i-1}_{i}R$. In order to eliminate the excessive floating point operations, we need further to decompose the relative orientation matrices. We will now develop a new set of recurrent formulas from (6) which are valid in the general case, and then derive simpler formulas for parallel and perpendicular axes.

As shown in subsection **??** we can write: ${}^{0}_{i}R = [{}^{0}\mathbf{x}_i \ {}^{0}\mathbf{y}_i \ {}^{0}\mathbf{z}_i]$, or if we drop the leading superscripts for simplicity: ${}^{0}_{i}R = [\mathbf{x}_i \ \mathbf{y}_i \ \mathbf{z}_i]$, then (6) can be rewritten as:

$$
\begin{aligned}
[\mathbf{x}_i \ \mathbf{y}_i \ \mathbf{z}_i] &= [\mathbf{x}_{i-1} \ \mathbf{y}_{i-1} \ \mathbf{z}_{i-1}]\, {}^{i-1}_{i}R \\
&= [\mathbf{x}_{i-1} \ \mathbf{y}_{i-1} \ \mathbf{z}_{i-1}]\, rot(\mathbf{e}_1, \alpha_{i-1})\, rot(\mathbf{e}_3, \theta_i) \\
&= [\mathbf{x}_{i-1} \ \mathbf{y}_{i-1} \ \mathbf{z}_{i-1}] \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_{\alpha_{i-1}} & -s_{\alpha_{i-1}} \\ 0 & s_{\alpha_{i-1}} & c_{\alpha_{i-1}} \end{bmatrix} rot(\mathbf{e}_3, \theta_i) \\
&= [\mathbf{x}_{i-1} \,|\, \mathbf{y}_{i-1}\, c_{\alpha_{i-1}} + \mathbf{z}_{i-1}\, s_{\alpha_{i-1}} \,|\, -\mathbf{y}_{i-1}\, s_{\alpha_{i-1}} + \mathbf{z}_{i-1}\, c_{\alpha_{i-1}}]\, rot(\mathbf{e}_3, \theta_i) \\
&= [\mathbf{x}_{i-1} \ \mathbf{v}_{i-1} \ \mathbf{w}_{i-1}] \begin{bmatrix} c_i & -s_i & 0 \\ s_i & c_i & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
&= [\mathbf{x}_{i-1}\, c_i + \mathbf{v}_{i-1}\, s_i \,|\, -\mathbf{x}_{i-1}\, s_i + \mathbf{v}_{i-1}\, c_i \,|\, \mathbf{w}_{i-1}]
\end{aligned} \tag{7}
$$

where $\mathbf{x}_0 = \mathbf{e}_1,\ \mathbf{y}_0 = \mathbf{e}_2$ and $\mathbf{z}_0 = \mathbf{e}_3$.

If the frame $\{1\}$ is coincident with the base frame $\{B\} = \{0\}$, i.e. $a_0 = \alpha_0 = 0$, then the first iteration of (7) gives:

$$\mathbf{x}_1 = \begin{bmatrix} c_1 \\ s_1 \\ 0 \end{bmatrix}, \qquad \mathbf{y}_1 = \begin{bmatrix} -s_1 \\ c_1 \\ 0 \end{bmatrix}, \qquad \mathbf{z}_1 = \mathbf{e}_3. \tag{8}$$

Therefore we can reduce number of iterations in (7) and start from $i = 2$.

Furthermore it is easy to see that:

$$\begin{aligned} {}^{0}_{i-1}R\,{}^{i-1}\mathbf{p}_i &= {}^{0}_{i-1}R\,({}^{i-1}\mathbf{x}_{i-1}a_{i-1} + d_i\,{}^{i-1}\mathbf{z}_i) \\ &= {}^{0}\mathbf{x}_{i-1}\,a_{i-1} + d_i\,{}^{0}\mathbf{z}_i \\ &= \mathbf{x}_{i-1}\,a_{i-1} + d_i\,\mathbf{z}_i \end{aligned} \tag{9}$$

Combining now (7) and (9) the original recurrent equation (6) becomes:

$$\boxed{\begin{aligned} \mathbf{v}_{i-1} &= \mathbf{y}_{i-1}\,c_{\alpha_{i-1}} + \mathbf{z}_{i-1}\,s_{\alpha_{i-1}} \\ \mathbf{x}_i &= \mathbf{x}_{i-1}\,c_i + \mathbf{v}_{i-1}\,s_i \\ \mathbf{y}_i &= -\mathbf{x}_{i-1}\,s_i + \mathbf{v}_{i-1}\,c_i \qquad\qquad i = 2, 3, ..., n \\ \mathbf{z}_i &= -\mathbf{y}_{i-1}\,s_{\alpha_{i-1}} + \mathbf{z}_{i-1}\,c_{\alpha_{i-1}} \\ \mathbf{p}_i &= \mathbf{p}_{i-1} + a_{i-1}\,\mathbf{x}_{i-1} + d_i\,\mathbf{z}_i \end{aligned}} \tag{10}$$

This is a generic formula which is valid for any serial manipulator. Note that all vectors are with reference to the base frame.

For manipulators with parallel and/or perpendicular joint axes these formulas can be further simplified:

$$\boxed{\begin{array}{lcl} \mathbf{x}_i = \mathbf{x}_{i-1}\, c_i \pm \mathbf{y}_{i-1}\, s_i & & \text{for parallel axes} \\ \mathbf{y}_i = -\mathbf{x}_{i-1}\, s_i \pm \mathbf{y}_{i-1}\, c_i & & (\alpha_{i-1} = 0^0/180^0) \\ \mathbf{z}_i = \pm\mathbf{z}_{i-1} & & \\ & i = 3, 4, ..., n & \\ \mathbf{x}_i = \mathbf{x}_{i-1}\, c_i \pm \mathbf{z}_{i-1}\, s_i & & \text{for perpendicular axes} \\ \mathbf{y}_i = -\mathbf{x}_{i-1}\, s_i \pm \mathbf{z}_{i-1}\, c_i & & (\alpha_{i-1} = \pm 90^0) \\ \mathbf{z}_i = \mp\mathbf{y}_{i-1} & & \end{array}} \tag{11}$$

Note that the iterations here start at $i = 3$, since in the second iteration there are more savings possible. The closed-form values for the second iteration are as follows:

$$\textit{First two axes parallel } (\alpha_{i-1} = 0^0/180^0)\textit{:}$$

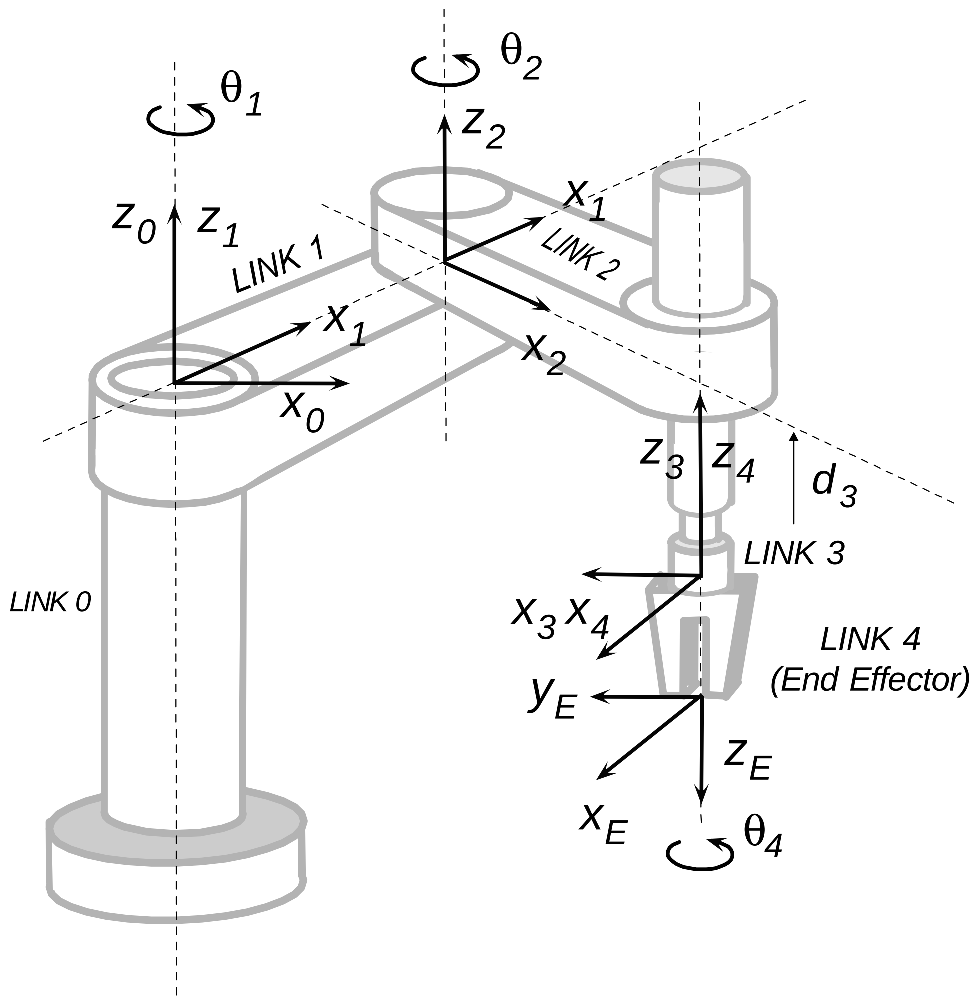
$$\mathbf{x}_2 = \begin{bmatrix} c_{12} \\ s_{12} \\ 0 \end{bmatrix}, \qquad \mathbf{y}_2 = \pm \begin{bmatrix} -s_{12} \\ c_{12} \\ 0 \end{bmatrix}, \qquad \mathbf{z}_2 = \pm\mathbf{e}_3 \tag{12}$$

where: $c_{12} = \cos(\theta_1 \pm \theta_2),\ s_{12} = \sin(\theta_1 \pm \theta_2)$.

$$\textit{First two axes perpendicular } (\alpha_{i-1} = \pm 90^0)\textit{:}$$

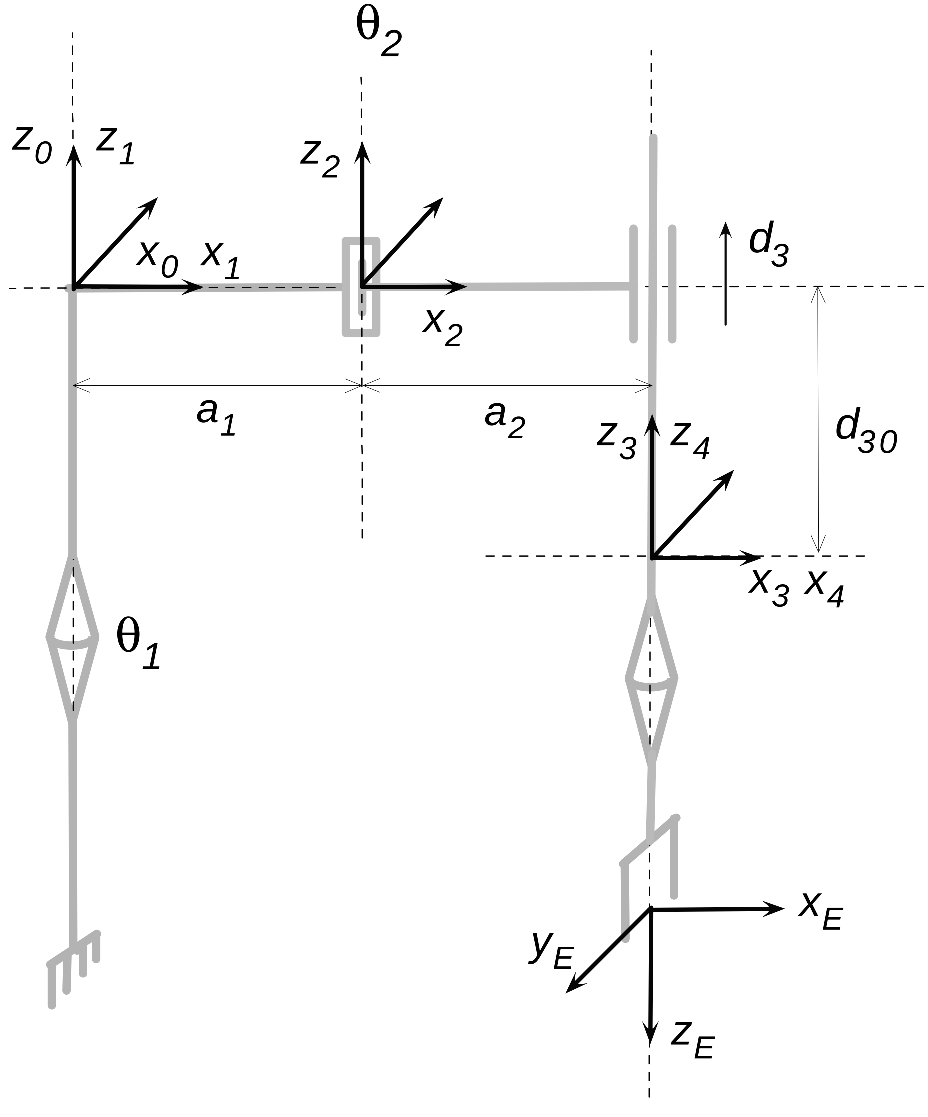
$$\mathbf{x}_2 = \begin{bmatrix} c_1 c_2 \\ s_1 c_2 \\ \pm s_2 \end{bmatrix}, \qquad \mathbf{y}_2 = \begin{bmatrix} -c_1 s_2 \\ -s_1 s_2 \\ \pm c_2 \end{bmatrix}, \qquad \mathbf{z}_2 = \mp\mathbf{y}_1 \tag{13}$$

1.7.4 Example 1: AdeptOne Robot



Here we consider a popular four DOF manipulator which belongs to the family of SCARA-type robots, called AdeptOne Robot. As seen, the robot is RRPR, which means that the first, second and the forth joint are revolute, while the third joint is prismatic. The configuration in the picture above is not the <u>*home configuration*</u> (i.e. the values of joint displacements are not all zero). We will assume that the origin of the frame {3} is off-set by a constant value, d_{30} along the z-axis.

The simplified kinematic diagram of the AdeptOne Robot is given below. The figure shows the home configuration:



The simplified kinematic diagram is convenient to determine the DH parameters by a straightforward application of the rules given in (1.4). The resulting DH table is as follows:

i	α_{i-1}	a_{i-1}	θ_i	d_i	σ_i
1	0	0	θ_1	0	0
2	0	a_1	θ_2	0	0
3	0	a_2	0	$d_3 - d_{30}$	1
4	0	0	θ_4	0	0

The relative HT matrices can be directly derived by using (1):

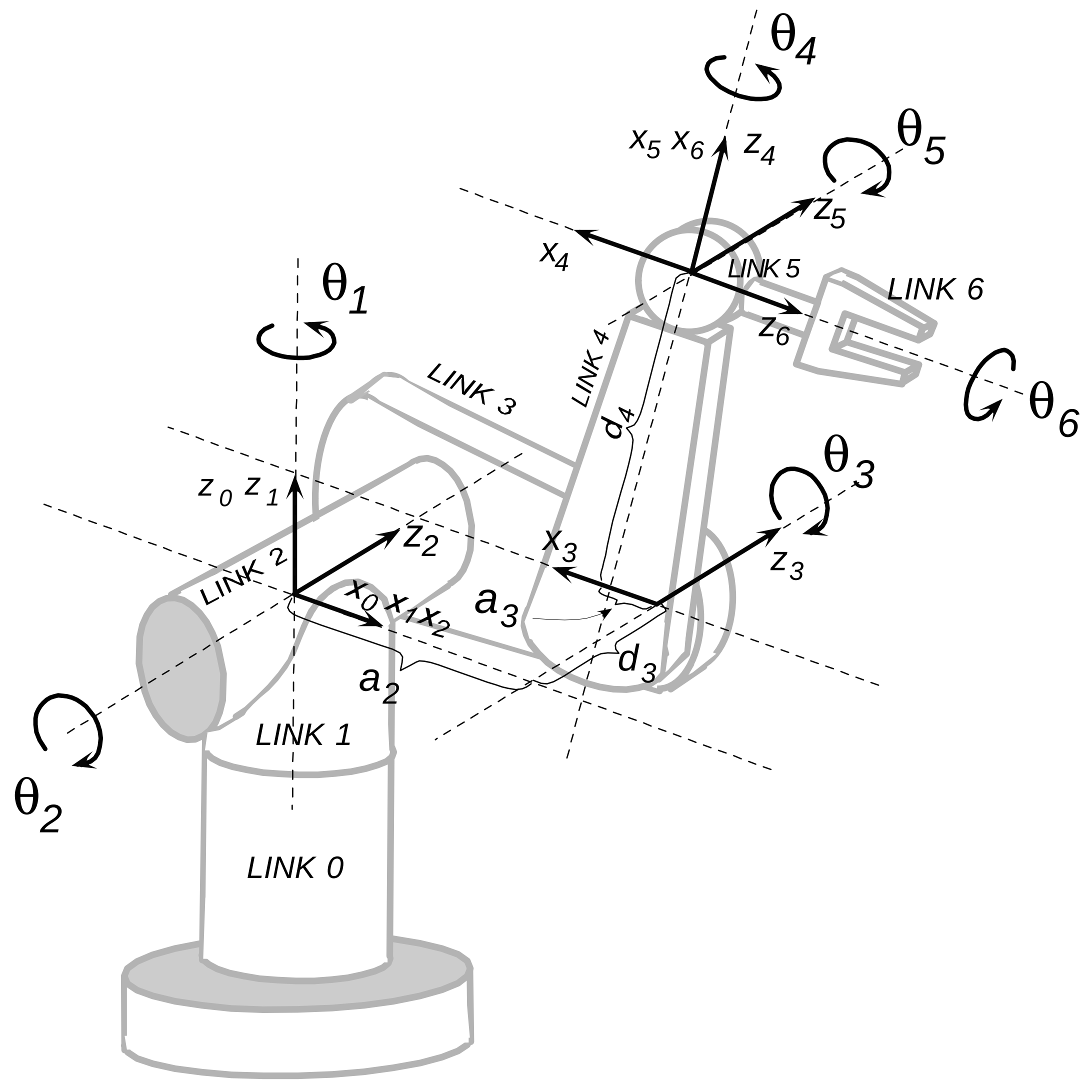
$$ {}^0_1T = \begin{bmatrix} c_1 & -s_1 & 0 & 0 \\ s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad {}^1_2T = \begin{bmatrix} c_2 & -s_2 & 0 & a_1 \\ s_2 & c_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^2_3T = \begin{bmatrix} 1 & 0 & 0 & a_2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_3 - d_{30} \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad {}^3_4T = \begin{bmatrix} c_4 & -s_4 & 0 & 0 \\ s_4 & c_4 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

The HT matrices for all frames, with respect to the base can be obtained by multiplying the relative HT matrices:

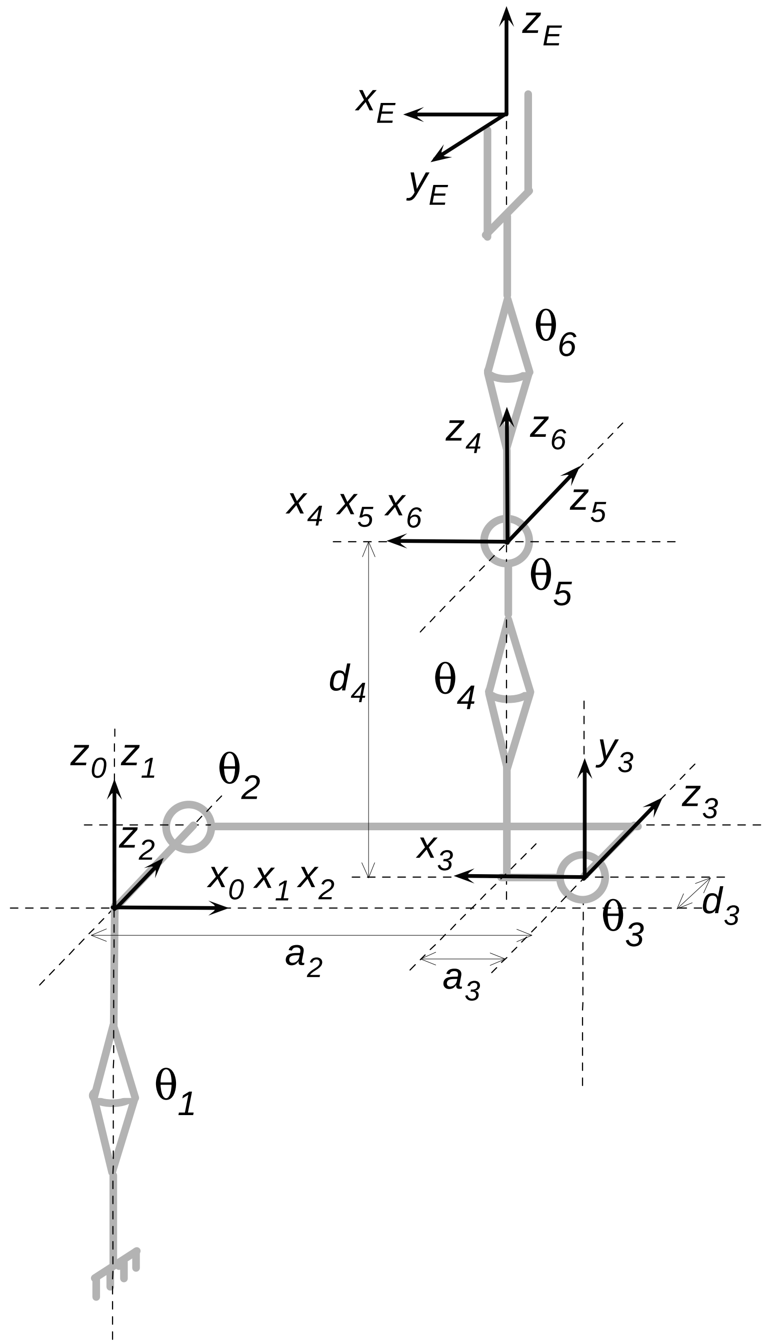
$$
\begin{aligned}
{}^0_1T &= \begin{bmatrix} c_1 & -s_1 & 0 & 0 \\ s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
{}^0_2T &= \begin{bmatrix} c_{12} & -s_{12} & 0 & c_1\, a_1 \\ s_{12} & c_{12} & 0 & s_1\, a_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
{}^0_3T &= \begin{bmatrix} c_{12} & -s_{12} & 0 & c_1\, a_1 + c_{12}\, a_2 \\ s_{12} & c_{12} & 0 & s_1\, a_1 + s_{12}\, a_2 \\ 0 & 0 & 1 & d_3 - d_{30} \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
{}^0_4T &= \begin{bmatrix} c_{124} & -s_{124} & 0 & c_1\, a_1 + c_{12}\, a_2 \\ s_{124} & c_{124} & 0 & s_1\, a_1 + s_{12}\, a_2 \\ 0 & 0 & 1 & d_3 - d_{30} \\ 0 & 0 & 0 & 1 \end{bmatrix}
\end{aligned}
\tag{14}
$$

1.7.5 Example 2: Puma 560 Manipulator



PUMA 560 manipulator is a popular 6 DOF, all revolute manipulator which is commonly used in research and academic studies. Note that the manipulator on this sketch is also not in the home configuration.

The simplified kinematic diagram of the Puma 560 manipulator in home configuration is given below:



z_E
x_E
y_E
θ6
z4
z6
x4 x5 x6
z5
θ5
d4
θ4
y3
z0 z1
θ2
z3
z2
x3
x0 x1 x2
d3
θ3
a2
a3
θ1

The DH table which corresponds to the frame assignments in the figures above:

i	α_{i-1}	a_{i-1}	θ_i	d_i	σ_i
1	0	0	θ_1	0	0
2	-90^0	0	θ_2	0	0
3	0	a_2	θ_3	d_3	0
4	-90^0	a_3	θ_4	d_4	0
5	90^0	0	θ_5	0	0
6	-90^0	0	θ_6	0	0

Nominal values for link distances and link offsets for PUMA 560 manipulator are:

$$\begin{array}{ll} a_2 = 0.43180\ m & (17\ in) \\ a_3 = 0.02032\ m & (0.8\ in) \\ d_3 = 0.12446\ m & (4.9\ in) \\ d_4 = 0.43180\ m & (17\ in) \end{array}$$

The joint limits for the PUMA 560 have the following values:

$$\begin{array}{rcr} -170^0 & \leq \theta_1 \leq & 170^0 \\ -225^0 & \leq \theta_2 \leq & 45^0 \\ -250^0 & \leq \theta_3 \leq & 75^0 \\ -135^0 & \leq \theta_4 \leq & 135^0 \\ -100^0 & \leq \theta_5 \leq & 100^0 \\ -180^0 & \leq \theta_6 \leq & 180^0 \end{array}$$

The relative HT matrices derived by using (1):

$$
{}^{0}_{1}T = \begin{bmatrix} c_1 & -s_1 & 0 & 0 \\ s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad {}^{1}_{2}T = \begin{bmatrix} c_2 & -s_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -s_2 & -c_2 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
{}^{2}_{3}T = \begin{bmatrix} c_3 & -s_3 & 0 & a_2 \\ s_3 & c_3 & 0 & 0 \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad {}^{3}_{4}T = \begin{bmatrix} c_4 & -s_4 & 0 & a_3 \\ 0 & 0 & 1 & d_4 \\ -s_4 & -c_4 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
{}^{4}_{5}T = \begin{bmatrix} c_5 & -s_5 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ s_5 & c_5 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad {}^{5}_{6}T = \begin{bmatrix} c_6 & -s_6 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -s_6 & -c_6 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

As will be discussed later, it is useful sometimes to consider arm and wrist separately if the manipulator has a *spherical wrist*. A spherical wrist has all joint axes intersecting in a single point. Thus, the arm part is defined as the part of the manipulator which contributes to the position of the wrist, while the spherical wrist only changes its orientation (wrist does not affect the position). In case of Puma 560, the arm part consists of links #0-#3 and a part of the link #4, which includes screw about x-axis and the translation part of the screw about z-axis, excluding rotation of the 4-th joint θ_4. Since the wrist does not have any length parameters ($a_i = d_i = 0$, $i = 4, 5, 6$), its relative HT matrices have only pure rotations.. Consequently the HT matrices of the arm and wrist of the Puma 560 manipulator are:

$$\begin{aligned} T_A &= {}^0_1T\ {}^1_2T\ {}^2_3T\ Screw(\mathbf{e}_1, \alpha_3, a_3)\ Trans(\mathbf{e}_3, d_4) \\ T_W &= Rot(\mathbf{e}_3, \theta_4)\ Rot(\mathbf{e}_1, \tfrac{\pi}{2})\ Rot(\mathbf{e}_3, \theta_5)\ Rot(\mathbf{e}_1, -\tfrac{\pi}{2})\ Rot(\mathbf{e}_3, \theta_6) \end{aligned} \tag{15}$$

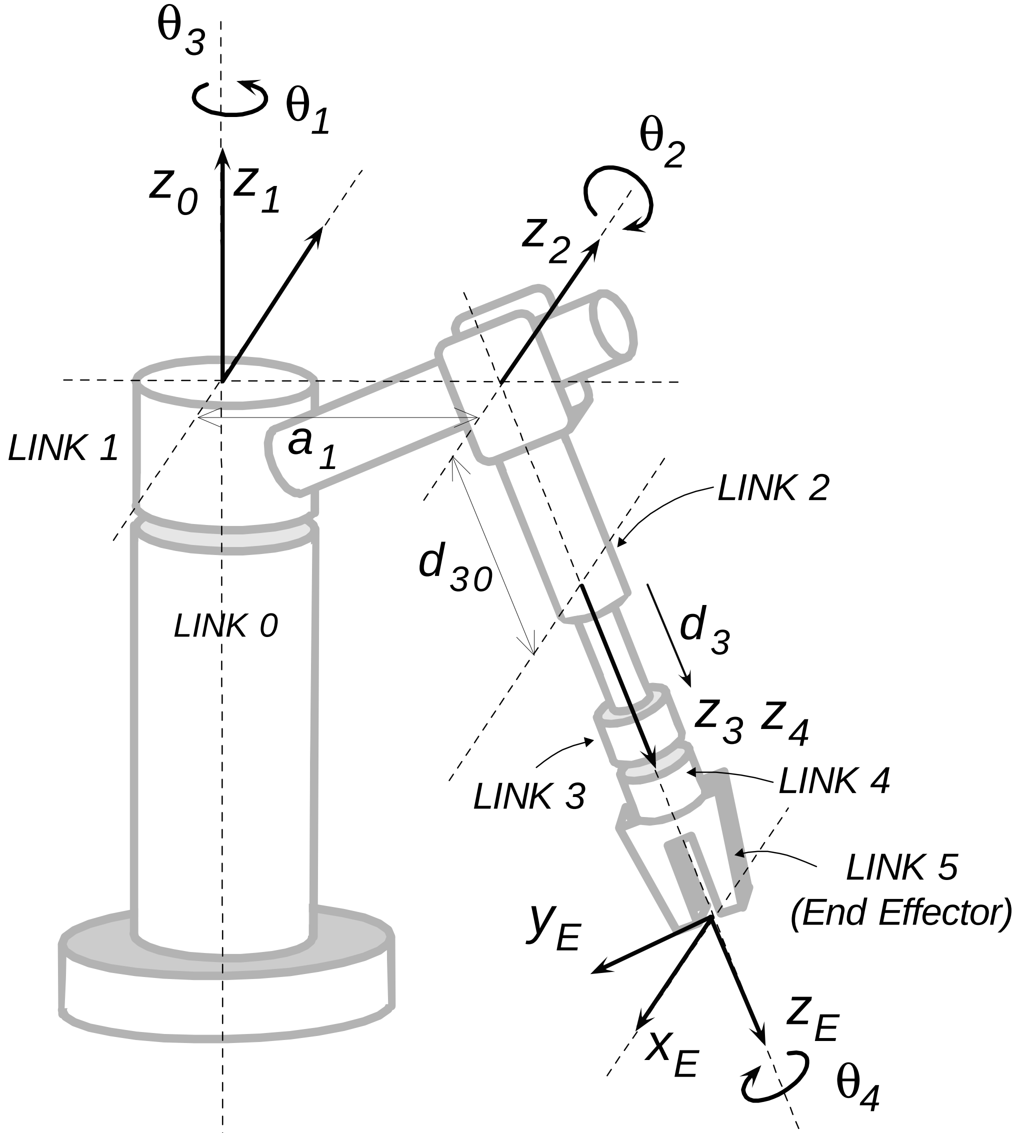
After multiplying the parts, we can get for the arm:

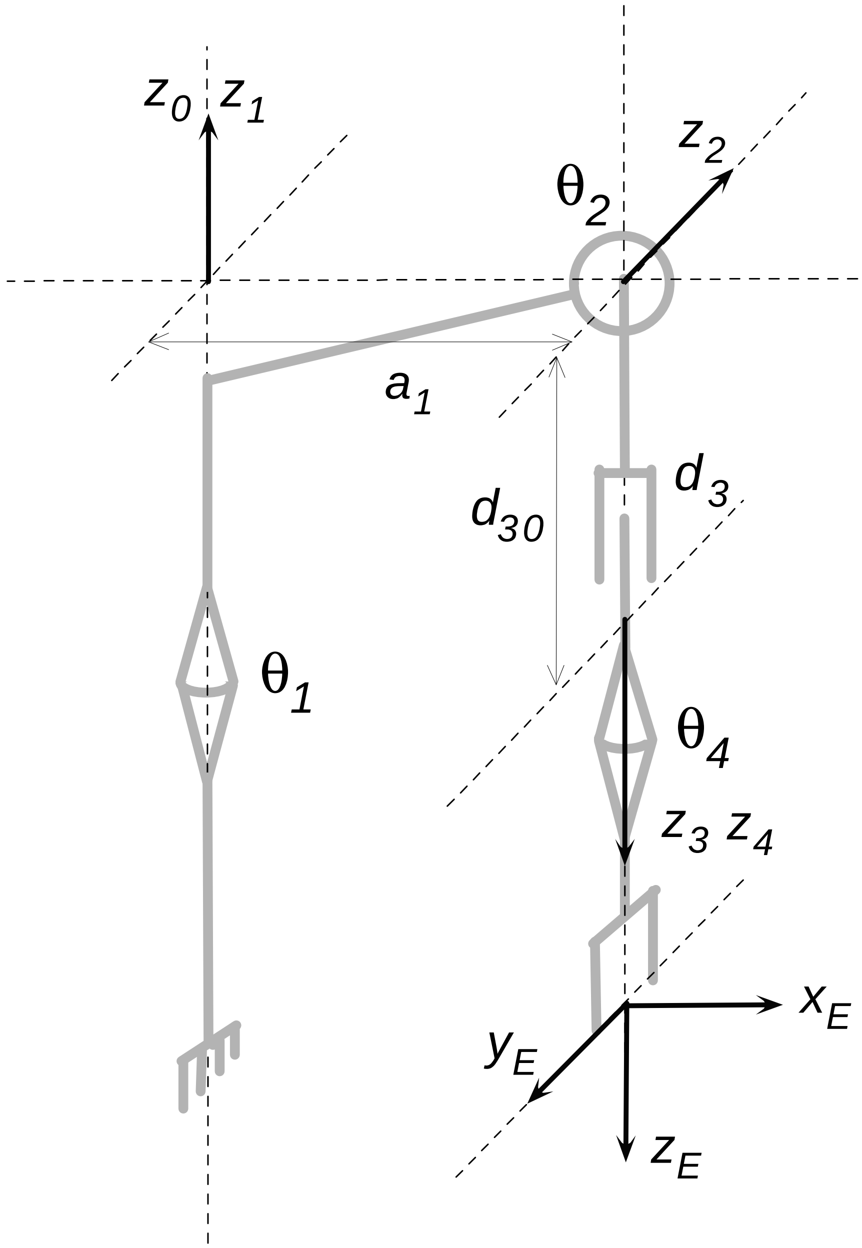
$$\begin{aligned} R_A &= \begin{bmatrix} c_1\, c_{23} & s_1 & -c_1\, s_{23} \\ s_1\, c_{23} & -c_1 & -s_1\, s_{23} \\ -s_{23} & 0 & -c_{23} \end{bmatrix} \\ p_A &= \begin{bmatrix} c_1\,(c_{23}\, a_3 - s_{23}\, d_4 + c_2\, a_2) - s_1\, d_3 \\ s_1\,(c_{23}\, a_3 - s_{23}\, d_4 + c_2\, a_2) + c_1\, d_3 \\ -s_{23}\, a_3 - c_{23}\, d_4 - s_2\, a_2 \end{bmatrix} \end{aligned} \tag{16}$$

Exercise:

Given is a four DOF manipulator shown in the figures on the following two pages. The home configuration is shown in the second figure. The z-axes in both figures indicate the positive orientation of the joint displacements. Suppose a link offset for the prismatic joint as indicated in the figures. Also suppose that the base frame is defined as shown. Do the following:

1. Write-in the x- and y- axes for all links.
2. Compose the DH table.
3. Derive all relative HT matrices.
4. Derive the HT matrix of the wrist with respect to the base.


θ3
θ1
θ2
z0
z1
z2
LINK 1
a1
LINK 2
d30
LINK 0
d3
z3 z4
LINK 3
LINK 4
LINK 5
(End Effector)
yE
zE
xE
θ4



z_0 z_1
z_2
θ_2
a_1
d_3
d_{30}
θ_1
θ_4
z_3 z_4
x_E
y_E
z_E

1.8 Inverse Kinematics

The inverse kinematics discussed in section 1.1 consist of finding the joint configuration $\mathbf{q}$ for a given Cartesian pose of the manipulator's wrist (or end-effector) ${}^{B}_{W}T = {}^{0}_{n}T$. There is no general approach for the inverse kinematics, as we had in forward kinematics. In general case there is not even a closed-form solution for it. Pieper[3] has shown that a manipulator has a closed form solution (in analytic form) if:

1. The manipulator has six DOF,
2. Its three consecutive axes intersect in a single point, or
3. Its three consecutive axes are parallel

Most of the industrial robots satisfy these conditions.. Such manipulators are called *simple manipulators*.

Here we will show examples of deriving an analytic solution for the inverse kinematics of the AdeptOne and Puma 560 manipulators. As will be shown, the approach is heavily based on the solution of the transcendental equations of the form $c_i\,p - s_i\,q = h$ or $s_i\,p + c_i\,q = g$, (see Appendix B).

[3] D. Pieper: "The Kinematics of Manipulators Under Computer Control," Ph.D. Thesis, Stanford University, 1968

1.8.1 Example 1: AdeptOne Robot

Closed-form solution

The HT matrix of the manipulator's wrist, according to (14), is:

$$
{}^0_4T = \begin{bmatrix} c_{124} & -s_{124} & 0 & c_1\,a_1 + c_{12}\,a_2 \\ s_{124} & c_{124} & 0 & s_1\,a_1 + s_{12}\,a_2 \\ 0 & 0 & 1 & d_3 - d_{30} \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

We start with the position first:

$$
\mathbf{p} = \begin{bmatrix} c_1\,a_1 + c_{12}\,a_2 \\ s_1\,a_1 + s_{12}\,a_2 \\ d_3 - d_{30} \end{bmatrix} \tag{17}
$$

where $\mathbf{p}$ is a given vector. The joint displacement d_3 can be found immediately:

$$
d_3 = p_3 + d_{30}
$$

For the x- and y- component of the position we can write:

$$
\begin{aligned} c_{12}\,a_2 &= p_1 - c_1\,a_1 \\ s_{12}\,a_2 &= p_2 - s_1\,a_1 \end{aligned} \tag{18}
$$

After squaring and adding these two equations we can eliminate the angle $\theta_{12} = \theta_1 + \theta_2$:

$$
a_2^2 = p_1^2 + p_2^2 + a_1^2 - 2\,a_1\,(s_1\,p_2 + c_1\,p_1)
$$

which is rearranged to become:

$$
s_1\,p_2 + c_1\,p_1 = b_1 \tag{19}
$$

where:

$$
b_1 = \frac{p_1^2 + p_2^2 + a_1^2 - a_2^2}{2\,a_1} \tag{20}
$$

We will add now to the equation (19) its accompanying equation $c_1\,p_2 - s_1\,p_1 = \sigma_1$ as shown in Appendix B. These equations written together are:

$$
\begin{aligned} c_1\,p_2 - s_1\,p_1 &= \sigma_1 \\ s_1\,p_2 + c_1\,p_1 &= b_1 \end{aligned} \tag{21}
$$

or in the matrix form:

$$
\begin{bmatrix} c_1 & -s_1 \\ s_1 & c_1 \end{bmatrix} \begin{bmatrix} p_2 \\ p_1 \end{bmatrix} = \begin{bmatrix} \sigma_1 \\ b_1 \end{bmatrix} \tag{22}
$$

As discussed in Appendix B, the latter equation represents a two-dimensional rotation about the z-axis by θ_1 which directly suggests the solution:

$$\theta_1 = atan2(b_1, \sigma_1) - atan2(p_1, p_2) \tag{23}$$

where[4] σ_1 can be obtained from $p_1^2 + p_2^2 = \sigma_1^2 + b_1^2$, i.e.:

$$\sigma_1 = \pm\sqrt{p_1^2 + p_2^2 - b_1^2}$$

After swapping the arguments of the $atan2()$ functions (23)can also be written :

$$\theta_1 = atan2(p_2, p_1) - atan2(\sigma_1, b_1) \tag{24}$$

Once we have θ_1, and therefore s_1 and c_1, we can use (18) to find θ_2:

$$\theta_2 = atan2(p_2 - s_1\, a_1, p_1 - c_1\, a_1) - \theta_1 \tag{25}$$

However, we will rewrite the solution for θ_2 in a more compact and structured form. By appropriate multiplications of equations (18) by p_1 and p_2 and by adding and subtracting them, we can get:

$$\begin{aligned} a_2(c_{12}\, p_2 - s_{12}\, p_1) &= -a_1(c_1\, p_2 - s_1\, p_1) \\ a_2(s_{12}\, p_2 + c_{12}\, p_1) &= p_1^2 + p_2^2 - a_1(s_1\, p_2 + c_1\, p_1) \end{aligned}$$

The expressions in parentheses at the right hand side can replaced by σ_1 and b_1 according to the equation (21):

$$\begin{aligned} c_{12}\, p_2 - s_{12}\, p_1 &= \frac{-a_1\sigma_1}{a_2} \\ s_{12}\, p_2 + c_{12}\, p_1 &= \frac{a_1 b_2}{a_2} \end{aligned}$$

where:

$$b_2 = \frac{p_1^2 + p_2^2 - a_1 b_1}{a_1} = \frac{p_1^2 + p_2^2 - a_1^2 + a_2^2}{2a_1} \tag{26}$$

which gives us:

$$\theta_1 + \theta_2 = atan2(b_2, -\sigma_1) - atan2(p_1, p_2)$$

or after reversing the arguments and other minor rearrangements:

$$\begin{aligned} \theta_2 &= atan2(\sigma_1, b_2) + atan2(p_2, p_1) - \theta_1 \\ &= atan2(\sigma_1, b_1) + atan2(\sigma_1, b_2) \end{aligned}$$

An important special case we have if the robot is <u>*symmetric*</u> with two elbow links with equal length, i.e. $a_1 = a_2 = a$, then:

$$\begin{aligned} b_1 &= b_2 = \frac{p_1^2 + p_2^2}{2\,a} \\ \theta_2 &= 2\, atan2(\sigma_1, b_1) \end{aligned}$$

[4] Note, we have used the symbol σ by purpose, since this quantity can have two signs (sigma sounds as "signum" = sign).

So far we have found angles θ_1, θ_2 and displacement d_3 for the given position of the wrist $\mathbf{p}$. Now we need to find angle θ_4, so as to achieve a desired orientation. Since this is not a full 3D manipulator (it has only four DOF), it can not have arbitrary orientations. Recall the orientation matrix

$$
{}^0_4R = \begin{bmatrix} c_{124} & -s_{124} & 0 \\ s_{124} & c_{124} & 0 \\ 0 & 0 & 1 \end{bmatrix}
$$

It is obvious that this robot can only have an angle about the z-axis (this is also apparent from the figure on page 19). If φ_z is the desired orientation, then the last joint becomes:

$$
\theta_4 = \varphi_z - \theta_1 - \theta_2
$$

The inverse kinematics for the AdeptOne Robot we can summarize as follows:

$$\boxed{\begin{array}{l} \rho^2 = p_1^2 + p_2^2 \\ b_1 = \dfrac{\rho^2 + (a_1^2 - a_2^2)}{2\,a_1} \\ b_2 = \dfrac{\rho^2 - (a_1^2 - a_2^2)}{2a_1} \\ \sigma_1 = \pm\sqrt{\rho^2 - b_1^2} \\ \\ \theta_1 = atan2(p_2, p_1) - atan2(\sigma_1, b_1) \\ \theta_2 = atan2(\sigma_1, b_1) + atan2(\sigma_1, b_2) \\ d_3 = p_3 + d_{30} \\ \theta_4 = \varphi_z - \theta_1 - \theta_2 \end{array}} \tag{27}$$

The symmetric case:

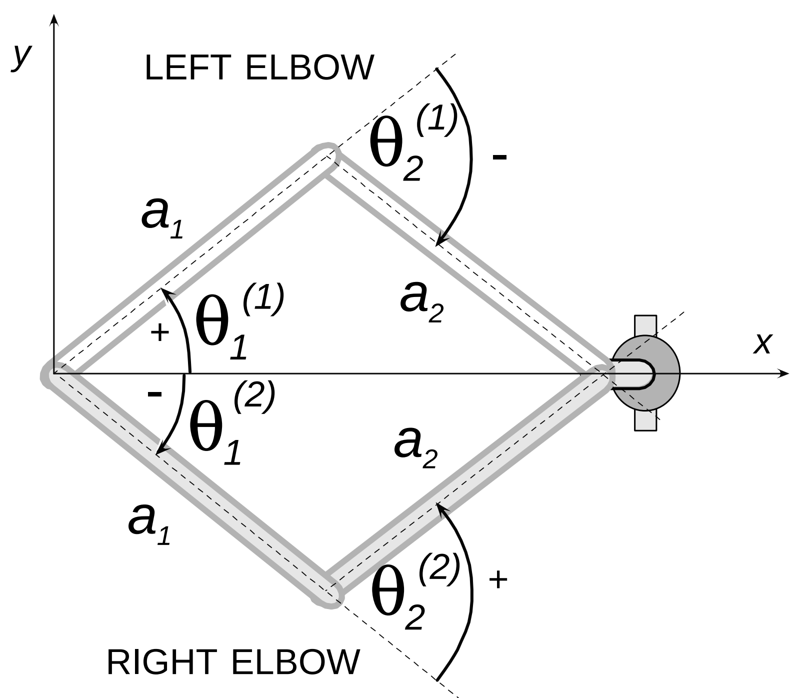
$$\boxed{\begin{array}{l} b_1 = \dfrac{\rho^2}{2\,a} \\ \sigma_1 = \pm\sqrt{\rho^2 - b_1^2} \\ \\ \theta_2 = 2\,atan2(\sigma_1, b_1) \\ \theta_1 = atan2(p_2, p_1) - \frac{1}{2}\theta_2 \\ d_3 = p_3 + d_{30} \\ \theta_4 = \varphi_z - \theta_1 - \theta_2 \end{array}}$$

The variable ρ denotes the reach of the manipulator in x-y plane, i.e. the projection of $\mathbf{p}$ onto the x-y plane

Multiplicity of solutions:

The double sign of σ_1 indicates the multiplicity of solutions for the inverse kinematics. In case of the AdeptOne Robot there are obviously two solutions (we'll consider the symmetric case only):

Left Elbow Solution:	*Right Elbow Solution:*
$\theta_2^{(1)} = 2\,atan2(-\left\|\sigma_1\right\|, b)$	$\theta_2^{(2)} = 2\,atan2(+\left\|\sigma_1\right\|, b)$
$\theta_1^{(1)} = atan2(p_2, p_1) - \frac{1}{2}\theta_2^{(1)}$	$\theta_1^{(2)} = atan2(p_2, p_1) - \frac{1}{2}\theta_2^{(2)}$
$\theta_4^{(1)} = \varphi_z - \theta_1^{(1)} - \theta_2^{(12)}$	$\theta_4^{(2)} = \varphi_z - \theta_1^{(2)} - \theta_2^{(2)}$



Existence of the solution

In order to have a real solution, the argument of the square root in equation (27) must be non-negative, i.e. $\rho - b_1 \geq 0$. This condition can be written:

$$\rho - \frac{\rho^2 + (a_1^2 - a_2^2)}{2\,a_1} \geq 0$$

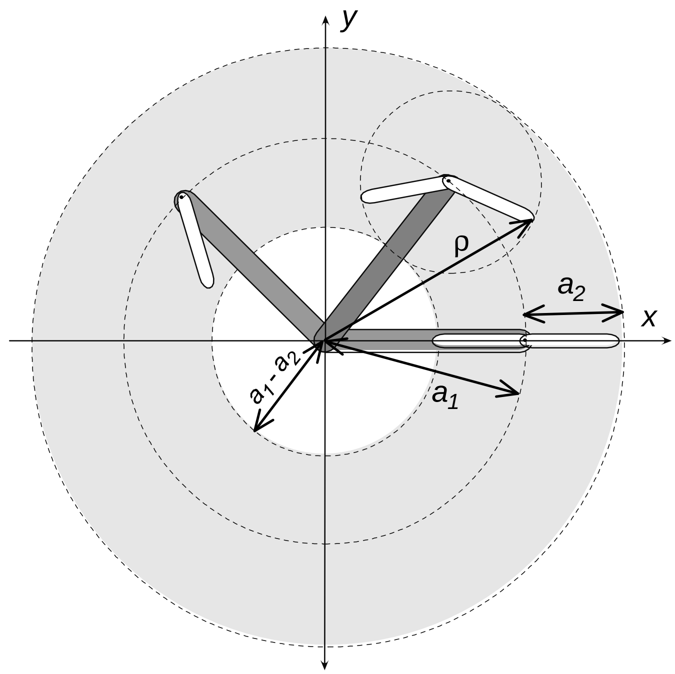
After rearranging we get the following polynomial inequality:

$$\rho^2 - 2a_1\rho + (a_1^2 - a_2^2) \leq 0 \tag{28}$$

The roots of the equation (28) are $\rho_1 = a_1 - a_2$ and $\rho_2 = a_1 + a_2$, therefore the condition (28) is satisfied if:

$$\boxed{a_1 - a_2 \leq \rho \leq a_1 + a_2} \tag{29}$$

This is illustrated in the figure on the next page.



The shaded area represents the subspace of all possible desired positions for which we have a real solution for the inverse kinematics equations (27). This space is called the *dexterous work space*. Note this is only a two-dimensional projection of the complete dexterous workspace, which obviously has a cylindrical shape. In the case of a symmetric robot, $a_1 = a_2$, the entire area around the center would be covered. This shows that the symmetrical robot is optimal from that point of view.

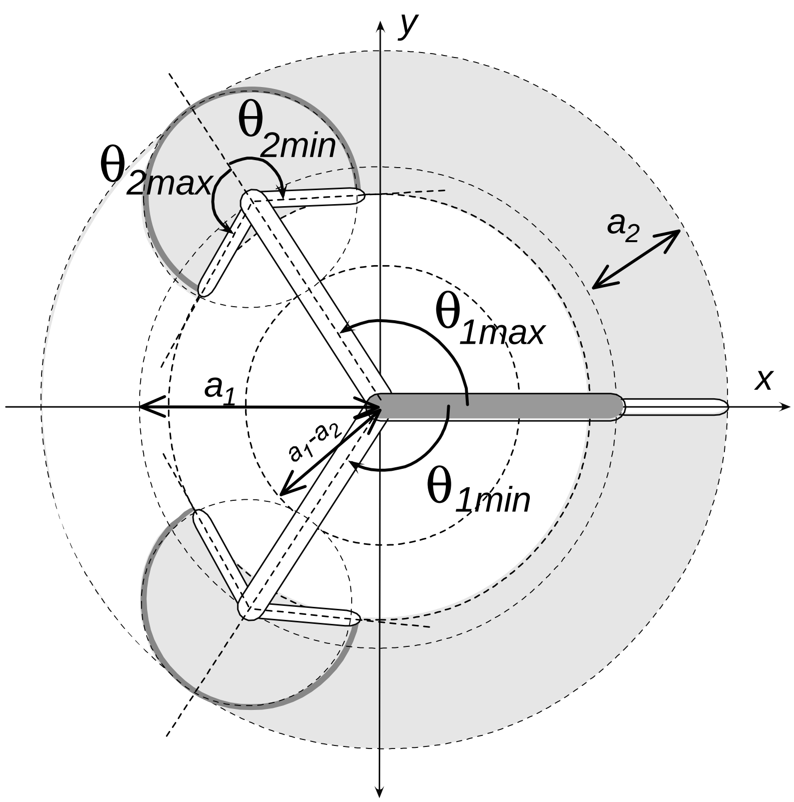
Remark 2 *The dexterous workspace considered here is not strictly dexterous, since the wrist (or end-effector) can not have arbitrary orientation in 3D space. However, since this robot is expected to have an arbitrary orientation in the x-y plane while the other orientation angles are ignored, we can call it "dexterous".*

Limitation of Joints

In real life, the joints don't have infinite freedom. For example, in the case of the AdeptOne Robot, the first two joints are limited:

$$\begin{aligned} \theta_{1\,\min} &\leq \theta_1 \leq \theta_{1\,\max} \\ \theta_{2\,\min} &\leq \theta_2 \leq \theta_{2\,\max} \end{aligned}$$

The effect of limiting joints is a severe reduction of the dexterous workspace, which is illustrated in the figure below.



1.8.2 Spherical Wrists

As mentioned earlier, most of the industrial manipulators are designed to be simple, with three perpendicular wrist joint axes which intersect in a single point - *the spherical wrist center*. In this subsection we show that the wrist kinematics of such manipulators can be represented in a unique form of z-y-z Euler angles. Therefore, the problem of the inversion of the wrist kinematics can be reduced to the inversion of Euler angles, which was discussed in subsection **??**.

The orientation of the wrist of the Puma 560 Manipulator (15), rewritten in non-homogenous form is:

$$R_W = rot(\mathbf{e}_3, \theta_4)\, rot(\mathbf{e}_1, \tfrac{\pi}{2})\; rot(\mathbf{e}_3, \theta_5)\, rot(\mathbf{e}_1, -\tfrac{\pi}{2})\, rot(\mathbf{e}_3, \theta_6) \tag{30}$$

By reviewing the figures in chapter 2, it can be seen that the same equation holds for the Stanford/JPL manipulator. The basic characteristics of these manipulators is that the joint axes 4 and 6 are aligned in the home configuration.

We will now make use of the property of the $rot()$ operator shown in subsection **??**:

$$R\, rot(\mathbf{k}, \varphi)\, R^T = rot(R\mathbf{k}, \varphi) \tag{31}$$

In the particular case: $R = rot(\mathbf{e}_1, \frac{\pi}{2})$ and $\mathbf{k} = \mathbf{e}_3, \varphi = \theta$ we have:

$$rot(\mathbf{e}_1, \tfrac{\pi}{2})\, rot(\mathbf{e}_3, \theta)\, rot(\mathbf{e}_1, -\tfrac{\pi}{2}) = rot(\, rot(\mathbf{e}_1, \tfrac{\pi}{2})\, \mathbf{e}_3, \theta) \tag{32}$$

Clearly, the rotation of $\mathbf{e}_3$ about $\mathbf{e}_1$ by $\frac{\pi}{2}$ gives $-\mathbf{e}_2$, and consequently:

$$rot(\mathbf{e}_1, \tfrac{\pi}{2})\, rot(\mathbf{e}_3, \theta)\, rot(\mathbf{e}_1, -\tfrac{\pi}{2}) = rot(\mathbf{e}_2, -\theta) \tag{33}$$

Replacing now the appropriate terms of (30) by (33) we get the z-y-z version of (30):

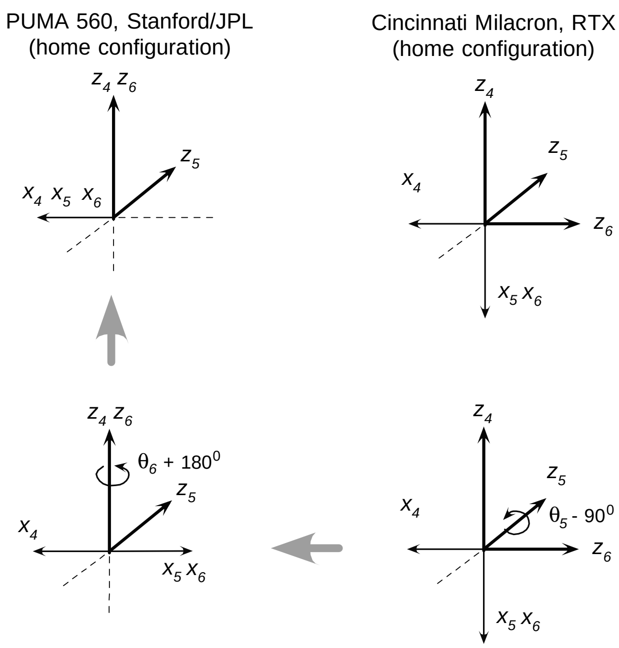
$$R_W = rot(\mathbf{e}_3, \theta_4)\ rot(\mathbf{e}_2, -\theta_5)\ rot(\mathbf{e}_3, \theta_6)$$

This can also be written in a more concise form:

$$R_W = ZYZ(\theta_4, -\theta_5, \theta_6) \tag{34}$$

where $ZYZ(\alpha, \beta, \gamma)$ represents the mapping of z-y-z Euler angles into the rotation matrix, discussed in subsection **??**

Other manipulators described in chapter 2 are Cincinnati Milacron (T^3) and RTX where the neighboring wrist axes are mutually perpendicular for the home configuration, as shown in the figure below.



However, by introducing the appropriate offsets to last two joints, the kinematic frames can be converted to the arrangement of Puma 560 and Stanford/JPL manipulators. The corresponding wrist orientation will then be:

$$R_W = \\ rot(\mathbf{e}_3, \theta_4)\, rot(\mathbf{e}_1, \tfrac{\pi}{2})\, rot(\mathbf{e}_3, \theta_5 - \tfrac{\pi}{2})\, rot(\mathbf{e}_1, -\tfrac{\pi}{2})\, rot(\mathbf{e}_3, \theta_6 + \pi) \tag{35}$$

which results in the following z-y-z Euler angle equivalent:

$$\boxed{R_W = ZYZ(\theta_4, \tfrac{\pi}{2} - \theta_5, \theta_6 + \pi\,)} \tag{36}$$

In general, all spherical wrists can be represented in the form:

$$R_W = ZYZ(\theta_4, \theta_5, \theta_6)$$

where the arguments θ_4, θ_5 and θ_6 are taken with an appropriate offset.

The total orientation of the manipulator is:

$$R = R_A R_W$$

where R_A is orientation contributed by the arm part, while R_W is orientation of the wrist which should compensate for R_A and to realize the desired total orientation R. Therefore the wrist equation becomes:

$$\boxed{ZYZ(\theta_4, \theta_5, \theta_6) = R_A^T R} \tag{37}$$

This equation can be solved analytically for θ_4, θ_5 and θ_6 by using expressions (**??**) presented in subsection **??**

1.8.3 Example 2: Puma 560 Manipulator

Inverse Kinematics of the Arm

It was established in section 1.7.5 that the manipulator distance is fully determined by the arm distance as shown in (16). Therefore we will use this equation to find first three joint angles for a given Cartesian position of the robot wrist $\mathbf{p} = (p_1, p_2, p_3)$:

$$\begin{aligned} p_1 &= c_1\,(c_{23}\,a_3 - s_{23}\,d_4 + c_2\,a_2) - s_1\,d_3 \\ p_2 &= s_1\,(c_{23}\,a_3 - s_{23}\,d_4 + c_2\,a_2) + c_1\,d_3 \\ p_3 &= -s_{23}\,a_3 - c_{23}\,d_4 - s_2\,a_2 \end{aligned} \tag{38}$$

If we introduce the substitution:

$$\sigma_1 = c_{23}\,a_3 - s_{23}\,d_4 + c_2\,a_2 \tag{39}$$

the first two equations of (38) become:

$$\begin{aligned} c_1\,\sigma_1 - s_1\,d_3 = p_1 \\ s_1\,\sigma_1 + c_1\,d_3 = p_2 \end{aligned} \tag{40}$$

which directly gives:

$$\sigma_1 = \pm\sqrt{p_1^2 + p_2^2 - d_3^2} \tag{41}$$

$$\theta_1 = atan2(\sigma_1, d_3) - atan2(p_1, p_2) \tag{42}$$

The third equation of (38) and the equation (39) written together:

$$\begin{aligned} c_{23}\,a_3 - s_{23}\,d_4 + c_2\,a_2 &= \sigma_1 \\ s_{23}\,a_3 + c_{23}\,d_4 + s_2\,a_2 &= -p_3 \end{aligned}$$

which after breaking s_{23} and c_{23}, and rearranging gives:

$$\begin{aligned} c_2\,(b_2 + a_2) - s_2\,\sigma_2 &= \sigma_1 \\ s_2\,(b_2 + a_2) + c_2\,\sigma_2 &= -p_3 \end{aligned} \tag{43}$$

where:

$$\begin{aligned} b_2 &= c_3\,a_3 - s_3\,d_4 \\ \sigma_2 &= s_3\,a_3 + c_3\,d_4 \end{aligned} \tag{44}$$

The first two equations (43) can be used to determine b_2. Since the equations represent a 2D rotation, it follows:

$$(b_2 + a_2)^2 + \sigma_2^2 = \sigma_1^2 + p_3^2 = p^2 - d_3^2 \tag{45}$$

where:

$$p = |\mathbf{p}| = \sqrt{p_1^2 + p_1^2 + p_3^2}$$

Similarly, the second two equations (44)give:

$$b_2^2 + \sigma_2^2 = a_3^2 + d_4^2 \tag{46}$$

Equations (45) and (46) can be solved for b_2 and σ_2:

$$b_2 = \frac{p^2 - a_2^2 - a_3^2 - d_3^2 - d_4^2}{2a_2} \tag{47}$$

$$\sigma_2 = \pm\sqrt{a_3^2 + d_4^2 - b_2^2} \tag{48}$$

Solutions for θ_2 and θ_3 directly follow from (43) and (44):

$$\theta_2 = atan2(-p_3, \sigma_1) - atan2(\sigma_2, b_2 + a_2) \tag{49}$$

$$\theta_3 = atan2(\sigma_2, b_2) - atan2(d_4, a_3)$$

Expression for θ_2 can also be written in more convenient form:

$$\theta_2 = atan2(\sigma_1, p_3) - atan2(\sigma_2, b_2 + a_2) - \frac{\pi}{2} \tag{50}$$

Inverse kinematics of the wrist

The general equation for the spherical wrist (37) applied to the Puma 560 Manipulator (see (34)) is:

$$ZYZ(\theta_4, -\theta_5, \theta_6) = G \tag{51}$$

where $G = R_A^T\, R$ can be evaluated for the given R and R_A derived in subsection 1.7.5. By using (16) it can easily be shown that the elements of G are:

$$\begin{aligned} g_{1i} &= (c_1\, c_{23})\, r_{1i} + (s_1\, c_{23})\, r_{2i} - s_{23}\, r_{3i} \\ g_{2i} &= s_1\, r_{1i} - c_1\, r_{2i} \\ g_{3i} &= -(c_1\, s_{23})\, r_{1i} - (s_1\, s_{23})\, r_{2i} - c_{23}\, r_{3i} \end{aligned}$$

Solution of (51) can now be directly obtained from (**??**) i.e. (**??**):

$$\boxed{\begin{array}{l} \underline{\textit{Regular case}}\ (\theta_5 \neq 0^o, 180^o) \\[1em] \sigma_3 = \pm\sqrt{g_{31}^2 + g_{32}^2} \\[1em] \theta_4 = \left\{ \begin{array}{ll} -\pi - \arctan 2(g_{23}, g_{13}) & \text{if } \ \sigma_3 < 0 \\ \arctan 2(g_{23}, g_{13}) & \text{if } \ \sigma_3 > 0 \end{array} \right. \\[1em] \theta_5 = -\arctan 2(\sigma_3, g_{33}) \\[1em] \theta_6 = \left\{ \begin{array}{ll} -\arctan 2(g_{32}, g_{31}) & \text{if } \ \sigma_3 < 0 \\ \pi - \arctan 2(g_{32}, g_{31}) & \text{if } \ \sigma_3 > 0 \end{array} \right. \end{array}} \tag{52}$$

Degenerate cases:

$\theta_5 = 0^o \qquad (\theta_5 = 0^o;\ \sigma_3 = 0;\ g_{33} \approx +1)$
$\theta_4 + \theta_6 = atan2(-g_{12}, g_{11})$

$\theta_5 = \pi \qquad (\theta_5 = 180^o;\ \sigma_3 = 0;\ g_{33} \approx -1)$
$\theta_4 - \theta_6 = atan2(g_{12}, -g_{11})$

Remark 3 *The configuration of the manipulator for which the solution is degenerate is called the* singular configuration. *Singular configurations and the related problems will be discussed in the following chapter.*

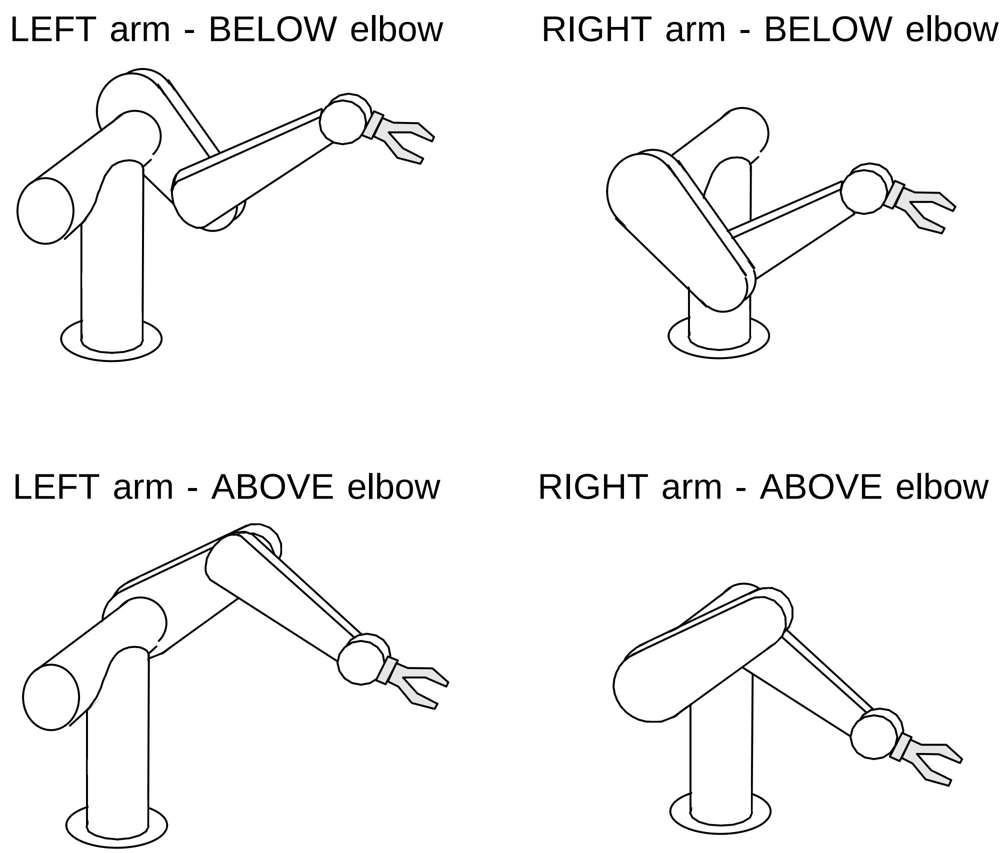
Remark 4 *A singular configuration can be recognized right after the computation of the arm joints, from the coefficients of the matrix G:* $g_{31} = g_{32} = 0$, *or* $g_{33} = \pm 1$. *These values correspond to* $\sigma_3 = 0$, *or* $\theta_5 = 0, \pi$.

Remark 5 *The singular configuration* $\theta_5 = \pi$ *is only theoretical. In real life, this value can never occur due to the limits of the 5-th joint (see page 26.)*

Remark 6 *In a singular configuration only the sum (difference) between* θ_4 *and* θ_6 *can be determined. In order to resolve the ambiguity, the most practical approach is to use the* most recent *value of* θ_4. *This means that if the manipulator moves into a new pose which is singular, then simply do not change the 4-th joint, i.e. keep its current value and determine the 6-th joint from* $\theta_6 = \text{atan2}(-g_{12}, g_{11}) - \theta_4$.

Multiplicity of solutions:

The multiplicity of the solutions for the Puma 560 Manipulator is the result of having two signs for the three variables: σ_1, σ_2, and σ_3. Therefore we have $2^3 = 8$ possible solutions for each nonsingular configuration (the arm has four and the wrist has two solutions.) The arm solutions are illustrated in the figure below:



Existence of the solution

The existence of real solutions for Puma 560 is determined by two square roots (41) and (48). The third square root (52) can never be imaginary and can be excluded from the discussion. The reality of the first two square roots is determined by the following inequalities:

$$\begin{aligned} p_1^2 + p_2^2 - d_3^2 &\geq 0 \\ a_3^2 + d_4^2 - \left(\frac{p^2 - a_2^2 - a_3^2 - d_3^2 - d_4^2}{2a_2} \right)^2 &\geq 0 \end{aligned}$$

which after some rearrangements become:

$$\boxed{\begin{aligned} p_1^2 + p_2^2 &\geq d_3^2 \\ p_1^2 + p_2^2 + p_3^2 &\leq a_2^2 + a_3^2 + d_3^2 + d_4^2 + 2a_2\sqrt{a_3^2 + d_4^2} \end{aligned}} \tag{53}$$

Remark 7 *These inequalities can be easily derived from the geometry of the manipulator shown in figures on pages 23 and 25.*

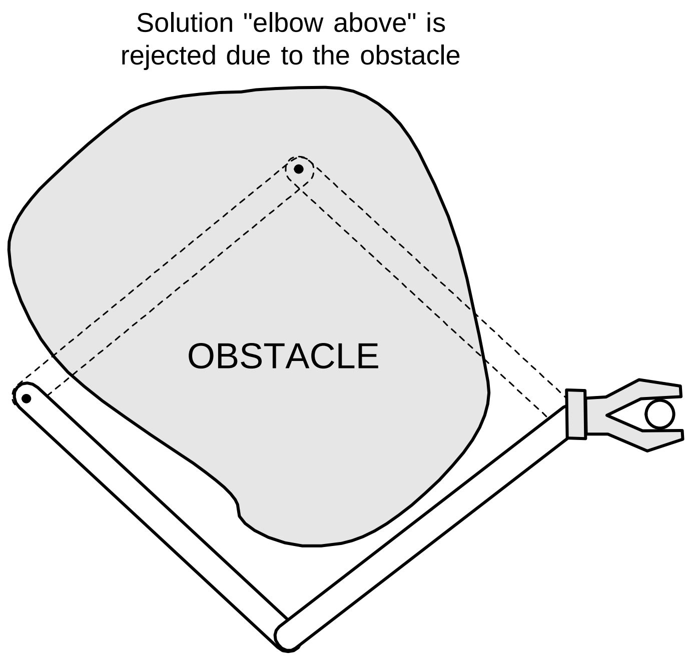
Remark 8 *The first inequality does not allow the wrist to approach too close to the manipulator's shoulder, while the second inequality prevents the manipulator from stretching beyond the maximum reach (when the arm is fully stretched).*

1.8.4 Resolving The Ambiguity

In order to choose a single solution from among the several possible solutions, some additional rules are required. There are basically three important criteria to be considered:

<u>*Obstacle avoidance*</u>

Sometimes the choice of an alternative solution can help to avoid an obstacle (see illustration)



Maximum distance from the joint limits

Joint limits are always a painful restriction when performing the manipulation tasks. Therefore it is a good practice to keep the joints away from the joint limits $q_{j\min}$, $q_{j\max}$, and close to the middle of the joint range $q_{jmid} = (q_{j\min} + q_{j\max})/2$. Consequently the solutions can be evaluated from the following criteria:

$$F\left(k\right) = \sum_{j=1}^{N} w_j \left(\frac{q_j^{(k)} - q_{jmid}}{q_{j\max} - q_{j\min}} \right)^2 \qquad k = 1, 2, ..., N_{sol} \tag{54}$$

where N is number of joints (DOF), N_{sol} is the total number of solutions, while w_j are weighting coefficients ($\sum w_j = 1$). These coefficients can be used to give a higher priority to some joints (if all joints are equally important then $w_j = 1$ for all j.)

After evaluating all solutions, we choose the one which has minimal value of $F\left(k\right)$, i.e.

$$F(k_{opt}) = \min_k F\left(k\right).$$

Minimal joint travel (The closest solution)

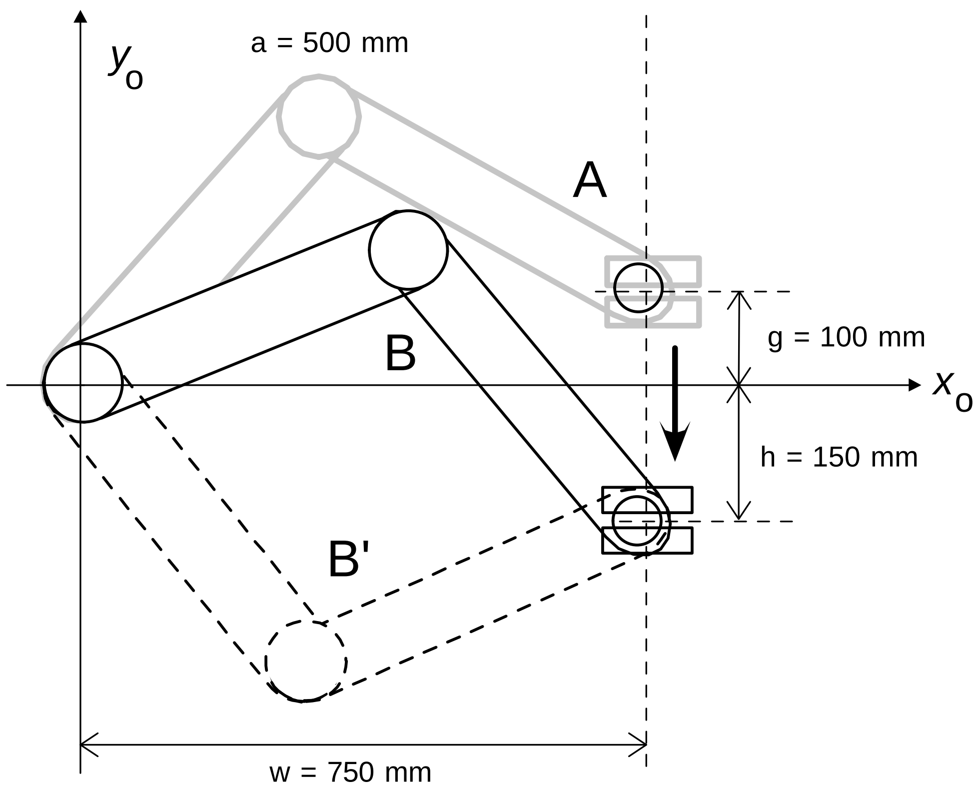
Another possible optimization criteria is to choose a solution which minimizes the total joint motion of the manipulator. This can make the motion more natural and will also minimize the wear of the movable parts in joint transmission (gears, ball bearings etc.). The minimization function in that case would be:

$$F(k) = \sum_{j=1}^{N} w_j \left| q_j^{(k)} - q_{jcurrent} \right| \qquad k = 1, 2, ..., N_{sol} \tag{55}$$

where $q_{jcurrent}$ are the current joint positions, i.e. $\left| q_j^{(k)} - q_{jcurrent} \right|$ is absolute value of the joint travel when going from some current position to the new position determined by the inverse kinematics. The weighting coefficients are usually defined according to the size of the joints. For example, the lower numbered joints are usually bigger since they carry all the other higher-numbered links and joints. Therefore the weights with smaller index j should generally have larger values.

Example

Given is a symmetric AdeptOne type robot with $a = 500$ mm and $d_{30} = -200$ mm. The robot is in configuration A as shown in the figure below. The robot is to move to configuration B or B' which are also shown in the figure. The latter two configurations have identical poses for the wrist. Suppose that all three configurations have the same elevation of the wrist, and that the orientation of the wrist in all configurations is $\varphi_z = 0^o$. (a) Find values of all joints for the three configurations A, B and B'. (b) Evaluate the configurations B and B' in terms of two criteria: minimal joint travel and maximal distance from joint limits. In both criteria use equal weights for all joints. Suppose that the joint limits are as follows: $|\theta_1| \leq 170^o, |\theta_2| \leq 150^o, |\theta_4| \leq 180^o$.



From the figure above we can determine the position of the wrist for the two configurations:

$$\mathbf{p}_A = \begin{bmatrix} 750 \\ 100 \\ \zeta \end{bmatrix}, \qquad \mathbf{p}_B = \begin{bmatrix} 750 \\ -150 \\ \zeta \end{bmatrix}$$

The elevation is not important in this exercise so we ignore it. By using formulas for inverse kinematics for AdeptOne Robot (27), we can obtain the following values:

$Conf.$	σ	θ_1	θ_2	θ_4
A	$-$	48.4^o	-81.7^o	33.2^o
$A^{'}$	$+$	-33.2^o	81.7^o	-48.4^o
B	$-$	28.8^o	-80.2^o	51.4^o
$B^{'}$	$+$	-51.4^o	80.2^o	-28.8^o

By comparing the angles from the table and those from the figure we can conclude that the actual initial configuration A is the one with the negative σ. Therefore, we ignore the second solution for the initial configuration $A^{'}$.

The configuration $\mathbf{q}_A$ we use now as the initial configuration needed to determine the total joint travel F_t by using the formula (55). Also, by using the formula (54) we can find the value for the criterion function for the maximal distance from the joint limits, F_l. The values for these criterion functions for two solutions, B and $B^{'}$ are as follows:

$Config.$	F_l	F_t
B	0.1817	39.4^o
$B^{'}$	0.1833	323.7^o

We can see that the solution B is superior over $B^{'}$ in terms of the minimal travel criteria. As far as joint limits are concerned, both solutions are about equal.